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CRITICAL YIELD NUMBERS OF RIGID PARTICLES SETTLING IN BINGHAM FLUIDS AND CHEEGER SETS*

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Abstract. We consider the fluid mechanical problem of identifying the critical yield number Y_c of a dense solid inclusion (particle) settling under gravity within a bounded domain of Bingham fluid, i.e., the critical ratio of yield stress to buoyancy stress that is sufficient to prevent motion. We restrict ourselves to a two-dimensional planar configuration with a single antiplane component of velocity. Thus, both particle and fluid domains are infinite cylinders of fixed cross-section. We then show that such yield numbers arise from an eigenvalue problem for a constrained total variation. We construct particular solutions to this problem by consecutively solving two Cheeger-type set optimization problems. Finally, we present a number of example geometries in which these geometric solutions can be found explicitly and discuss general features of the solutions.

Key words. Bingham fluid, exchange flow, settling, critical yield number, total variation, Cheeger sets

AMS subject classifications. 49Q10, 76A05, 76T20, 49Q20

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1. Introduction.

One hundred years ago Eugene Bingham [9] presented results of flow experiments through a capillary tube, measuring the flow rate and pressure drop for various materials of interest. Unlike with simple viscous fluids, he recorded a "friction constant" (a stress) that must be exceeded by the pressure drop in order for flow to occur, and thereafter postulated a linear relationship between applied pressure drop and flow rate. This empirical flow law evolved into the Bingham fluid: the archetypical yield stress fluid. However, it was not until the 1920s that ideas of visco-plasticity became more established [10] and other flow laws were proposed; e.g., [27]. These early works were empirical and focused largely on viscometric flows. Proper tensorial descriptions, general constitutive laws, and variational principles waited until Oldroyd [42] and Prager [44]. These constitutive models are now widely used in a range of applications, in both industry and nature; see [5] for an up-to-date review.

An essential feature of Bingham fluids flows is the occurrence of plugs; that is, regions within the flow containing fluid that moves as a rigid body. This occurs when the deviatoric stress falls locally below the yield stress, which is a physical property

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of the fluid. Plug regions may occur either within the interior of a flow or may be attached to the wall. In general, as the applied forcing decreases, the plug regions increase in size and the velocity decreases in magnitude. It is natural that at some critical ratio of the driving stresses to the resistive yield stress of the fluid, the flow stops altogether. This critical yield ratio or *yield number* is the topic of this paper.

Critical yield numbers are found for even the simplest one-dimensional (1D) flows, such as Poiseuille flows in pipes and plane channels or uniform film flows, e.g., paint on a vertical wall. These limits have been estimated and calculated exactly for flows around isolated particles, such as the sphere [8] (axisymmetric flow) and the circular disc [46, 48] (two-dimensional (2D) flow). Such flows have practical application in industrial non-Newtonian suspensions, e.g., mined tailings transport, cuttings removal in drilling of wells, etc.

The first systematic study of critical yield numbers was carried out by Mosolov and Miasnikov [40, 41], who considered antiplane shear flows, i.e., flows with velocity $\mathbf{u} = (0, 0, w(x_1, x_2))$ in the x_3-direction along ducts (infinite cylinders) of arbitrary cross-section Ω. These flows driven by a constant pressure gradient only admit the static solution ($w(x_1, x_2) = 0$) if the yield stress is sufficiently large. Amongst the many interesting results in [40, 41] the key contributions relate to exposing the strongly geometric nature of calculating the critical yield number Y_c. First, they show that Y_c can be related to the maximal ratio of area to perimeter of subsets of Ω. Second, they develop an algorithmic methodology for calculating Y_c for specific symmetric Ω, e.g., rectangular ducts. This methodology is extended further by [29].

Critical yield numbers have been studied for many other flows, using analytical estimates, computational approximations, and experimentation. Critical yield numbers to prevent bubble motion are considered in [18, 50]. Settling of shaped particles is considered in [31, 45]. Natural convection is studied in [32, 33]. The onset of landslides is studied in [28, 30, 26] (where the terminologies “load limit analysis” and “blocking solutions” have also been used). In [22, 23] we have studied two-fluid antiplane shear flows that arise in oil field cementing.

In this paper we study critical yield numbers for two-phase antiplane shear flows, in which a particulate solid region Ω_s settles under gravity in a surrounding Bingham fluid of smaller density. As the particle settles downwards the surrounding fluid moves upwards, with zero net flow: a so-called *exchange flow*. Our objective is to derive new results that set out an analytical framework and algorithmic methodology for calculating Y_c for this class of flows.

Our analysis naturally leads to the so-called Cheeger sets, that is, minimizers of the ratio of perimeter to volume inside a given domain. Recently, starting with [34], many of their properties have been studied, particularly regularity and uniqueness in the case of convex domains [35, 12]. These sets constitute examples of explicit solutions to the total variation flow, which has motivated their investigation [3, 6, 7].

A related line of research is the use of total variation regularization in image processing. In particular, set problems like those treated here appear in image segmentation [15] and as the problem solved by the level sets of minimizers [14, 1, 13] of the Rudin–Osher–Fatemi functional [47]. The analogy between antiplane shear flows of yield stress fluids and image processing techniques has been exploited previously by the authors in the context of nonlinear diffusion filtering using total variation flows or bounded variation type regularization. In our previous work [21, 24] we exploited physical insights from the fluid flow problem in order to derive optimal stopping times for diffusion filtering.

1.1. Summary and outline. First, in section 2 we write the simplified Navier–Stokes equations and corresponding variational formulation for the inclusion of a Newtonian fluid in a Bingham fluid, in geometries consisting of infinite cylinders, and antiplane velocities.

Section 3 is dedicated to the background theory for the exchange flow problem. After proving existence of solutions, we make the viscosity of the inclusion tend to infinity, that is, we study the flow of a solid inclusion into a Bingham fluid.

We then recall the usual notion of critical yield number, seen as the supremum of an eigenvalue quotient (3.8) in the standard Sobolev space H^1, which writes after simplification as a minimization of total variation with constraints. Since it is well known that such a problem does not necessarily have a solution in H^1, we relax it, enlarging the admissible space to functions with bounded variation, which ensures the existence of a minimizer.

In section 4 we study the relaxed problem and show that we can construct minimizers that attain only three values, and whose level-sets are solutions of simple geometrical problems closely related to the Cheeger problem (see Definition 2). We show how the geometrical properties of Cheeger sets are reflected in the structure of our three level-set minimizer, and give several explicit examples exhibiting the influence of the geometry of the domain and the particles in that of the solution. In particular, we emphasize the role of nonuniqueness of Cheeger sets in the nonuniqueness of our minimizers.

Finally, section 5 is dedicated to the explicit construction of three-valued solutions and computing the corresponding yield numbers in simple situations.

It has to be noticed that the restriction to antiplane flows and equal particle velocities is fundamental in all this work. The in-plane case remains an exciting challenge.

2. Modelling. As discussed in section 1 we study antiplane shear flows of particles within a Bingham fluid. Antiplane shear flows have velocity in a single direction and the velocity depends on the two other coordinate directions. We assume the solid is denser than the fluid ($\hat{\rho}_f < \hat{\rho}_s$) and align the flow direction $\hat{x}_3$ with gravity. In the antiplane shear flow context, particles (solid regions) are infinite cylinders represented as $\hat{\Omega}_s \times \mathbb{R} \subseteq \mathbb{R}^3$ and moving uniformly in the $\hat{x}_3$-direction. The flows are thus described in a 2D region $(\hat{x}_1, \hat{x}_2) \in \hat{\Omega}$. The fluid is contained in $(\hat{\Omega}_f := \hat{\Omega}\backslash\hat{\Omega}_s) \times \mathbb{R}$, and is considered to be a Bingham fluid. The flow variables are the deviatoric stress $\hat{\tau}$, pressure $\hat{p}$, and velocity $\hat{w}$, all of which are independent of $\hat{x}_3$. Only steady flows are considered.

The fluid is characterized physically by its density, yield stress, and plastic viscosity: $\hat{\rho}_f$, $\hat{\tau}_Y$, and $\hat{\mu}_f$, respectively. We adopt a fictitious domain approach to modeling the solid phase, treating it initially as a fluid and then formally taking the solid viscosity to infinity. The solid phase density and viscosity are $\hat{\rho}_s$ and $\hat{\mu}_s$. These parameters are assumed constant.

The incompressible Navier–Stokes equations simplify to only the $\hat{x}_3$-momentum balance. This and the constitutive laws are

$$\hat{\operatorname{div}}\,\hat{\tau} = \begin{cases} \hat{p}_{x_3} - \hat{\rho}_f \hat{g} & \text{in } \hat{\Omega}_f\,, \\ \hat{p}_{x_3} - \hat{\rho}_s \hat{g} & \text{in } \hat{\Omega}_s\,, \end{cases} \qquad \hat{\tau} = \begin{cases} \left(\hat{\mu}_f + \frac{\hat{\tau}_Y}{|\hat{\nabla}\hat{w}|}\right)\hat{\nabla}\hat{w} & \text{in } \hat{\Omega}_f\,, \\ \hat{\mu}_s \hat{\nabla}\hat{w} & \text{in } \hat{\Omega}_s\,, \end{cases} \tag{2.1}$$

where $\hat{g}$ is the gravitational acceleration. Strictly speaking, the fluid constitutive law applies only to where $|\hat{\tau}| > \hat{\tau}_Y$.

The above model and variables are dimensional, for which we have adopted the convention of using the "hat" accent, e.g., $\hat{g}$. We now make the model dimensionless by scaling. In (2.1) the driving force for the motion is the density difference, which results in a buoyancy force that scales proportional to the size of the particle. Thus, we scale lengths with $\hat{L}$:

$$\hat{L} = \sqrt{\text{area}(\hat{\Omega}_s)}\,, \quad \mathbf{x} = (x_1, x_2) := \frac{1}{\hat{L}}(\hat{x}_1, \hat{x}_2)\,, \quad \nabla = \hat{L}\hat{\nabla}\,, \quad \text{div} = \hat{L}\,\hat{\text{div}}.$$

An appropriate measure of the buoyancy stress is $(\hat{\rho}_s - \hat{\rho}_f)\hat{g}\hat{L}$, which we use to scale $\hat{\tau} = (\hat{\rho}_s - \hat{\rho}_f)\hat{g}\hat{L}\tau$. For the pressure gradient in (2.1) we subtract the hydrostatic pressure gradient from the fluid phase and scale the modified pressure gradient with $(\hat{\rho}_s - \hat{\rho}_f)\hat{g}$, defining

$$f = \frac{\hat{p}_{x_3} - \hat{\rho}_f \hat{g}}{(\hat{\rho}_s - \hat{\rho}_f)\hat{g}}.$$

The scaled momentum equations are

$$\text{div}\,\tau = \begin{cases} f & \text{in } \Omega_f\,, \\ f - 1 & \text{in } \Omega_s\,. \end{cases} \tag{2.2}$$

For the constitutive laws, we define a velocity scale $\hat{w}_0$ by balancing the buoyancy stress with a representative viscous stress in the fluid:

$$(\hat{\rho}_s - \hat{\rho}_f)\hat{g}\hat{L} = \frac{\hat{\mu}_f \hat{w}_0}{\hat{L}}.$$

Scaled constitutive laws are

$$\tau = \frac{1}{\varepsilon}\nabla w \text{ in } \Omega_s; \quad \begin{cases} \tau = \left(1 + \dfrac{Y}{|\nabla w|}\right)\nabla w, & |\tau| > Y, \\ |\nabla w| = 0, & |\tau| \le Y, \end{cases} \quad \text{in } \Omega_f. \tag{2.3}$$

We note that there are two dimensionless parameters, ε and Y, defined as

$$\varepsilon := \frac{\hat{\mu}_f}{\hat{\mu}_s}\,, \qquad Y := \frac{\hat{\tau}_Y}{(\hat{\rho}_s - \hat{\rho}_f)\hat{g}\hat{L}}.$$

Evidently, ε is a viscosity ratio. Soon we shall consider the solid limit $\varepsilon \to 0$, and thereafter, ε plays no role in our study.

The parameter Y is called the *yield number* and is central to our study. We see that physically Y balances the yield stress and the buoyancy stress. As buoyancy is the only driving force for motion, it is intuitive that there will be no flow if Y is large enough. The smallest Y for which the motion is stopped is called the *critical yield number*, Y_c, although this will be defined rigorously later.[1]

In terms of w the momentum equation is

$$\begin{aligned} \text{div}\left(\left(1 + \tfrac{Y}{|\nabla w|}\right)\nabla w\right) &= f && \text{in } \Omega_f, \\ \text{div}\left(\tfrac{1}{\varepsilon}\nabla w\right) &= f - 1 && \text{in } \Omega_s\,. \end{aligned} \tag{2.4}$$

[1] The yield number is sometimes referred to as the yield gravity number or yield buoyancy number. As the viscous stresses are also driven by buoyancy, an alternate interpretation would be as a ratio of yield stress to viscous stress, which is referred to as the Bingham number.

It is assumed that Ω has finite extent and at the stationary boundary we assume the no-slip condition:

$$w \equiv 0 \text{ on } \partial\Omega\,. \tag{2.5}$$

At the interface between the two phases the shear stresses are assumed continuous, leading to the transmission condition:

$$\frac{1}{\epsilon}\nabla w \cdot \mathbf{n}_s + \left(1 + \frac{Y}{|\nabla w|}\right)\nabla w \cdot \mathbf{n}_f = 0 \quad \text{on } \partial\Omega_s. \tag{2.6}$$

Here $\mathbf{n}_s,\ \mathbf{n}_f$ denote the outer unit-normals on $\partial\Omega_s,\ \partial\Omega_f$, and the equality has to hold in a weak sense.

We note that for given f and $\varepsilon > 0$ fixed, the solution w_f of (2.4), (2.6), (2.5) is equivalently characterized as the minimizer of the functional

$$\begin{aligned} &\mathcal{F}_{\epsilon,f}(w) := \mathcal{G}_\epsilon(w) + \int_\Omega f w \text{ with} \\ &\mathcal{G}_\epsilon(w) := \frac{1}{2}\int_{\Omega_f} |\nabla w|^2 + \frac{1}{2\varepsilon}\int_{\Omega_s} |\nabla w|^2 + Y\int_{\Omega_f} |\nabla w| - \int_{\Omega_s} w \end{aligned} \tag{2.7}$$

over the space $H_0^1(\Omega)$.

3. Exchange flow problem. Physically, as a solid particle settles in a large expanse of incompressible fluid, its downward motion causes an equal upward motion such that the net volumetric flux is zero. Here we wish to mimic this same scenario in the antiplane shear flow context.

Therefore, we are interested in the *exchange flow problem*, which consists of finding the pair (w, f) that satisfies the following:

- equation (2.4) and condition (2.6) in a suitable variational sense,
- the homogeneous boundary conditions (2.5),
- and the *exchange flow condition*

$$\int_\Omega w(x)\, dx = 0\,. \tag{3.1}$$

Note that (3.1) states that the antiplane flow is divergence free. Therefore, we identify f with a scalar. Two equivalent formulations of this problem are possible:

1. Finding a saddle point of the functional
$$\mathcal{F}_\epsilon(w, f) := \mathcal{F}_{\epsilon,f}(w) \tag{3.2}$$
on $H_0^1(\Omega)\times\mathbb{R}$, with $\mathcal{F}_{\epsilon,f}$ from (2.7). In other words, f is a Lagrange multiplier in the saddle point problem for satisfying the constraint (3.1).
2. Incorporating the constraint (3.1) as part of the domain of definition. Thus we consider minimization of the functional
$$\mathcal{G}_\epsilon^\diamond(w) := \begin{cases} \mathcal{G}_\epsilon(w) & \text{if } w \in H_\diamond^1(\Omega) := \left\{w \in H_0^1(\Omega) : \int_\Omega w = 0\right\}, \\ +\infty & \text{for } w \in H_0^1(\Omega)\backslash H_\diamond^1(\Omega)\,. \end{cases} \tag{3.3}$$
We show in Lemma 3.1 that a minimizer of $\mathcal{G}_\epsilon^\diamond$ exists.

In the rest of this paper we focus on the second formulation.

LEMMA 3.1. *The functionals $\mathcal{F}_{\epsilon,f}(\cdot)$ and $\mathcal{G}_\epsilon^\diamond(\cdot)$ attain their minimum. If the minimizer w^* of $\mathcal{F}_{\epsilon,f}(\cdot)$ satisfies $\int_\Omega w^* = 0$, then it is also a minimizer of $\mathcal{G}_\epsilon^\diamond(\cdot)$.*

Proof. In order to prove the existence of a minimizer of $w \mapsto \mathcal{F}_{\epsilon,f}(w)$ for f fixed, we show that the functional is coercive and lower semicontinuous:

(i) *The functional $\mathcal{F}_{\epsilon,f}(w)$ is coercive with respect to w.* For all $\delta > 0$, and denoting by $|\Omega|$ the Lebesgue measure of Ω, it follows from Poincaré's and Jensen's inequalities that

$$f\int_\Omega w \geqslant -\frac{1}{2\delta^2}f^2 - \frac{\delta^2}{2}\left(\int_\Omega |w|\right)^2 \geqslant -\frac{1}{2\delta^2}f^2 - \frac{\delta^2}{2}|\Omega|\int_\Omega |w|^2 \geqslant -\frac{1}{2\delta^2}f^2 - C\frac{\delta^2}{2}|\Omega|\int_\Omega |\nabla w|^2 . \tag{3.4}$$

Similarly, we have

$$-\int_{\Omega_s} w \geqslant -\frac{1}{2\delta^2} - \frac{\delta^2}{2}|\Omega_s|\int_{\Omega_s}|w|^2 \geqslant -\frac{1}{2\delta^2} - \frac{\delta^2}{2}|\Omega|\int_\Omega |w|^2 \geqslant -\frac{1}{2\delta^2} - C\frac{\delta^2}{2}|\Omega|\int_\Omega |\nabla w|^2 \ . \tag{3.5}$$

Summing (3.4) and (3.5) yields

$$f\int_\Omega w - \int_{\Omega_s} w \geqslant -\frac{1}{2\delta^2}(f^2+1) - C\delta^2|\Omega|\int_\Omega |\nabla w|^2 \ .$$

Now, choosing $\delta > 0$ such that

$$0 < C\delta^2|\Omega| < \frac{1}{2}\min\left\{1, \frac{1}{\epsilon}\right\} ,$$

the coercivity with respect to w follows.

(ii) For $\epsilon < 1$, we now have $2C|\Omega| < 1/\delta^2$ and thus we see that $\mathcal{F}_{\epsilon,f}$ is bounded from below by $-C\,(f^2+1)|\Omega|$.

(iii) *The functional $\mathcal{F}_{\epsilon,f}$ is weakly lower semicontinuous:* $\mathcal{F}_{\epsilon,f}$ can be rewritten as

$$\mathcal{F}_{\epsilon,f}(w) = \int_\Omega g(x, w(x), \nabla w(x))dx ,$$

where $p \to g(s,z,p)$ is convex. Since $\mathcal{F}_{\epsilon,f}$ is also bounded below, we have (see, for instance, [4, Thm. 13.1.2]) that $\mathcal{F}_{\epsilon,f}(w)$ is weakly lower semicontinuous.

With this (coercivity, boundedness, and weak lower semicontinuity) existence of a minimizer of $w \to \mathcal{F}_{\epsilon,f}(w)$ follows immediately (see [4, Thm. 3.2.1]).

The proof of existence of minimizer of $\mathcal{F}_\epsilon^\diamond$ also requires showing that $H^1_\diamond(\Omega)$ is weakly closed. Therefore, note first that the set $H^1_\diamond(\Omega)$ is convex (linearity of the constraint) and closed with respect to the norm topology on $H^1_\diamond(\Omega)$. From this we can conclude that $H^1_\diamond(\Omega)$ is weakly closed, so that (see [4, Thm. 3.3.2]) the functional attains a minimum on this subset. ∎

3.1. Solid limit. Now we want to study the behavior of the problem when $\hat{\mu}_s \to \infty$ (so that $\hat{\Omega}_s$ becomes rigid), that is, $\epsilon \to 0$. We will see that it leads to minimization of the functional

$$\begin{aligned}&\mathcal{G}^\diamond : H^1_0(\Omega) \to \mathbb{R} \cup \{+\infty\}\ ,\\ &w \to \begin{cases} \frac{1}{2}\int_{\Omega_f} |\nabla w|^2 + Y\int_{\Omega_f} |\nabla w| - \int_{\Omega_s} w & \text{if } w \in H^1_{\diamond,c}(\Omega),\\ +\infty & \text{else,}\end{cases}\end{aligned} \tag{3.6}$$

where we define

$$H^1_{\diamond,c}(\Omega) := \left\{ w \in H^1_0(\Omega) : \int_\Omega w = 0,\ \nabla w = 0 \text{ in } \Omega_s \right\}\ .$$

Lemma 3.2. *The functionals* $\mathcal{G}^\diamond_\epsilon$ *defined in* (3.3) Γ*-converge to* $\mathcal{G}^\diamond$ *in* $H^1_0(\Omega)$*, that is, for all* $w \in H^1_0(\Omega)$ *and all sequences* $\{\epsilon_j\}_{j\in\mathbb{N}}$ *converging to* 0 *we have the following:*

(i) *(lim inf inequality) For every sequence* $\{w_j\}_{j\in\mathbb{N}}$ *converging to* w *in* H^1 *we have*

$$\mathcal{G}^\diamond(w) \leqslant \liminf_{j\to\infty} \mathcal{G}^\diamond_{\epsilon_j}(w_j).$$

(ii) *(lim sup inequality) There exists a sequence* $\{w_j\}_{j\in\mathbb{N}}$ *converging to* w *in* H^1 *with*

$$\mathcal{G}^\diamond(w) \geqslant \limsup_{j\to\infty} \mathcal{G}^\diamond_{\epsilon_j}(w_j)\ . \tag{3.7}$$

Proof. Let $w \in H^1_0(\Omega)$, and let $\epsilon_j \to 0+$ be a decreasing sequence with limit 0.

(i) For every sequence w_j converging to w in $H^1_0(\Omega)$, we have

$$\lim_{j\to\infty}\int_\Omega |w_j| = \int_\Omega |w|\,, \quad \lim_{j\to\infty}\int_\Omega |\nabla w_j|^2 = \int_\Omega |\nabla w|^2\,,$$
$$\lim_{j\to\infty}\int_\Omega w_j = \int_\Omega w, \quad \lim_{j\to\infty}\int_{\Omega_s} w_j = \int_{\Omega_s} w,$$

such that for all $w \in H^1_{\diamond,c}(\Omega)$

$$\begin{aligned}\mathcal{G}^\diamond(w) &= \frac{1}{2}\int_{\Omega_f} |\nabla w|^2 + Y\int_{\Omega_f} |\nabla w| - \int_{\Omega_s} w\\ &\leqslant \liminf_{j\to\infty}\left(\frac{1}{\epsilon_j}\int_{\Omega_s} |\nabla w_j|^2 + \frac{1}{2}\int_{\Omega_f} |\nabla w_j|^2 + Y\int_{\Omega_f} |\nabla w_j| - \int_{\Omega_s} w_j\right)\\ &\leqslant \liminf_{j\to\infty} \mathcal{G}^\diamond_{\epsilon_j}(w_j)\ .\end{aligned}$$

If w is not constant in Ω_s, $\mathcal{G}^\diamond(w) = +\infty$ and also $\liminf_{j\to\infty} \mathcal{G}^\diamond_{\epsilon_j}(w_j) \to \infty$ since $\lim_j \int_{\Omega_s} |\nabla w_j|^2 \neq 0$ so that $\frac{1}{\epsilon_j}\int_{\Omega_s} |\nabla w_j|^2 \to \infty$.

(ii) In the case where $w \notin H^1_{\diamond,c}(\Omega)$, we have

$$\limsup \mathcal{G}^\diamond_{\epsilon_j}(w) = \infty = \mathcal{G}^\diamond(w).$$

For $w \in H^1_{\diamond,c}(\Omega)$ we have that $\int_{\Omega_s} |\nabla w|^2 = 0$. This shows that the constant sequence $w_j \equiv w$ satisfies (3.7). ∎

Since $\mathcal{G}^\diamond_\epsilon(\cdot) \geqslant \mathcal{G}^\diamond_2(\cdot)$, they are equicoercive and we get the following corollary (see [11, Thm. 1.21]).

COROLLARY 3.3. *The sequence of minimizers of* $\mathcal{G}^\diamond_\epsilon(\cdot)$ *converges strongly in* H^1_0 *to the minimizer of* $\mathcal{G}^\diamond(\cdot)$ *as* $\epsilon \to 0$.

3.2. Critical yield numbers and total variation minimization. We now want to identify the limiting yield number Y such that the solution of the exchange flow problem satisfies $w \equiv 0$ in Ω, i.e., both solid and fluid motions are stagnating.

DEFINITION 1. *The critical yield number is defined to be*

$$Y_c := \sup_{H^1_{\diamond,c}(\Omega)} \frac{\int_{\Omega_s} v}{\int_\Omega |\nabla v|} \,. \tag{3.8}$$

Assume that w_c minimizes $\mathcal{G}^\diamond$, defined in (3.6). Since $u \mapsto \frac{1}{2}\int |Du|^2$ is Gâteaux differentiable in H^1_0 and convex, we have that for any $v \in H^1_{\diamond,c}(\Omega)$,

$$\int_\Omega \nabla w_c \cdot (\nabla v - \nabla w_c) + Y \int_\Omega |\nabla v| - Y \int_\Omega |\nabla w_c| - \int_{\Omega_s} f(v - w_c) \geqslant 0.$$

Using $v = 2w_c$ and $v = 0$ (as in [19, sections I.3.5.4 and VI.8.2]), we obtain

$$\begin{aligned}\int_\Omega |\nabla w_c|^2 = \int_{\Omega_f} |\nabla w_c|^2 &= \int_{\Omega_s} w_c - Y \int_{\Omega_f} |\nabla w_c| \\ &\leqslant \int_{\Omega_f} |\nabla w_c| \left[\sup_{H^1_{\diamond,c}(\Omega)} \frac{\int_{\Omega_s} v}{\int_{\Omega_f} |\nabla v|} - Y\right] = (Y_c - Y) \int_{\Omega_f} |\nabla w_c| \,.\end{aligned}$$

Thus $w_c \equiv 0$ if $Y \geqslant Y_c$.

ASSUMPTION 3.4. *Even if functions in* $H^1_{\diamond,c}(\Omega)$ *could take different values in different connected components of* Ω_s*, in what follows we restrict ourselves to functions which are constant in* Ω_s*. This assumption covers the cases in which* Ω_s *is connected (Examples* 5.2, 5.3, 5.5*), when there are two connected components arranged symetrically (Example* 5.6*), or when a physical assumption can be made that the particles are linked and have the same possible velocities (Example* 5.7*).*

Under Assumption 3.4 we set $v = 1$ in Ω_s, and therefore, we need to minimize the total variation over the set

$$H^1_{\diamond,1}(\Omega) := \left\{ v \in H^1_0(\Omega) : \int_\Omega v = 0\,,\ v \equiv 1 \text{ in } \Omega_s \right\} \,. \tag{3.9}$$

It is easy to see that this functional does not necessarily attain a minimum. Hence we use standard relaxation techniques.

Relaxation. A function $u \in L^1(\mathbb{R}^2)$ is said to be of bounded variation if its distributional gradient Du is a vector valued Radon measure with finite mass, that is,

$$TV(u) := |Du|(\mathbb{R}^2) = \sup\left\{ \int_\Omega u \ \operatorname{div} z \, dx : z \in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2), \|z\|_{L^\infty} \leqslant 1 \right\} < +\infty.$$

The class of such functions is denoted by $BV(\mathbb{R}^2)$. The relaxation of minimizing TV in $H^1_{\diamond,1}(\Omega)$ with respect to strong convergence in L^1 (note that the constraints are preserved) turns out to be [4, Prop. 11.3.2] minimizing total variation over the set

$$\mathrm{BV}_{\diamond,1} := \left\{ v \in BV(\mathbb{R}^2) : \int_\Omega v = 0\,,\ v \equiv 1 \text{ in } \Omega_s, v \equiv 0 \text{ in } \mathbb{R}^2 \setminus \Omega \right\} \,. \tag{3.10}$$

Since $\mathrm{BV}_{\diamond,1} \subseteq \mathrm{BV}(\widetilde{\Omega})$ and $\mathrm{BV}(\widetilde{\Omega}) \subseteq L^1(\widetilde{\Omega})$ with compact embedding (see [2, Cor. 3.49]) for every bounded $\widetilde{\Omega} \supseteq \Omega$ with $\mathrm{dist}(\partial\Omega, \partial\widetilde{\Omega}) > 0$, the condition $\int_\Omega v = 0$ and compactness in the weak-* topology of $\mathrm{BV}(\widetilde{\Omega})$ (see [2, Thm. 3.23]) imply that there exists at least one minimizer of TV in $\mathrm{BV}_{\diamond,1}$.

Remark 3.5. Note that the total variation appearing in the relaxed problem is in $\mathbb{R}^2$, meaning that jumps at the boundary of Ω are counted. Likewise, in the rest of the paper, every time we speak of total variation with Dirichlet boundary conditions on the boundary of a set A, we mean the total variation in $\mathbb{R}^2$ of functions with their values fixed on $\mathbb{R}^2 \setminus A$.

In what follows, we will repeatedly use the relation between total variation and perimeter of sets. A measurable set $E \subseteq \mathbb{R}^2$ is said to be of finite perimeter in $\mathbb{R}^2$ if $1_E \in BV(\mathbb{R}^2)$, where 1_E is the indicatrix (or characteristic function) of the set E. The perimeter of E is defined as $\operatorname{Per} E := TV(1_E)$.

When E is a set of finite perimeter with Lipschitz boundary, its perimeter $\operatorname{Per} E$ coincides with $\mathcal{H}^1(\partial E)$, where $\mathcal{H}^1$ is the 1D Hausdorff measure. Moreover, we denote the Lebesgue measure of E by $|E|$, so that $|E| := \int_{\mathbb{R}^2} 1_E$.

We recall the so-called *coarea formula* for $u \in BV(\mathbb{R}^2)$ compactly supported

$$TV(u) = \int_{-\infty}^{\infty} \operatorname{Per}(u > t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \operatorname{Per}(u < t)\,\mathrm{d}t, \tag{3.11}$$

as well as the *layer cake formula*, valid for any nonnegative $u \in L^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} u = \int_0^{\infty} |\{u > t\}|\,\mathrm{d}t. \tag{3.12}$$

For more details on BV-functions and finite perimeter sets, we refer the reader to [2].

Particularly important for our analysis are *Cheeger sets*.

Definition 2 (see [43]). *Let Ω_0 be a set of finite perimeter. A set E_0 minimizing the ratio*

$$E \mapsto \frac{\operatorname{Per} E}{|E|}$$

over subsets of Ω_0 is called a Cheeger set *of Ω_0. The quantity*

$$\lambda = \frac{\operatorname{Per} E_0}{|E_0|}$$

is called the Cheeger constant *of Ω_0. If $\hat{\Omega}$ is open and bounded, at least one Cheeger set exists* [36, Prop. 3.5, iii)]. *Since being a Cheeger set is stable by union* [36, Prop. 3.5, vi)]*, there exists a unique maximal (with respect to $\subset$) Cheeger set.*

4. Piecewise constant minimizers. We search now for simple minimizers of TV over $\mathrm{BV}_{\diamond,1}$. We prove that one can find a minimizer that attains only three values, one of them being zero. After investigation of the particularly simple case where Ω_s is convex, we tackle the general case in four steps.

- Starting from a generic minimizer, in Proposition 4.2 we construct a minimizer whose negative part is constant.
- Based on the minimizer with a constant negative part, we then construct a minimizer with constant positive part (Theorem 4.3). Thus there exists a minimizer with three different values, a negative one, a positive one (which is constrained to be 1), and 0.

- We formulate the total variation minimization for three-level functions as a geometrical problem for optimizing the characteristic sets of the positive and negative value and study the curvature of the corresponding interfaces.
- Finally, we show that we can obtain these optimized characteristic sets by solving two consecutive Cheeger-type problems (Theorem 4.10).

4.1. Particular case: Ω_s is convex.

PROPOSITION 4.1. *If Ω_s is convex, then the function*

$$u_0 := 1_{\Omega_s} - \alpha 1_{\Omega_-},$$

where Ω_- is a Cheeger set of $\Omega \setminus \Omega_s$ and $\alpha = \frac{|\Omega_s|}{|\Omega_-|}$, is a minimizer of TV in $BV_{\diamond,1}$.

Proof. Let u be a minimizer. We write

$$u = u^+ - u^-, \text{ with } u^+, u^- \geqslant 0.$$

Then, we have (by the coarea formula, for example)

$$TV(u) = TV(u^+) + TV(u^-). \tag{4.1}$$

First, note that $u \leqslant 1$: indeed, if $|\{u > 1\}| > 0$, then the function

$$\hat{u} := u \cdot 1_{\{0<u<1\}} + 1_{\{u \geqslant 1\}} - \frac{\int u \cdot 1_{\{0<u<1\}} + 1_{\{u\geqslant 1\}}}{\int u^+} u^-.$$

satisfies $\int \hat{u} = 0$ because $\int u^- = \int u^+$, and moreover,

$$\begin{aligned} TV(\hat{u}) &= TV(u \cdot 1_{\{0<u<1\}} + 1_{\{u\geqslant 1\}}) + \frac{\int u \cdot 1_{\{0<u<1\}} + 1_{\{u\geqslant 1\}}}{\int u^+} TV(u^-) \\ &< TV(u^+) + TV(u^-), \end{aligned}$$

which contradicts that u is a minimizer.

Then, let us prove that we can choose $u^+ = 1_{\Omega_s}$. Thanks to the coarea formula,

$$TV(u^+) = \int_{t=0}^{1} \operatorname{Per}(u > t)\, \mathrm{d}t.$$

Since $u = 1$ on Ω_s, for every $0 < t < 1$, we have $\{u \geqslant t\} \supset \Omega_s$ which implies that $\operatorname{Per}(u > t) \geqslant \operatorname{Per} \Omega_s$ by the convexity of Ω_s (since the projection onto a convex set is a contraction). As a result, we reduce the total variation of u^+ by replacing it with 1_{Ω_s}. Replacing then u^- by ηu^- where $\eta = \frac{|\Omega_s|}{\int u^+} < 1$, we produce a competitor $\tilde{u} = 1_{\Omega_s} - \eta u^-$, which has, since u is a minimizer, the same total variation as u.

Now, notice that $\tilde{u}^-$ minimizes total variation with constraints

$$u = 0 \text{ on } (\mathbb{R}^2 \setminus \Omega) \cup \Omega_s, \quad \int \tilde{u}^- = |\Omega_s|.$$

We can link this to the Cheeger problem in $\Omega \setminus \Omega_s$. We denote

$$\lambda = \min_{E \subset (\Omega \setminus \Omega_s)} \frac{\operatorname{Per} E}{|E|}$$

and E_0 a minimizer of this ratio. Then, one can write, observing that for $t \leqslant 0$, $\{\tilde{u} < t\} \subset (\Omega \setminus \Omega_s)$,

$$TV(\tilde{u}^-) = \int_{-\infty}^{0} \mathrm{Per}(\tilde{u} < t)\,\mathrm{d}t \geqslant \lambda \int_{-\infty}^{0} |\tilde{u} < t|\,\mathrm{d}t = \lambda \int \tilde{u}^-$$
$$= \lambda |\Omega_s| = \frac{\mathrm{Per}\, E_0}{|E_0|} |\Omega_s| = TV\left(\frac{|\Omega_s|}{|E_0|} 1_{E_0}\right).$$

Finally, (4.1) implies that the function

$$u_0 := 1_{\Omega_s} - \frac{|\Omega_s|}{|E_0|} 1_{E_0}$$

is a minimizer of TV which has the expected form. ∎

4.2. General case (Ω_s not convex). For any minimizer u on TV in $\mathrm{BV}_{\diamond,1}$, there exists a (possibly different) minimizer in which u^- is replaced by a constant function on the characteristic set of the negative part of u^-.

PROPOSITION 4.2. *Let* $\Theta_+ := \mathrm{Supp}\, u^+$. *Then,*

$$u_0 := u^+ - \frac{\int u^+}{|\Omega_-|} 1_{\Omega_-}, \tag{4.2}$$

where Ω_- *is a Cheeger set of* $\Omega \setminus \Theta_+$ *and is a minimizer of* TV *on* $BV_{\diamond,1}$. *In addition, for every* $t \leqslant 0$, *the level-sets* $\{u < t\}$ *are also Cheeger sets of* $\Omega \setminus \Theta_+$.

Proof. First, we notice that u^- minimizes TV with constraints $\int u^- = \int u^+$ and $u^- = 0$ on $\Theta_+ \cup (\mathbb{R}^2 \setminus \Omega)$. Let us show that u^- minimizes $\frac{TV(v)}{\int v}$ among all functions supported in $\overline{\Omega \setminus \Theta_+}$. Indeed, if we have for such a v

$$\frac{TV(u^-)}{\int u^-} > \frac{TV(v)}{\int v},$$

then $v^- := \frac{\int u^+}{\int v} v$ satisfies $TV(v^-) = \frac{\int u^+}{\int v} TV(v) < TV(u^-)$, which is a contradiction. Then, it is well known (see, once again, [43]) that the minimizer v can be chosen as an indicatrix of a Cheeger set Ω_- of $\Omega \setminus \Theta_+$. That shows that u_0 is a minimizer.

Now, just introduce $\lambda = \frac{\mathrm{Per}\,\Omega_-}{|\Omega_-|}$ and use the previous computations to write

$$\lambda \int u^+ = TV(u^-) = \int_{-\infty}^{0} \mathrm{Per}(u < t)\,\mathrm{d}t = \int_{-\infty}^{0} \frac{\mathrm{Per}(u < t)}{|u < t|} |u < t|\,\mathrm{d}t$$
$$\geqslant \int_{-\infty}^{0} \lambda |u < t|\,\mathrm{d}t = \lambda \int u^-.$$

Since $\int u^+ = \int u^-$, all of these inequalities are equalities and for almost every (a.e.) t, we have $\frac{\mathrm{Per}(u<t)}{|u<t|} = \lambda$, and $\{u < t\}$ is therefore a Cheeger set of $\Omega \setminus \Theta_+$. ∎

In the following, starting from u_0, we show that there exists another minimizer of TV if we replace u_0^+ by the indicatrix of a set Ω_1.

THEOREM 4.3. *There exists a minimizer of* TV *in* $BV_{\diamond,1}$ *which has the form*

$$u_c := 1_{\Omega_1} - \frac{|\Omega_1|}{|\Omega_-|} 1_{\Omega_-}, \tag{4.3}$$

where Ω_1 *is a minimizer of the functional*

$$\mathcal{T}(E) := \mathrm{Per}(E) + \frac{\mathrm{Per}(\Omega_-)}{|\Omega_-|}|E| \tag{4.4}$$

over Borel sets E *with* $\Omega_s \subset E \subset \Omega \setminus \Omega_-$*. In fact, for every* $0 \leqslant t < 1$*, the level-sets* $E_t := \{u > t\}$ *of every minimizer* u *minimize* $\mathcal{T}$*.*

Proof. Let u_0 be the minimizer of TV in $\mathrm{BV}_{\diamond,1}$ from (4.2). Then

$$TV(u_0) = TV(u_0^+) + TV(u_0^-) = TV(u_0^+) + \frac{\mathrm{Per}(\Omega_-)}{|\Omega_-|}\int u_0^+.$$

Then from (3.11), (3.12), and (4.4) it follows that

$$TV(u_0) = \int_0^1 \mathrm{Per}(u_0 > t) + \frac{\mathrm{Per}(\Omega_-)}{|\Omega_-|}|u_0 > t|\,\mathrm{d}t = \int_0^1 \mathcal{T}(u_0 > t)\,\mathrm{d}t \geq \mathcal{T}(\Omega_1).$$

That means that if we replace u^+ by 1_{Ω_1}, TV is decreased and thus

$$TV(u_c) \leqslant TV(u_0) \leqslant TV(u).$$

Because u_c satisfies $\int u_c = 0$ we see from the last inequality that u_c is a minimizer of TV in $\mathrm{BV}_{\diamond,1}$. As before, since u is a minimizer, the inequalities are equalities and we infer the last statement. □

4.3. Geometrical properties of three-valued minimizers. We introduce the class

$$M := \left\{(E_1, E_-) \subset \Omega \;\middle|\; \mathring{E}_1 \cap \mathring{E}_- = \emptyset,\ \Omega_s \subset E_1\right\},$$

and the functional

$$\mathcal{S}(E_1, E_-) = \mathrm{Per}(E_1) + \frac{|E_1|}{|E_-|}\,\mathrm{Per}(E_-).$$

In addition, for $(E_1, E_-) \in M$ we define the function

$$u_c(E_1, E_-) = 1_{E_1} - \frac{|E_1|}{|E_-|}1_{E_-}\ .$$

Proposition 4.4. *$\mathcal{S}$ has a minimizer in M. In addition, the second part of every minimizer has positive Lebesgue measure.*

Proof. Let (E_1^n, E_-^n) be a minimizing sequence for $\mathcal{S}$ in M. The conditions $\Omega_s \subset E_1$ and $E_- \subset \Omega$ ensure that $\mathrm{Per}(E_1^n) + \mathrm{Per}(E_-^n) \leqslant C$, so that standard compactness and lower semicontinuity results for sets of finite perimeter [2] imply existence of a minimizer. Note that nonempty interiors have positive measure, so the class M is preserved by L^1 convergence. Moreover, using the isoperimetric inequality we get

$$\mathrm{Per}(E) \geqslant \sqrt{4\pi}|E|^{\frac{1}{2}},\ \text{so that}\ \frac{\mathrm{Per}(E)}{|E|} \geqslant \sqrt{4\pi}|E|^{-\frac{1}{2}}.$$

Therefore $|E_-^n|$ is bounded away from zero and the corresponding part of the minimizer has positive measure. □

Using Theorem 4.3, we see that the connection between minimizing TV in $\mathrm{BV}_{\diamond,1}$ and minimizing $\mathcal{S}$ is as follows.

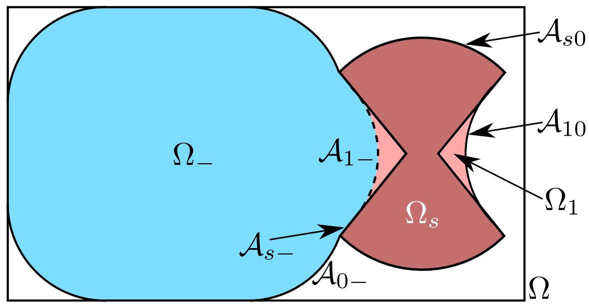

FIG. 1. *Interfaces present in minimizers of* $\mathcal{S}$.

PROPOSITION 4.5. *If the function* $u_c := u_c(\Omega_1, \Omega_-)$ *minimizes* TV *in* $BV_{\diamond,1}$, *then* (Ω_1, Ω_-) *minimizes* $\mathcal{S}$ *in* M. *Conversely, if* (Ω_1, Ω_-) *minimizes* $\mathcal{S}$ *in* M, *then* $u_c(\Omega_1, \Omega_-)$ *minimizes* TV *in* $BV_{\diamond,1}$.

Remark 4.6. The above proposition explains why, in the following, we consider the shape optimization problem of minimizing $\mathcal{S}$ in M.

We remark that this produces minimizers of TV in $\mathrm{BV}_{\diamond,1}$ of a certain (geometric) form, which are not necessarily all of them.

In what follows, we consider small perturbations of a minimizer (Ω_1, Ω_-) of $\mathcal{S}$ in which only one of the sets is changed. This will be enough to determine the curvature of their boundaries, which we split as follows (see Figure 1):

$$\begin{aligned} \mathcal{A}_{1-} &= \{x \in \Omega : x \in \partial\Omega_1, x \in \partial\Omega_-\}, \quad \mathcal{A}_{10} = \{x \in \Omega : x \in \partial\Omega_1, x \notin \partial\Omega_-\}, \\ \mathcal{A}_{0-} &= \{x \in \Omega : x \notin \partial\Omega_1, x \in \partial\Omega_-\}, \quad \mathcal{A}_{s-} = \{x \in \Omega : x \in \partial\Omega_s, x \in \partial\Omega_-\}, \\ & \qquad \mathcal{A}_{s0} = \{x \in \Omega : x \in \partial\Omega_s, x \notin \partial\Omega_-\}. \end{aligned}$$

We denote by κ_1, κ_- the curvature functions of Ω_1, Ω_-, defined in $\partial\Omega_1, \partial\Omega_-$ through their outer normals n_1, n_- (i.e., a circle has positive curvature).

For a generic set of finite perimeter in $\mathbb{R}^2$ only a distributional curvature is available [38, Rem. 17.7]. However, since Ω_1 and Ω_- minimize the functionals $\mathcal{S}(\cdot, \Omega_-)$ and $\mathcal{S}(\Omega_1, \cdot)$, respectively, regularity theorems for Λ-minimizers of the perimeter [38, Thm. 26.3] are applicable to them. In consequence, $\mathcal{A}_{1-}$, $\mathcal{A}_{0-}$ and $\mathcal{A}_{10} \setminus \mathcal{A}_{s0}$ are locally graphs of $C^{1,\gamma}$ functions. Combined with standard regularity theory for uniformly elliptic equations [25], one obtains higher regularity, so that, in particular, the curvatures κ_1, κ_- are defined classically on those interfaces (on $\partial\Omega_s \cap \partial\Omega_1$, no information is provided).

PROPOSITION 4.7. *Let* (Ω_1, Ω_-) *be a minimizer of* $\mathcal{S}$. *Then, the curvatures* κ_-, κ_1 *of the interfaces* $\mathcal{A}_{0-}$ *and* $\mathcal{A}_{10} \setminus \mathcal{A}_{s0}$ *are given by*

$$\kappa_- = \frac{\operatorname{Per}\Omega_-}{|\Omega_-|} \text{ on } \mathcal{A}_{0-} \quad \text{and} \quad \kappa_1 = -\frac{\operatorname{Per}\Omega_-}{|\Omega_-|} \text{ on } \mathcal{A}_{10} \setminus \mathcal{A}_{s0}.$$

In consequence, $\mathcal{A}_{0-}$ *and* $\mathcal{A}_{10} \setminus \mathcal{A}_{s0}$ *are composed of pieces of circles of radius* $\frac{|\Omega_-|}{\operatorname{Per}\Omega_-}$.

Proof. For every $x \in \mathcal{A}_{10} \setminus \mathcal{A}_{s0}$ we consider a perturbed domain Ω_1^w (see Figure 1), such that $\Omega_1^w = (I + \overrightarrow{w})(\Omega_1)$, where $\overrightarrow{w}$ is supported in a neighborhood of x. Calling $w := \overrightarrow{w} \cdot n_1$ and thanks to the first variation formula [38, Thm. 17.5 and Rem. 17.6], we can develop the first variation of $\mathcal{S}(\cdot, \Omega_-)$ at a minimizer Ω_1 in direction w and obtain

$$\int_{\mathcal{A}_{10} \setminus \mathcal{A}_{s0}} \kappa_1 w + w \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|} \, d\mathcal{H}^1 = 0.$$

Since w was arbitrary, we get the optimality condition for Ω_1:

$$\kappa_1 + \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|} = 0 \quad \text{ in } \mathcal{A}_{10} \setminus \mathcal{A}_{s0}.$$

Proceeding similarly for Ω_- we obtain

$$\frac{1}{|\Omega_1|} \left(\frac{\kappa_-}{|\Omega_-|} - \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|^2} \right) = 0 \quad \text{ in } \mathcal{A}_{0-}.$$

This shows that the curvatures of $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}$ and $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}$ are constant with values $\kappa_1 = -\kappa_- = \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|}$. This, in particular, shows that these interfaces are composed of circles of radii $\frac{|\Omega_-|}{\operatorname{Per}(\Omega_-)}$. ∎

Proposition 4.8. *Let (Ω_1, Ω_-) be a minimizer of $\mathcal{S}$. Then*

$$\kappa_- = \frac{\operatorname{Per} \Omega_-}{|\Omega_-|} = -\kappa_1 \text{ on } \mathcal{A}_{1-} \setminus \mathcal{A}_{s-}.$$

Thus, $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}$ consists of pieces of circle with the same radius of Proposition 4.7.

Proof. First, we note that since $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-} \subset \partial\Omega_1 \cap \partial\Omega_-$, we must have

$$\kappa_1 = -\kappa_- \text{ on } \mathcal{A}_{1-} \setminus \mathcal{A}_{s-}.$$

Now, we perturb Ω_1 while keeping Ω_- fixed. In this context, Ω_1 is a minimizer of $E \mapsto \mathcal{S}(E, \Omega_-)$ with constraints $E \subset \Omega$ and $\Omega_1 \cap E = \emptyset$. Since Ω_- is fixed the second constraint allows only inward perturbations. We, therefore, perturb Ω_1 in its exterior normal direction with a function $w \leqslant 0$ supported in $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}$. The variation formula for Ω_1 in direction w provides

$$\int_{\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}} \kappa_1 w + \int w \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|} \, d\mathcal{H}^1 \geqslant 0,$$

which yields

$$\kappa_1 \leqslant -\frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|} \text{ on } \mathcal{A}_{1-} \setminus \mathcal{A}_{s-}.$$

Now, we fix Ω_1 and perturb Ω_- similarly with $w \leqslant 0$, again supported in $\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}$ (so the perturbation goes inside Ω_-). Since Ω_- now minimizes $\mathcal{S}(\Omega_1, \cdot)$, we get

$$\int_{\mathcal{A}_{1-} \setminus \mathcal{A}_{s-}} w\kappa_- \frac{|\Omega_1|}{|\Omega_-|} - w \frac{|\Omega_1|}{|\Omega_-|^2} \operatorname{Per}(\Omega_-) \, d\mathcal{H}^1 \geqslant 0,$$

which gives

$$\kappa_- \leqslant \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|} \text{ on } \mathcal{A}_{1-} \setminus \mathcal{A}_{s-}.$$

Proposition 4.9. *Let E be a connected component of $\Omega \setminus (\Omega_- \cup \Omega_1)$ such that $\partial E \cap \partial\Omega = \emptyset$. Then, $(\Omega_1 \cup E, \Omega_-)$ and $(\Omega_1, \Omega_- \cup E)$ belong to M and minimize $\mathcal{S}$.*

Proof. We abbreviate $\lambda = \frac{\operatorname{Per}\Omega_-}{|\Omega_-|}$. Then because $E \cap \Omega_- = E \cap \Omega_1 = \emptyset$, the pairs $(\Omega_1 \cup E, \Omega_-)$ and $(\Omega_1, \Omega_- \cup E)$ both belong to M and we have

$$\operatorname{Per}(\Omega_1 \cup E) + \lambda|\Omega_1 \cup E| \geqslant \operatorname{Per}(\Omega_1) + \lambda|\Omega_1|,$$

which, because $E \cap \Omega_1 = \emptyset$, implies

$$\lambda|E| \geqslant \operatorname{Per}(\Omega_1) - \operatorname{Per}(\Omega_1 \cup E). \tag{4.5}$$

Because Ω_- is a Cheeger set of $\Omega \setminus \Omega_1$, we have

$$\frac{\operatorname{Per}(\Omega_- \cup E)}{|\Omega_- \cup E|} \geqslant \frac{\operatorname{Per}(\Omega_-)}{|\Omega_-|},$$

which, because $E \cap \Omega_- = \emptyset$, implies

$$\operatorname{Per}(\Omega_- \cup E)|\Omega_-| \geqslant \operatorname{Per}(\Omega_-)(|\Omega_-| + |E|)\,,$$

which yields

$$\operatorname{Per}(\Omega_- \cup E) - \operatorname{Per}(\Omega_-) \geqslant \lambda|E|. \tag{4.6}$$

In summary, we have shown in (4.5) and (4.6) that

$$\operatorname{Per}(\Omega_- \cup E) - \operatorname{Per}(\Omega_-) \geqslant \lambda|E| \geqslant \operatorname{Per}(\Omega_1) - \operatorname{Per}(\Omega_1 \cup E).$$

Since $\partial E \cap \partial\Omega = \emptyset$ and $E \cap \Omega_- = E \cap \Omega_1 = \emptyset$, we know $\partial E \subset \partial\Omega_1 \cup \partial\Omega_-$. Furthermore, $E \cap \Omega_- = E \cap \Omega_1 = \emptyset$ also implies that the common boundaries between E and Ω_- and between E and Ω_1 have opposite-pointing outer normals, and one can write [38, Thm. 16.3]

$$\operatorname{Per}(\Omega_- \cup E) - \operatorname{Per}(\Omega_-) = \operatorname{Per}(\Omega_1) - \operatorname{Per}(\Omega_1 \cup E),$$

which implies that all the inequalities above are equalities, and the set E can be joined to Ω_- or Ω_1 without changing the value of $\mathcal{S}$. ∎

In the following we show that one may obtain minimizers of $\mathcal{S}$ (and, therefore, minimizers of TV in $\mathrm{BV}_{\diamond,1}$ with three values) in two simpler steps:

1. Solve the Cheeger problem for $\Omega \setminus \Omega_s$. Let Ω_c be the maximal Cheeger set and $\lambda_c := \frac{\operatorname{Per}\Omega_c}{|\Omega_c|}$ its Cheeger constant.
2. Obtain the minimal (with respect to $\subset$) minimizer Ω_{1c} of

$$\operatorname{Per}(E) + \lambda_c|E| \text{ over } E \supset \Omega_s.$$

Note that minimizers of the second problem exist by an argument similar to Proposition 4.4.

THEOREM 4.10. *The pair (Ω_{1c}, Ω_c) minimizes $\mathcal{S}$.*

Proof. Let $\lambda := \frac{\operatorname{Per}\Omega_-}{|\Omega_-|}$ (by definition of the Cheeger set Ω_c, we have $\lambda \geqslant \lambda_c$). Let also E be the smallest (with respect to $\subset$) minimizer of

$$\hat{E} \mapsto \operatorname{Per}(\hat{E}) + \lambda|\hat{E}| \text{ subject to } \Omega_s \subset \hat{E}. \tag{4.7}$$

We want to show that $E \cap \Omega_- = \emptyset$; that is, E is also a minimizer of $\operatorname{Per}(\cdot) + \lambda|\cdot|$ with respect to the constraints $E \cap \Omega_- = \emptyset$ and $\Omega_s \subset E$.

Because $E \setminus \Omega_-$ is admissible in (4.7),

$$\operatorname{Per}(E \setminus \Omega_-) + \lambda |E \setminus \Omega_-| \geqslant \operatorname{Per}(E) + \lambda |E|.$$

On the other hand, Ω_-, as a Cheeger set of $\Omega \setminus \Omega_1$, is a minimizer of

$$\hat{E} \mapsto \operatorname{Per}(\hat{E}) - \lambda |\hat{E}| \text{ subject to } \hat{E} \cap \Omega_1 = \emptyset. \tag{4.8}$$

Then $\Omega_- \setminus E$ is a competitor for (4.8),

$$\operatorname{Per}(\Omega_- \setminus E) - \lambda |\Omega_- \setminus E| \geqslant \operatorname{Per}(\Omega_-) - \lambda |\Omega_-|.$$

Summing these two inequalities and using that (see [38, Ex. 16.5])

$$\operatorname{Per}(E \setminus \Omega_-) + \operatorname{Per}(\Omega_- \setminus E) \leqslant \operatorname{Per}(E) + \operatorname{Per}(\Omega_-),$$

we obtain

$$\lambda \left(|E \setminus \Omega_-| - |\Omega_- \setminus E| \right) \geqslant \lambda \left(|E| - |\Omega_-| \right).$$

Since this last inequality is an equality, it is also true for the two previous ones, and we can conclude that

$$\operatorname{Per}(E \setminus \Omega_-) + \lambda |E \setminus \Omega_-| = \operatorname{Per}(E) + \lambda |E|,$$

which implies, since E is minimal with respect to the inclusion, that $E \cap \Omega_- = \emptyset$.

Similarly, if E_c is a minimizer of

$$\hat{E} \mapsto \operatorname{Per} \hat{E} + \lambda_c |\hat{E}| \text{ with constraint } \Omega_s \subset \hat{E}, \tag{4.9}$$

one can prove that $E_c \cap \Omega_c = \emptyset$.

We have proved that Ω_1, Ω_{1c} minimize $\operatorname{Per}(\cdot) + \lambda\,|\cdot|$, $\operatorname{Per}(\cdot) + \lambda_c\,|\cdot|$ with the same constraint (containing Ω_s). Hence, $\Omega_1 \cap \Omega_{1c}$ is admissible in (4.7) and $\Omega_1 \cup \Omega_{1c}$ is admissible for (4.9), which implies

$$\operatorname{Per}(\Omega_1 \cap \Omega_{1c}) + \lambda |\Omega_1 \cap \Omega_{1c}| \geqslant \operatorname{Per} \Omega_1 + \lambda |\Omega_1|,$$

$$\operatorname{Per}(\Omega_1 \cup \Omega_{1c}) + \lambda_c |\Omega_1 \cup \Omega_{1c}| \geqslant \operatorname{Per} \Omega_{1c} + \lambda_c |\Omega_{1c}|.$$

Summing these inequalities and recalling that [38, Lem. 12.22]

$$\operatorname{Per}(\Omega_1 \cap \Omega_{1c}) + \operatorname{Per}(\Omega_1 \cup \Omega_{1c}) \leqslant \operatorname{Per}(\Omega_1) + \operatorname{Per}(\Omega_{1c}),$$

we get

$$\lambda_c |\Omega_1 \setminus \Omega_{1c}| \geqslant \lambda |\Omega_1 \setminus \Omega_{1c}|.$$

Then, if $\lambda_c < \lambda$, we obtain $\Omega_{1c} \supset \Omega_1$, and if $\lambda = \lambda_c$, all the inequalities above are equalities, which implies once again (using the minimality of Ω_1) that $\Omega_{1c} \supset \Omega_1$. Then, $\Omega_c \cap \Omega_1 = \emptyset$, hence Ω_c is also a Cheeger set of $\Omega \setminus \Omega_1$. ◻

Remark 4.11. By the statements in the previous section about level sets of the generic minimizer u, we infer that the only lack of uniqueness present in the minimization of TV in $\mathrm{BV}_{\diamond,1}$ is that of the corresponding geometric problems. More precisely, if the Cheeger set of $\Omega \setminus \Omega_s$ is unique, (which is shown in [12, Thm. 1] to be a generic situation), then the minimizer of TV in $\mathrm{BV}_{\diamond,1}$ is unique as well. Indeed, with the same arguments as in the proof of Proposition 4.9, one sees that the minimizer of (4.4) is also unique, which implies by Proposition 4.2 and Theorem 4.3 that the level-sets of u are all uniquely determined.

4.4. Behavior of Y_c as Ω grows large.

PROPOSITION 4.12. *Let Ω_0 be a convex set and $\Omega_s \subset \Omega_0$, both containing the origin, and assume that $|\Omega_s| = 1$. For $\alpha \geqslant 1$, let $\Omega = \alpha\Omega_0$, i.e., we consider the domain to be a rescaling of Ω_0 (note that $\Omega_s \subset \alpha\Omega_0$). Then*

$$\lim_{\alpha\to\infty} Y_c(\alpha) = \frac{1}{\min\limits_{E\supset\Omega_s} \operatorname{Per} E}.$$

Proof. We recall that

$$Y_c(\alpha) = \frac{|\Omega_s|}{\inf_{M_\alpha} \mathcal{S}},$$

where

$$M_\alpha := \left\{(E_1, E_-) \subset \alpha\Omega_0 \mid \overset{\circ}{E}_1 \cap \overset{\circ}{E}_- = \emptyset,\ \Omega_s \subset E_1\right\}.$$

Then, noticing that for every $\tilde{\Omega}$ such that $\Omega_s \subset \tilde{\Omega} \subset \alpha\Omega_0$ we have $(\tilde{\Omega}, \alpha\Omega_0 \setminus \tilde{\Omega}) \in M_\alpha$, one can write

$$\begin{aligned}\inf_{M_\alpha} \mathcal{S} \leqslant \mathcal{S}\left(\tilde{\Omega}, \alpha\Omega_0 \setminus \tilde{\Omega}\right) &= \operatorname{Per}(\tilde{\Omega}) + \frac{\operatorname{Per}(\alpha\Omega_0) + \operatorname{Per}(\tilde{\Omega})}{|\alpha\Omega_0| - |\tilde{\Omega}|}|\tilde{\Omega}| \\ &\leqslant \operatorname{Per}(\tilde{\Omega}) + \frac{\alpha \operatorname{Per}(\Omega_0) + \operatorname{Per}(\tilde{\Omega})}{\alpha^2|\Omega_0| - |\tilde{\Omega}|}|\tilde{\Omega}| \xrightarrow[\alpha\to\infty]{} \operatorname{Per}(\tilde{\Omega}).\end{aligned}$$

On the other hand, since $(\tilde{\Omega}, \alpha\Omega_0 \setminus \tilde{\Omega}) \in M_\alpha$,

$$\mathcal{S}(\tilde{\Omega}, \alpha\Omega_0 \setminus \tilde{\Omega}) \geqslant \operatorname{Per}(\tilde{\Omega}).$$

Optimizing in $\tilde{\Omega}$ establishes the result. ∎

Remark 4.13. If Ω_s is indecomposable (i.e., "connected" in an adequate sense for this framework), we have by [20, Prop. 5] that

$$\min_{E\supset\Omega_s} \operatorname{Per} E = \operatorname{Per}(\operatorname{Co}(\Omega_s)),$$

where $\operatorname{Co}(X)$ is the convex envelope of X.

Remark 4.14. As may be seen in Examples 5.2 and 5.3, the above limit is not attained at a finite α. There is no "critical size" at which the boundary of Ω stops playing a role. We see that the limiting Y_c is approached at least as $O(1/\alpha)$ as $\alpha \to \infty$.

5. Application examples. In the previous section, we have seen that the free boundaries of the optimal sets are composed of pieces of circles of the same radius, which suggests that one might be able to use morphological operations to construct these minimizers. We introduce these now.

DEFINITION 3 (opening, closing). *For a set X and $r > 0$, we define the* opening *of X with radius r by*

$$Open_r\left(X\right) := \bigcup_{x:B_r(x)\subset X} B_r(x)\ ,$$

where $B_r(x)$ is the disk with radius r and center x. Additionally, we define the closing *of X with radius r as*

$$Close_r\left(X\right) := \mathbb{R}^2 \setminus \left(Open_r\left(\mathbb{R}^2 \setminus X\right)\right).$$

5.1. Morphological operations and Cheeger sets. The Cheeger problem is far from being entirely understood. Nonetheless, it is for convex sets. As a result, if Ω is convex and $\Omega_s = \emptyset$, the Cheeger set Ω_- of Ω satisfies

- Ω_- is unique,
- Ω_- is convex and $\mathcal{C}^{1,1}$,
- $\Omega_- = Open_r(\Omega)$ where r is the Cheeger constant of Ω.

In the general case, for a Cheeger set Ω_- of $\Omega \setminus \Omega_s$, few results are available [36]:

- The boundaries of Ω_- are pieces of circles of radius $\frac{1}{\lambda}$ (λ is the Cheeger constant of $\Omega \setminus \Omega_s$) which are shorter than half the corresponding circle.
- If x_0 is a smooth point of $\partial(\Omega \setminus \Omega_s)$ and belongs to $\partial\Omega_-$, then $\partial\Omega_-$ is $\mathcal{C}^{1,1}$ around x_0 [12, Thm. 2].
- We also have [36, Lem. 2.14], which basically tells that if the maximal Cheeger set of $\Omega \setminus \Omega_s$ contains a ball of radius $\frac{1}{\lambda}$, then it also contains all the balls of radius $\frac{1}{\lambda}$ obtained by rolling the first ball inside $\Omega \setminus \Omega_s$.

Remark 5.1. Let Ω and Ω_s be convex, and let λ be the Cheeger constant of Ω. If $d(\Omega_s, \partial\Omega) \geqslant \frac{2}{\lambda}$, then the maximal Cheeger set of $\Omega \setminus \Omega_s$ can be obtained rolling a ball of radius $\frac{1}{\lambda_0} < \frac{1}{\lambda}$ around Ω_s ($\lambda_0 \geqslant \lambda$ being the Cheeger constant of $\Omega \setminus \Omega_s$). In particular, it fills a neighborhood of $\partial\Omega_s$ in $\Omega \setminus \Omega_s$.

5.2. Single convex particles. We start with two simple examples in which a single convex particle is placed centrally within a larger convex domain.

Example 5.2 (circular Ω).

(a) Let Ω_s, Ω be two circles with radii $\frac{1}{\sqrt{\pi}}$, R, ensuring that $|\Omega_s| = 1$. Since in this case $Open_r(\Omega) = \Omega$ for all $r \leqslant R$, $\Omega_- = \Omega \setminus \Omega_s$ minimizes $S(\Omega_s, \cdot)$.

Thus, $\Omega_c = \Omega \setminus \Omega_s$ and $\Omega_{1c} = \Omega_s$. We have

$$\lambda_c = \frac{\operatorname{Per}\Omega_c}{|\Omega_c|} = \frac{2\pi R + 2\sqrt{\pi}}{\pi R^2 - 1}$$

and

$$Y_c = \frac{|\Omega_{1c}|}{\operatorname{Per}(\Omega_{1c}) + \lambda_c|\Omega_{1c}|} = \frac{1}{2\sqrt{\pi} + \frac{2\pi R + 2\sqrt{\pi}}{\pi R^2 - 1}}.$$

We may also construct the minimizer of TV over $\mathrm{BV}_{\diamond,1}$, given (in cylindrical coordinates) by $v_0 : [0,\infty] \times [0,\pi] \to \mathbb{R}$:

$$v_0(r,\phi) := \begin{cases} |\Omega_s| = 1 & \text{for } \ 0 \leqslant r \leqslant \frac{1}{\sqrt{\pi}}\,, \\ -\frac{|\Omega_s|}{|\Omega| - |\Omega_s|} = -\frac{1}{R^2\pi - 1} & \text{for } \ \frac{1}{\sqrt{\pi}} < r \leqslant R\,, \\ 0 & \text{for } \ R < r < \infty \end{cases}$$

(evidently axisymmetric). The total variation is

$$\begin{aligned} |Dv_0|(\Omega) &= \operatorname{Per}\Omega_s + (\operatorname{Per}\Omega_s + \operatorname{Per}\Omega)\frac{|\Omega_s|}{|\Omega| - |\Omega_s|} \\ &= 2\sqrt{\pi} + \frac{2\sqrt{\pi} + 2R\pi}{R^2\pi - 1} = \frac{1}{Y_c}\,. \end{aligned}$$

For $R \to \infty$ the limit is $\operatorname{Per}\Omega_s = 2\sqrt{\pi}$ and Y_c approaches $\frac{1}{2\sqrt{\pi}}$.

(b) As a slight variation on the above now let Ω_s be the unit square. Again we find $\Omega_c = \Omega \setminus \Omega_s$ and $\Omega_{1c} = \Omega_s$, and hence

$$\lambda_c = \frac{\operatorname{Per}\Omega_c}{|\Omega_c|} = \frac{2\pi R + 4}{\pi R^2 - 1}$$

and

$$Y_c = \frac{1}{Per(\Omega_{1c}) + \lambda_c|\Omega_{1c}|} = \frac{1}{4 + \frac{2\pi R+4}{\pi R^2-1}} \to 0.25 \text{ as } R \to \infty.$$

Example 5.3 (square Ω). We now consider Ω to be a square of side L. In the absence of Ω_s the optimal set Ω_- is given by $Open_{r_\infty}(\Omega)$ for $r_\infty = L/(2+\sqrt{\pi}) = 1/\lambda_c$; see [40].

(a) Now consider a centrally positioned unit square Ω_s, within Ω of side $L > 1$. The optimal set Ω_- is given by $Open_r(\Omega) \setminus \Omega_s$ for some $r > 0$. We have $|Open_r(\Omega)| = |\Omega| + r^2(\pi - 4)$, $\operatorname{Per} Open_r(\Omega) = \operatorname{Per}\Omega + r(2\pi - 8)$, and to find $r = r(L)$ we use Propositions 4.7 and 4.8:

$$\frac{1}{r} = \frac{\operatorname{Per}(Open_r(\Omega) \setminus \Omega_s)}{|Open_r(\Omega) \setminus \Omega_s|} = \frac{4L + 4 + 2r(\pi - 4)}{L^2 - 1 + r^2(\pi - 4)}.$$

The resulting quadratic equation gives the optimal $r(L)$:

$$r(L) = \frac{L}{2}\frac{1 + 1/L}{1 - \pi/4}\left(1 - \sqrt{1 - (1 - \pi/4)\frac{1 - 1/L}{(1 + 1/L)}}\right).$$

We find that $r(L) < r_\infty$ with $r(L) \to r_\infty$ as $L \to \infty$ and $r(L) \to 0$ as $L \to 1^+$, as expected. Consequently, $\Omega_c = Open_{r(L)}(\Omega) \setminus \Omega_s$ and the Cheeger constant $\lambda_c(L)$ is

$$\lambda_c(L) = \frac{\operatorname{Per}(Open_{r(L)}(\Omega) \setminus \Omega_s)}{\left|Open_{r(L)}(\Omega) \setminus \Omega_s\right|} = \frac{4L + 4 + 2r(L)(\pi - 4)}{L^2 - 1 + r(L)^2(\pi - 4)}.$$

Again we have $\Omega_{1c} = \Omega_s$, and

$$Y_c(L) = \frac{1}{\operatorname{Per}(\Omega_{1c}) + \lambda_c(L)|\Omega_{1c}|} = \frac{1}{4 + \lambda_c(L)}.$$

The minimizer of TV over $\mathrm{BV}_{\diamond,1}$ is constructed from the optimal sets:

$$u_{r(L)} := 1_{\Omega_s} - \frac{|\Omega_s|}{\left|Open_{r(L)}(\Omega)\right| - |\Omega_s|} 1_{Open_{r(L)}(\Omega)\setminus\Omega_s}$$

with total variation:

$$\begin{aligned}\left|Du_{r(L)}\right|(\Omega) &= \operatorname{Per}\Omega_s + \frac{(\operatorname{Per}\Omega_s + \operatorname{Per}\Omega + r(L)(2\pi - 8))\,|\Omega_s|}{|\Omega| + r(L)^2(\pi - 4) - |\Omega_s|} \\ &= 4 + \frac{(4 + 4L + r(L)(2\pi - 8))}{L^2 + r(L)^2(\pi - 4) - 1}.\end{aligned}$$

(b) We replace Ω_s by circle of radius $1/\sqrt{\pi}$, ensuring $|\Omega_s| = 1$, and consider $L > 2/\sqrt{\pi}$. The calculations are similar. Again the optimal set Ω_- is $Open_r(\Omega) \setminus \Omega_s$ with $r = r(L)$ determined from Propositions 4.7 and 4.8. We now find

$$r(L) = \frac{L}{2}\frac{1 + \sqrt(\pi)/(2L)}{1 - \pi/4}\left(1 - \sqrt{1 - (1 - \pi/4)\frac{1 - 1/L^2}{(1 + \sqrt(\pi)/(2L))^2}}\right).$$

Thus, $\Omega_c = Open_{r(L)}(\Omega) \setminus \Omega_s$, $\Omega_{1c} = \Omega_s$, and

$$\lambda_c(L) = \frac{\operatorname{Per}(Open_{r(L)}(\Omega) \setminus \Omega_s)}{\left|Open_{r(L)}(\Omega) \setminus \Omega_s\right|} = \frac{4L + 2\sqrt{\pi} + 2r(L)(\pi - 4)}{L^2 - 1 + r(L)^2(\pi - 4)},$$

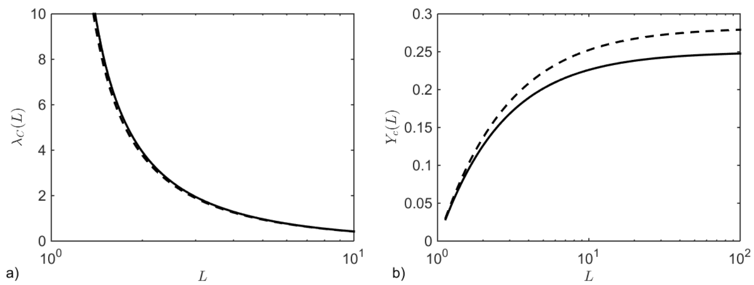

FIG. 2. *Comparison of results of Example* 5.3 *at different* L*:* (a) $\lambda_c(L)$*;* (b) $Y_c(L)$*. Circular* Ω_s *is marked with the broken line and square* Ω_s *is marked with the solid line.*

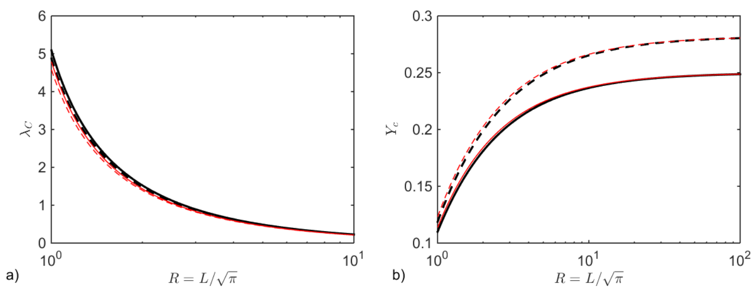

FIG. 3. *Comparison of results of Examples* 5.2 *and* 5.3 *at different* $R = L/\sqrt{\pi}$*:* (a) $\lambda_c(L)$*;* (b) $Y_c(L)$*. Circular* Ω_s *is marked with the broken line and square* Ω_s *is marked with the solid line. Circular* Ω *marked in red and square* Ω *in black. (Figure in color online.)*

$$Y_c(L) = \frac{1}{\mathrm{Per}(\Omega_{1c}) + \lambda_c(L)|\Omega_{1c}|} = \frac{1}{2\sqrt{\pi} + \lambda_c(L)}.$$

Figure 2(a) plots the results of Example 5.3 at different L. Interestingly, although $\lambda_c(L)$ is smaller for the circular Ω_s, it is only very marginally so. Figure 2(b) plots the yield limit $Y_c(L)$ for both Ω_s. Here we see a significant difference: the circular Ω_s requires a larger yield stress to prevent motion. As we have seen that $\lambda_c(L)$ is similar for both Ω_s, this difference in Y_c stems almost entirely from $\mathrm{Per}(\Omega_{1c}) = \mathrm{Per}(\Omega_s)$ (in these examples). We may deduce from the expressions derived that $\lambda_c(L) \sim O(1/L)$ as $L \to \infty$ and hence that $Y_c(L) \to 1/\mathrm{Per}(\Omega_s) + O(1/L)$ as $L \to \infty$; see also Proposition 4.12. The same behaviors are observed with the earlier Example 5.2, in a circle of radius R, i.e., little difference in $\lambda_c(R)$, significant difference in $Y_c(R)$, stemming primarily from $\mathrm{Per}(\Omega_s)$, and similar asymptotic trends as $R \to \infty$.

We might also seek to compare Examples 5.2 and 5.3 directly. The scaling introduced ensures $|\Omega_s| = 1$, matching the buoyancy force felt by each particle. By setting $L^2 = \pi R^2$ we also match the area of fluid within $\Omega \setminus \Omega_s$. Figure 3(a) plots $\lambda_c(R)$ and $\lambda_c(L(R))$. Figure 3(b) plots $Y_c(R)$ and $Y_c(L(R))$. We observe that $\lambda_c(R) < \lambda_c(L(R))$, for the same Ω_s, but again the effect is marginal and λ_c is very close for all four cases. Interestingly, in Figure 3(b) we see that by scaling $L^2 = \pi R^2$ the effects of the shape of Ω are minimized: $Y_c(R)$ and $Y_c(L(R))$ are very close for the same Ω_s, whether it be circular or square.

To summarize, these simple examples suggest that for (centrally placed convex) particles, when we have the same area of solid and the same area of fluid, the main differences in yield behavior come from the different perimeters of the particle. The

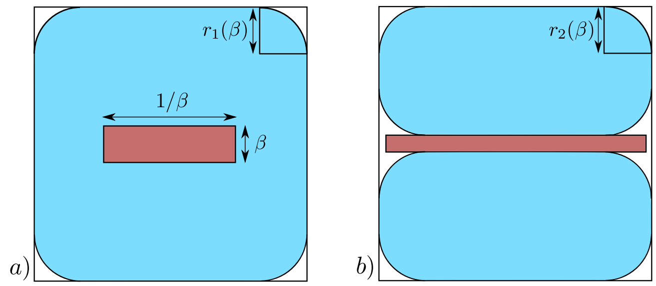

FIG. 4. *Schematic of two different configurations for the rectangle with aspect ratio β:* (a) *configuration* 1; (b) *configuration* 2.

optimal sets in $\Omega \setminus \Omega_s$ are selected such that λ_c varies primarily with the area of Ω (and less significantly with its shape). For the same size of Ω (and Ω_s) the particle with smaller perimeter has larger Y_c. An illustration of the optimal sets for the square in square case is shown in Figure 6 (left) for $L = 3.33$, for which we obtain $r = 0.600$ and $|Du_r|(\Omega) = 5.67$.

Example 5.4 (influence of the aspect ratio and boundary). We revise Example 5.3, keeping Ω as a square of side L and replacing Ω_s by a centrally positioned rectangle of aspect ratio β^2, i.e., the rectangle has height β and width $1/\beta \le L$. Provided that β is sufficiently large, there is a single Cheeger set in $\Omega \setminus \Omega_s$, given by $Open_r(\Omega) \setminus \Omega_s$ for some $r > 0$. However, for sufficiently small β,

$$\frac{1}{L} \le \beta \le \frac{L}{2}\left(\sqrt{1 + \frac{8}{L^2}} - 1\right),$$

and there may be a second Cheeger set configuration, as illustrated in Figure 4.

For the first configuration we use Propositions 4.7 and 4.8 to find the radius $r_1(\beta) = 1/\lambda_{c,1}(\beta)$:

$$r_1(\beta) = \frac{L}{2}\frac{1 + \frac{\beta+1/\beta}{2L}}{1 - \pi/4}\left(1 - \sqrt{1 - (1 - \pi/4)\frac{1 - 1/L^2}{(\frac{\beta+1/\beta}{2L})^2}}\right).$$

The second configuration gives radius $r_2(\beta) = 1/\lambda_{c,2}(\beta)$:

$$r_2(\beta) = \frac{3L - \beta}{8(1 - \pi/4)}\left(1 - \sqrt{1 - 8(1 - \pi/4)\frac{L(L - \beta)}{(3L - \beta)^2}}\right).$$

It is found that for a small band of β the second configuration gives $\lambda_{c,2}(\beta) < \lambda_{c,1}(\beta)$. In both cases we have $\Omega_{1c} = \Omega_s$, and the yield limit is

$$Y_c(\beta) = \frac{1}{Per(\Omega_{1c}) + \min\{\lambda_{c,k}(\beta)\}|\Omega_{1c}|} = \frac{1}{2(\beta + 1/\beta) + \min\{\lambda_{c,k}(\beta)\}}.$$

The variation of λ_c and Y_c is illustrated in Figure 5 for $L = 3$. Note that $Y_c(\beta)$ approaches the square in square results at $\beta = 1$. The difference between the two potential Y_c in Figure 5(b) is relatively small because for small β, $\text{Per}(\Omega_s)$ becomes relatively large.

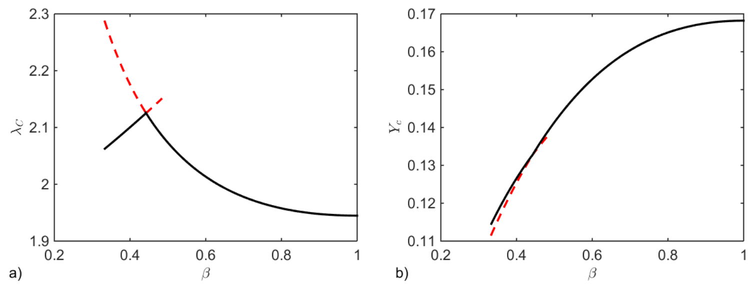

FIG. 5. *Different mechanisms for the rectangle as β is varied for $L = 3$:* (a) $\lambda_c(\beta)$*;* (b) $Y_c(\beta)$*. The optimal values are in solid black and suboptimal are in broken red. (Figure in color online.)*

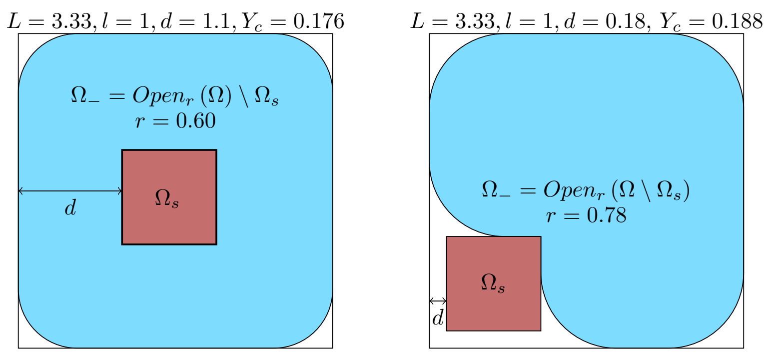

FIG. 6. *In this case, area and perimeter of Ω, Ω_s are constant. We change the distance between $\partial\Omega$ and Ω_s. The critical yield number is larger if the inner set Ω_s is close to $\partial\Omega$.*

This example also serves to demonstrate geometric nonuniqueness. In the case that $\lambda_{c,2}(\beta) < \lambda_{c,1}(\beta)$, either of the shaded regions above or below Ω_s in Figure 4(b) is a Cheeger set, as is the union. We may construct a minimizer of TV over $\mathrm{BV}_{\diamond,1}$ using the characteristic functions of either set, or any linear combination that satisfies the condition of zero flux. As commented earlier this nonuniqueness in $\mathrm{BV}_{\diamond,1}$ stems from the geometric nonuniqueness.

Interestingly, if one were to return to the original Bingham fluid problem and approach $Y \to Y_c^-$, the velocity solution is unique and can be shown to be symmetric, i.e., the effect of viscosity here is to select a symmetric minimizer for $Y < Y_c$.

Example 5.5 (influence of the position of Ω_s with respect to the boundary). We revise Example 5.3 with Ω_s again being a square with length 1. This time we move the inner square Ω_s in the direction of $\partial\Omega$ and denote $d := d(\Omega_s, \partial\Omega)$. The possible minimizers have $\Omega_- = Open_r(\Omega) \setminus \Omega_s$ or $\Omega_- = Open_r(\Omega \setminus \Omega_s)$ for some r, depending on d. We illustrate this phenomenon in Figure 6.

5.3. Multiple particles. We now consider multiple particles. In the first example, we retain the fixed $|\Omega_s| = 1$ and consider the effects of increasing the number of particles. Intuitively, this increases the ratio of perimeter to area and hence we expect that Y_c will reduce, as is indeed found to be the case.

Example 5.6 (a case with nontrivial Ω_1). We consider the two setups of Figure 7, where for simplicity we keep Ω circular. The flat regions correspond to the case

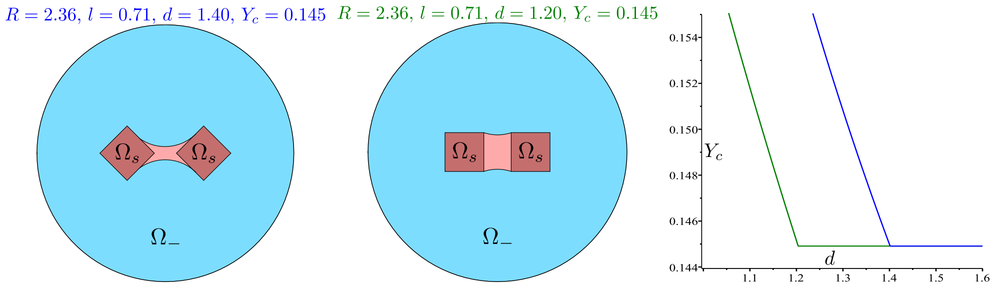

FIG. 7. *Left and center: Two different arrangements of squares at the corresponding transition points. Here, the trivial and nontrivial solutions coexist and the same critical yield number appears for both orientations of the square. Right: Critical yield numbers, with respect to the distance d between the centers of the squares. The corners in the graph represent the transition between* $\Omega_- = Open_r(\Omega \setminus \Omega_s)$ *and* $\Omega_- = \Omega \setminus \Omega_s$.

where the optimal set Ω_- is equal to $\Omega \setminus \Omega_s$.

We see that the orientation has an influence on the behavior of the minimizer as well as on the critical yield number. As d is decreased below a critical value Ω_{1c} incorporates a *bridge* between the two particles. The occurrence of the bridge clearly depends on orientation of the particles, and would also vary for different shaped particles. The phenomena of bridging between particles and of particles essentially acting independently beyond a critical distance have been studied computationally in the case of two spheres [37, 39] (axisymmetric flows) and two cylinders [49] (planar 2D flows). Aside from computed examples we know of no general theoretical results related to these phenomena, e.g., what the maximal distances for bridging are.

Example 5.7 (periodic arranged circles inside a square tube). As a second example, we consider large arrays of particles, as illustrated in Figure 8, i.e., Ω is a square with length L, and Ω_s is the union of N^2 small circles with radius δ, the outermost of which are at distance a from $\partial\Omega$. Here the intention is to illustrate particle size and separation effects. Therefore, we emphasize that in this case $|\Omega_s|$ is not constant for different δ.

Two types of optimal sets appear: For δ small (left), we have $\Omega_1 = \Omega_s$, $\Omega_- = Open_{\lambda^{-1}}(\Omega) \setminus \Omega_s$. For bigger δ (right), one gets $\Omega_1 = Close_{\lambda^{-1}}(\Omega_s)$, and $\Omega_- = Open_{\lambda^{-1}}(\Omega \setminus \Omega_s) = Open_{\lambda^{-1}}(\Omega) \setminus \Omega_1$ for λ the corresponding Cheeger constant. One could think of a third configuration in which isolated components of Ω_- appear between the circles of Ω_s, but it is easy to see that such a configuration has higher energy. Figure 8 (top right) shows the variation in Y_c with δ for a particular choice of parameters ($L = 12$, $N = 12$, and $a = 0.4$). The observable *kink* is where the transition between the two configurations occurs.

Although this example is quite theoretical, this type of phenomenon occurs commonly in non-Newtonian suspension flows. In hydraulic fracturing, proppant suspensions are pumped along narrow fractures. For critical flow rates the individual dense proppant particles may act together in settling: so-called *convection*; see, e.g., [16]. This represents a serious risk for the process in that in *convective* settling the group of particles settles faster than when individually settling, as in the latter case secondary flows are induced on a more local scale. It is interesting that these features (local and global) are captured by the simple model here, where the yield stress fluid definitively couples the particles via bridging. Convective settling is, however, not, in general, reliant on the yield stress.

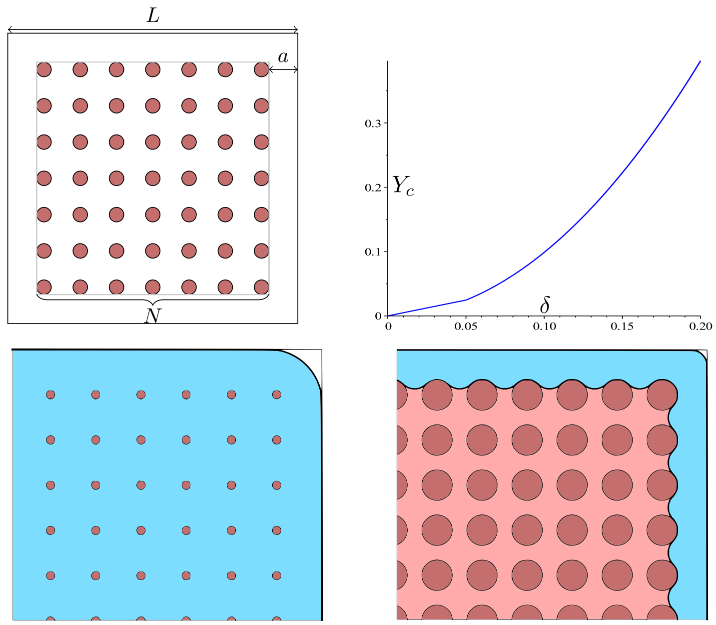

Fig. 8. *Upper row, left: Setup for the periodic case. Upper row, right: Dependence of the critical yield number on δ, for $L = 12$, $N = 12$, and $a = 0.4$. The corner in the graph corresponds to the transition from trivial to bridged optimal sets. Lower row: Optimal sets for $\delta = 0.04$ and $\delta = 0.2$, when $L = 12$, $N = 12$, and $a = 0.4$.*

These examples also expose an interesting question concerning individual particle behavior. Dense suspensions in shear-thinning fluids often exhibit interesting settling patterns, e.g., the column-like patterns in [17]. Such patterns are excluded in our study as we have assumed that the speed of Ω_s is uniform. There is a rich vein of interesting problems here to study. For example, if we remove the constraint of equal particle velocities, do particle arrays such as that considered above admit other optimal solutions that select patterns amongst the particles, e.g., stripes moving at different speeds, or are slight perturbations from the regular lattice favorable?

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