Reconstructing the Optical Parameters of a Layered Medium with Optical Coherence Elastography

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Abstract In this work we consider the inverse problem of reconstructing the optical properties of a layered medium from an elastography measurement where optical coherence tomography is used as the imaging method. We hereby model the sample as a linear dielectric medium so that the imaging parameter is given by its electric susceptibility, which is a frequency- and depth-dependent parameter. Additionally to the layered structure (assumed to be valid at least in the small illuminated region), we allow for small scatterers which we consider to be randomly distributed, a situation which seems more realistic compared to purely homogeneous layers. We then show that a unique reconstruction of the susceptibility of the medium (after averaging over the small scatterers) can be achieved from optical coherence tomography measurements for different compression states of the medium.

Keywords: Optical Coherence Tomography, Optical Coherence Elastography, Inverse Problem, Parameter Identification

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1 Introduction

Optical Coherence Tomography is an imaging modality producing high resolution images of biological tissues. It measures the magnitude of the back-scattered light of a focused laser illumination from a sample as a function of depth and provides cross-sectional or volumetric data by performing a series of multiple axial scans at different positions. Initially, it used to operate in time where a movable mirror was giving the depth information. Later on, frequency-domain optical coherence tomography was introduced where the detector is replaced by a spectrometer and no mechanical movement is needed. We refer to [3, 4] for an overview of the physics of the experiment and to [6] for a mathematical description of the problem.

Only lately, the inverse problems arising in optical coherence tomography have attracted the interest from the mathematical community, see, for example, [2, 7, 11, 13]. For many years, the proposed and commonly used reconstruction method was just the inverse Fourier transform. This approach is valid only if the properties of the medium are assumed to be frequency-independent in the spectrum of the light source. However, the less assumptions one takes, the more mathematically interesting but also difficult the problem becomes.

The main assumption, we want to make is that the medium can be (at least locally in the region where the laser beam illuminates the object) well described by a layered structure. Since there are in real measurement images typically multiple small particles visible inside these layers, we will additionally include small, randomly distributed scatterers into the model and calculate the averaged contribution of these particles to the measured fields.

To obtain a reconstruction of the medium, that is, of its electric susceptibility, we consider an elastography setup where optical coherence tomography is used as the imaging system. This so-called optical coherence elastography is done by recording optical coherence tomography data for different compression states of the medium, see [1, 5, 9, 12] for some recent works dealing with this interesting problem.

Under the assumption that the sample can be described as a linear elastic medium, we show that these measurements can be used to achieve a unique reconstruction of the electric susceptibility of the layered medium.

The paper is organised as follows: In Section 2 we review the main equations describing mathematically how the data in optical coherence tomography is collected and their relation to the optical properties of the medium. In Section 3, we show that the calculation of the back-scattered field can be decomposed into the corresponding subproblems for the single layers, for which we derive the resulting formulæ in Section 4. Finally, we present in Section 5 that from the measurements at different compression states a unique reconstruction of the susceptibility becomes feasible.

2 Modelling the optical coherence tomography measurement

We model the sample by a dispersive, isotropic, non-magnetic, linear dielectric medium characterised by its scalar electric susceptibility. To include randomly distributed scatterers in the model, we introduce the susceptibility as a random variable; so let $(\mathscr{X},\mathscr{A},P)$ be a probability space and write

$$\chi : \mathscr{X} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R},\ (\sigma,t,x) \mapsto \chi_\sigma(t,x)$$

for the electric susceptibility of the medium in the state σ. (Hereby, to have a causal model, we require an electric susceptibility $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ to be a function fulfilling $\chi(t,x) = 0$ for all $t < 0$.)

The object (in a certain realisation state $\sigma \in \mathscr{X}$) is then probed with a laser beam, described by an incident electric field $E^{(0)} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ which we choose as a solution in the homogeneous susceptibility outside the object that does not interact with the object before the time $t = 0$.

Definition 1. We call $E^{(0)} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ an incident wave for a given susceptibility $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ in a homogeneous background $\chi_0 : \mathbb{R} \to \mathbb{R}$ if it is a solution of Maxwell's equations for χ_0, that is,

$$\Delta E^{(0)}(t,x) = \frac{1}{c^2}\partial_{tt} D^{(0)}(t,x),$$

where c denotes the speed of light in vacuum and

$$D^{(0)}(t,x) = E^{(0)}(t,x) + \int_{\mathbb{R}} \chi_0(\tau) E^{(0)}(t-\tau,x)\,\mathrm{d}\tau,$$

and $E^{(0)}$ does not interact with the inhomogeneity for negative times, meaning that

$$E^{(0)}(t,x) = 0 \text{ for all } t \in (-\infty,0),\ x \in \Omega \tag{1}$$

with $\Omega = \{x \in \mathbb{R}^3 \mid \chi(\cdot,x) \neq \chi_0\}$.

We then measure the resulting electric field $E_\sigma : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ induced by the incident field $E^{(0)}$ in the presence of the dielectric medium described by the susceptibility χ_σ.

Definition 2. Let $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be a susceptibility and $E^{(0)} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be an incident wave for χ. Then, we call E the electric field induced by $E^{(0)}$ in the presence of χ if E is a solution of the equation system

$$\operatorname{curl}\operatorname{curl} E(t,x) + \frac{1}{c^2}\partial_{tt} D(t,x) = 0 \text{ for all } t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{2}$$

$$E(t,x) - E^{(0)}(t,x) = 0 \text{ for all } t \in (-\infty,0),\ x \in \mathbb{R}^3 \tag{3}$$

with c being the speed of light in vacuum and with the electric displacement field $D : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ being related to the electric field via

$$D(t,x) = E(t,x) + \int_{\mathbb{R}} \chi(\tau,x)E(t-\tau,x)\,\mathrm{d}\tau.$$

Remark 1. The fact that $E^{(0)}$ does not interact with the object before time $t = 0$, see (1), guarantees that $E^{(0)}$ is a solution of (2) and thus the initial condition in (3) is compatible with (2).

Remark 2. We do not want to specify the solution concept for solving (2) here (since we are going for a layered and therefore discontinuous susceptibility, there exists only a weak solution), but will silently assume that the susceptibility and the incident field are such that they induce an electric field with sufficient regularity and the appearing integrals and Fourier transforms are well-defined.

Equation (2) is more conveniently written in Fourier space, where we use the convention

$$\mathscr{F}[f](k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \mathrm{d}x$$

for the Fourier transform of an integrable function $f : \mathbb{R}^n \to \mathbb{R}$. For convenience, we also use the shorter notation

$$\check{F}(\omega,x) = \sqrt{2\pi}\,\mathscr{F}^{-1}[t \mapsto F(t,x)](\omega) = \int_{\mathbb{R}} F(t,x)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t$$

for this rescaled inverse Fourier transformation of a sufficiently regular function of the form $F : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^n$ with respect to the time variable.

Lemma 1. *Let $\chi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ be a susceptibility, $E^{(0)} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ be an incident wave for χ, and E be the induced electric field. Then, $\check{E}$ solves the vector Helmholtz equation*

$$\operatorname{curl}\operatorname{curl}\check{E}(\omega,x) - \frac{\omega^2}{c^2}(1+\check{\chi}(\omega,x))\check{E}(\omega,x) = 0 \textit{ for all } \omega\in\mathbb{R},\ x\in\mathbb{R}^3, \tag{4}$$

with the constraint

$$\check{E} \in \mathscr{H}(\check{E}^{(0)}), \tag{5}$$

where $\mathscr{H}(\check{E}^{(0)})$ is the space of all functions $F : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ so that the map $\omega \mapsto (F-\check{E}^{(0)})(\omega,x)$ can be holomorphically extended to the space $\mathbb{H}\times\mathbb{R}^3$, where $\mathbb{H} = \{z\in\mathbb{C} \mid \Im\mathrm{m}\, z > 0\}$ denotes the upper half complex plane, and the extension fulfils

$$\sup_{\lambda>0} \int_{\mathbb{R}} |(F-\check{E}^{(0)})(\omega+\mathrm{i}\lambda,x)|^2 \mathrm{d}\omega < \infty$$

for every $x\in\mathbb{R}^3$.

Proof. Equation (4) is obtained directly from the application of the Fourier transform to (2). The condition (5) is according to the Paley–Wiener theorem, see, for example, [10, Theorem 9.2], equivalent to the condition (3), which states that $t \mapsto (E-E^{(0)})(t,x)$ has for every $x\in\mathbb{R}^3$ only support in $[0,\infty)$.

In frequency-domain optical coherence tomography, we detect with a spectrometer at a position $x_0 \in \mathbb{R}^3$ outside the medium the intensity of the Fourier components of the superposition of the back-scattered light from the sample and the reference beam, which is the reflection of the incident laser beam from a mirror at some fixed position.

Here, we consider two independent measurements for two different positions of the mirror in order to overcome the problem of phase-less data, see [8]. Thus, we record for some realisation $\sigma \in \mathscr{X}$ and all $\omega \in \mathbb{R}$ the data

$$m_{0,\sigma}(\omega) = |\check{E}_\sigma(\omega,x_0)| \text{ and } m_{i,\sigma}(\omega) = |\check{E}_\sigma(\omega,x_0) + \check{E}_i^{(\mathrm{r})}(\omega,x_0)|,\ i \in \{1,2\},$$

where $E_1^{(\mathrm{r})} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ and $E_2^{(\mathrm{r})} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3$ denote the two known reference waves, which are solutions of Maxwell's equations in the homogeneous background medium (usually well approximated by the vacuum).

We can uniquely recover from this data the (complex-valued) Fourier transform $\check{E}(\omega,x_0)$ of the electric field for every $\omega \in \mathbb{R}$ by intersecting the three circles

$$\partial B_{m_{0,\sigma}(\omega)}(0) \cap \partial B_{m_{1,\sigma}(\omega)}(-\check{E}_1^{(\mathrm{r})}(\omega,x_0)) \cap \partial B_{m_{2,\sigma}(\omega)}(-\check{E}_2^{(\mathrm{r})}(\omega,x_0))$$

provided that the points 0, $\check{E}_1^{(\mathrm{r})}(\omega,x_0)$, and $\check{E}_2^{(\mathrm{r})}(\omega,x_0)$ in the complex plane do not lie on a single straight line. In the following, we assume that the fields $E_1^{(\mathrm{r})}$ and $E_2^{(\mathrm{r})}$ are chosen such that this condition is satisfied and we can recover the function

$$m_\sigma(\omega) = \check{E}_\sigma(\omega,x_0) \text{ for all } \omega \in \mathbb{R}.$$

However, this information is still not enough for reconstructing the material parameter χ_σ, see, for example, [6]. Thus, we make the a priori assumption that the illuminated region of the medium can be well approximated by a layered medium. Since the layers are typically not completely homogeneous, we also allow for randomly distributed small inclusions in every layer.

Thus, we describe χ to be of the form

$$\chi_\sigma(t,x) = \chi_j(t) + \psi_{j,\sigma_j}(t,x) \tag{6}$$

in the j-th layer $\{x \in \mathbb{R}^3 \mid z_{j+1} < x_3 < z_j\}$, $j \in \{1,\dots,J\}$, where we write the measure space as a product $\mathscr{X} = \prod_{j=1}^J \mathscr{X}_j$ with each factor representing the state of one layer. Here, χ_j is the homogeneous background susceptibility of the layer and ψ_j is the random contribution caused by some small particles in the layer. Outside these layers, we set $\chi_\sigma(t,x) = \chi_0(t)$ for some homogeneous background susceptibility χ_0.

To simplify the analysis, we will assume that the scatterers in the j-th layer only occur at some distance to the layer boundaries z_j and z_{j+1}, say between Z_j and ζ_j, where $z_{j+1} < Z_j < \zeta_j < z_j$. Moreover, we choose the particles independently, identically, uniformly distributed on the part $U_{j,L_j} = [-\frac{1}{2}L_j, \frac{1}{2}L_j] \times [-\frac{1}{2}L_j, \frac{1}{2}L_j] \times [Z_j, \zeta_j]$ of the layer for some width $L_j > 0$. Concretely, we assume that we have in the j-th layer for some number N_j of particles the probability measure P_{j,N_j,L_j} on the probability space $\mathscr{X}_j = (U_{j,L_j})^{N_j}$ given by

$$P_{j,N_j,L_j}(\prod_{\ell=1}^{N_j} A_\ell) = \prod_{\ell=1}^{N_j} \frac{|A_\ell|}{L_j^2(\zeta_j - Z_j)} \tag{7}$$

for all measurable subsets $A_\ell \subset U_{j,L_j}$, where $|A_\ell|$ denotes the three dimensional Lebesgue measure of the set A_ℓ.

The full probability measure $P = P_{N,L}$ is consistently chosen as the direct product $P_{N,L} = \prod_{j=1}^{J} P_{j,N_j,L_j}$ on $\mathscr{X} = \prod_{j=1}^{J} \mathscr{X}_j$.

The particles themselves, we model in each layer as identical balls with a sufficiently small radius R and a homogeneous susceptibility $\chi_j^{(\mathrm{p})}$. Thus, we define for a realisation $\sigma_j \in \mathscr{X}_j$ of the j-th layer the contribution of the particles to the susceptibility by

$$\psi_{j,\sigma_j}(t,x) = \sum_{\ell=1}^{N_j} \boldsymbol{\chi}_{B_R(\sigma_{j,\ell})}(x)\,(\chi_j^{(\mathrm{p})}(t) - \chi_j(t)), \tag{8}$$

where we ignore the problem of overlapping particles. Here, we denote by $\boldsymbol{\chi}_A$ the characteristic function of a set A and by $B_r(y)$ the open ball with radius r around a point y.

3 Domain decomposition of the solution

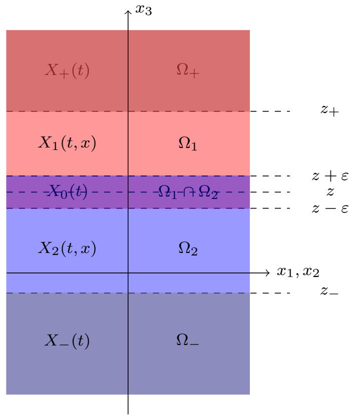
The layered structure of the medium allows us to decompose the solution as a series of solution operators for the single layers. To do so, we split the medium at a horizontal stripe where the medium is homogeneous and consider the two subproblems where once the region above and once the region below is replaced by the homogeneous susceptibility $X_0 : \mathbb{R} \to \mathbb{R}$ in the stripe. We write the stripe as the set $\{x \in \mathbb{R}^3 \mid z - \varepsilon < x_3 < z + \varepsilon\}$ for some $z \in \mathbb{R}$ and some height $\varepsilon > 0$ and parametrise the electric susceptibility in the form

$$\chi(t,x) = \begin{cases} X_1(t,x) & \text{if } x \in \Omega_1 = \{y \in \mathbb{R}^3 \mid y_3 > z - \varepsilon\}, \\ X_2(t,x) & \text{if } x \in \Omega_2 = \{y \in \mathbb{R}^3 \mid y_3 < z + \varepsilon\}. \end{cases} \tag{9}$$

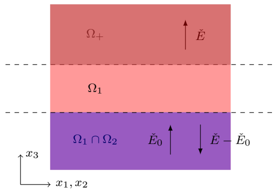
with the necessary compatibility condition that X_1 and X_2 coincide in the intersection $\Omega_1 \cap \Omega_2$, where they should both be equal to the homogeneous susceptibility X_0.

Additionally, we have the assumption that the medium is bounded in vertical direction. We can therefore assume that for some $z_- < z_+$, the susceptibilities X_1 and X_2 are homogeneous in $\Omega_+ = \{x \in \mathbb{R}^3 \mid x_3 > z_+\} \subset \Omega_1$ and $\Omega_- = \{x \in \mathbb{R}^3 \mid x_3 < z_-\} \subset \Omega_2$, respectively. We set

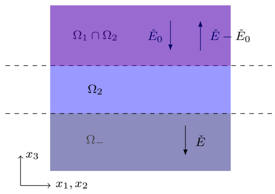
$$X_1(t,x) = X_+(t) \text{ for all } x \in \Omega_+ \text{ and } X_2(t,x) = X_-(t) \text{ for all } x \in \Omega_-.$$



(a) The subdomains and the corresponding optical parameters.



(b) The fields related to the operator $\mathscr{G}_1$.



(c) The fields related to the operator $\mathscr{G}_2$.

Fig. 1: The geometry and the notation used in this section.

Since we are solving Maxwell's equations on the whole space, we extend X_1 and X_2 by the homogeneous susceptibility X_0:

$$X_1(t,x) = X_0(t) \text{ for all } x \in \Omega_2 \text{ and } X_2(t,x) = X_0(t) \text{ for all } x \in \Omega_1,$$

see Picture (a) in Figure 1 for an illustration of the notation.

The aim is then to reduce the calculation of the electric field in the presence of χ to the subproblems of determining the electric fields in the presence of X_1 and X_2, independently. To do so, we consider the solution in the intersection $\Omega_1 \cap \Omega_2$ and split it there into waves moving in the positive and negative e_3 direction.

Lemma 2. *Let a homogeneous susceptibility $\chi : \mathbb{R} \to \mathbb{R}$ be given on a stripe $\Omega_0 = \{x \in \mathbb{R}^3 \mid x_3 \in (z_0 - \varepsilon, z_0 + \varepsilon)\}$. Then, every solution $\check{E} : \mathbb{R} \times \Omega_0 \to \mathbb{C}^3$ of*

$$\operatorname{curl}\operatorname{curl}\check{E}(\omega,x) - \frac{\omega^2}{c^2}(1+\check{\chi}(\omega))\check{E}(\omega,x) = 0 \text{ for all } \omega \in \mathbb{R},\ x \in \Omega_0, \tag{10}$$

admits the form

$$\begin{aligned}\check{E}(\omega,x) = &\int_{\mathbb{R}^2} e_1(k_1,k_2)\mathrm{e}^{-\mathrm{i}x_3\sqrt{\frac{\omega^2}{c^2}(1+\check{\chi}(\omega))-k_1^2-k_2^2}}\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)}\mathrm{d}(k_1,k_2) \\ &+ \int_{\mathbb{R}^2} e_2(k_1,k_2)\mathrm{e}^{\mathrm{i}x_3\sqrt{\frac{\omega^2}{c^2}(1+\check{\chi}(\omega))-k_1^2-k_2^2}}\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)}\mathrm{d}(k_1,k_2)\end{aligned} \tag{11}$$

for all $\omega \in \mathbb{R}$ and $x \in \Omega_0$ with some coefficients e_1, $e_2 : \mathbb{R}^2 \to \mathbb{C}^3$.

Proof. Taking the divergence of (10), we see that we have $\operatorname{div}\check{E} = 0$ on the stripe Ω_0 with homogeneous susceptibility. Then, equation (10) reduces to the three independent Helmholtz equations

$$\Delta\check{E}(\omega,x) + \frac{\omega^2}{c^2}(1+\check{\chi}(\omega))\check{E}(\omega,x) = 0 \text{ for all } \omega \in \mathbb{R},\ x \in \Omega_0.$$

Applying the Fourier transform with respect to x_1 and x_2 and solving the resulting ordinary differential equation in x_3 gives us (11).

Definition 3. Let $\check{E}$ be a solution of the equation (10) on some stripe Ω_0, written in the form (11). We then call $\check{E}$ a downwards moving solution if $e_2 = 0$ and an upwards moving solution if $e_1 = 0$.

Moreover, we define the solution operators $\mathscr{G}_1$ and $\mathscr{G}_2$. To avoid having to define an incident wave on the whole space, we replace the condition (5) by radiation conditions of the form that we specify the upwards moving part on a stripe below the region and the downwards moving part on a stripe above the region.

Definition 4. Let χ be given as in (9) and $\check{E}_0$ be an upwards moving solution in $\Omega_1 \cap \Omega_2$. Then, we define $\mathscr{G}_1\check{E}_0$ as a solution $\check{E}$ of the equation

$$\operatorname{curl}\operatorname{curl}\check{E}(\omega,x) - \frac{\omega^2}{c^2}(1+\check{X}_1(\omega,x))\check{E}(\omega,x) = 0$$

fulfilling the radiation condition that $\check{E} - \check{E}_0$ is a downwards moving solution in $\Omega_1 \cap \Omega_2$ and that $\check{E}$ is an upwards moving solution in Ω_+, see Picture (b) in Figure 1.

Analogously, we define $\mathscr{G}_2\check{E}_0$ for a downwards moving solution $\check{E}_0$ in $\Omega_1 \cap \Omega_2$ as a solution $\check{E}$ of the equation

$$\operatorname{curl}\operatorname{curl}\check{E}(\omega,x) - \frac{\omega^2}{c^2}(1+\check{X}_2(\omega,x))\check{E}(\omega,x) = 0$$

fulfilling the radiation condition that $\check{E} - \check{E}_0$ is an upwards moving solution in $\Omega_1 \cap \Omega_2$ and that $\check{E}$ is a downwards moving solution in Ω_-, see Picture (c) in Figure 1.

Remark 3. We do not consider the uniqueness of these solutions at this point, since we will only need the result for particular, simplified problems where the verification that this gives the desired solution can be done directly.

Instead we will simply assume that the susceptibilities χ, X_1, and X_2 are such that the only solution $\check{E}$ in the presence of this susceptibility for which $\check{E}$ is upwards moving on Ω_+ and downwards moving on Ω_- is the trivial solution $\check{E} = 0$, meaning that there is only the trivial solution in the absence of an incident wave.

Lemma 3. *Let χ be given by* (9) *and denote by $\mathscr{G}_1$, $\mathscr{G}_2$ the solution operators as in Definition 4. Let further $E^{(0)}$ be an incident wave on χ which is moving downwards and E_1 be the induced electric fields in the presence of X_1.*

Then, provided the following series converge, we have that the function E defined by

$$\check{E}(\omega,x)=\begin{cases}\check{E}_1(\omega,x)+\sum_{j=0}^{\infty}\mathscr{G}_1(\tilde{\mathscr{G}}_2\tilde{\mathscr{G}}_1)^j\tilde{\mathscr{G}}_2\check{E}_1(\omega,x) & \text{if } x\in\Omega_1,\\ \sum_{j=0}^{\infty}\mathscr{G}_2(\tilde{\mathscr{G}}_1\tilde{\mathscr{G}}_2)^j\check{E}_1(\omega,x) & \text{if } x\in\Omega_2,\end{cases}$$

where we set $\tilde{\mathscr{G}}_i=\mathscr{G}_i-\mathrm{id}$, $i\in\{1,2\}$, *is an electric field in the presence of* χ *fulfilling the radiation conditions that* $\check{E}-\check{E}^{(0)}$ *is an upwards moving wave in* Ω_+ *and* $\check{E}$ *is a downwards moving wave in* Ω_-.

Proof. First, we remark that the composition of the operators is well defined, since $\check{E}_1\in\mathscr{H}(\check{E}^{(0)})$ is a downwards moving solution in $\Omega_1\cap\Omega_2$, see Lemma 1, the range of $\tilde{\mathscr{G}}_2$ consists of upwards moving solutions, and the range of $\tilde{\mathscr{G}}_1$ consists of downwards moving solutions.

The field $\check{E}$ is seen to satisfy (4) in Ω_1 by using the definitions of E_1 and the solution operator $\mathscr{G}_1$ on Ω_1. Similarly, using the definition of $\mathscr{G}_2$, we get that the function $\check{E}$ satisfies (4) in Ω_2.

Therefore, it only remains to check that the two formulas coincide in the intersection $\Omega_1\cap\Omega_2$. Using that $\mathscr{G}_i=\tilde{\mathscr{G}}_i+\mathrm{id}$, $i\in\{1,2\}$, we find that

$$\begin{aligned}\check{E}_1+\sum_{j=0}^{\infty}\mathscr{G}_1(\tilde{\mathscr{G}}_2\tilde{\mathscr{G}}_1)^j\tilde{\mathscr{G}}_2\check{E}_1&=\check{E}_1+\sum_{j=0}^{\infty}\tilde{\mathscr{G}}_1(\tilde{\mathscr{G}}_2\tilde{\mathscr{G}}_1)^j\tilde{\mathscr{G}}_2\check{E}_1+\sum_{j=0}^{\infty}(\tilde{\mathscr{G}}_2\tilde{\mathscr{G}}_1)^j\tilde{\mathscr{G}}_2\check{E}_1\\&=\sum_{j=0}^{\infty}(\tilde{\mathscr{G}}_1\tilde{\mathscr{G}}_2)^j\check{E}_1+\sum_{j=0}^{\infty}\tilde{\mathscr{G}}_2(\tilde{\mathscr{G}}_1\tilde{\mathscr{G}}_2)^j\check{E}_1=\sum_{j=0}^{\infty}\mathscr{G}_2(\tilde{\mathscr{G}}_1\tilde{\mathscr{G}}_2)^j\check{E}_1.\end{aligned}$$

Moreover, we have that $\check{E}-\check{E}_1$ is by construction an upwards moving wave in Ω_+, and therefore so is $\check{E}-\check{E}^{(0)}$. Similarly, the wave $\check{E}$ is a downwards moving wave in Ω_-.

If we are in a case where our uniqueness assumption mentioned in Remark 3 holds, then Lemma 3 allows us to iteratively reduce the problem of determining the electric field in the presence of the susceptibility χ_σ, defined in (6), to problems of simpler susceptibilities. To this end, we could, for example, successively apply the result to values $z\in(\zeta_j,z_j)$ and $z\in(z_{j+1},Z_j)$, $j=1,\dots,J$, where each successive step is only used to further simplify the operator $\mathscr{G}_2$ from the previous step. This then leads to a sort of layer stripping algorithm, see, for example, [8], where a similar argument was presented.

4 Wave propagation through a scattering layer

Using the above analysis, we can calculate the electric field in the presence of a layered medium of the form (6) as a combination of the solutions of the following two subproblems.

Problem 1. Let $j \in \{0,\dots,J-1\}$. Find the electric field induced by some incident field in the presence of the piecewise homogeneous susceptibility χ given by

$$\chi(t,x) = \begin{cases} \chi_j(t) & \text{if } x_3 > z_{j+1}, \\ \chi_{j+1}(t) & \text{if } x_3 < z_{j+1}. \end{cases} \tag{12}$$

Problem 2. Let $\sigma \in \mathscr{X}$ and $j \in \{1,\dots,J\}$. Find the electric field induced by some incident field in the presence of the susceptibility χ given by

$$\chi(t,x) = \chi_j(t) + \psi_{j,\sigma}(t,x), \tag{13}$$

where the function ψ_j is described by (8).

We thus fix a layer $j \in \{0,\dots,J\}$, and to simplify the calculations, we restrict ourselves in both subproblems to an illumination by a downwards moving plane wave of the form

$$\check{E}^{(0)}(\omega,x) = \check{f}(\omega)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)x_3}\eta \tag{14}$$

for some function $f : \mathbb{R} \to \mathbb{R}$ and a polarisation vector $\eta \in \mathbb{S}^1 \times \{0\}$. Here we define the complex-valued refractive indices for all $j \in \{0,\dots,J\}$ by

$$n_j : \mathbb{R} \to \mathbb{H},\ n_j(\omega) = \sqrt{1+\check{\chi}_j(\omega)}. \tag{15}$$

Then, the solution of Problem 1 can be explicitly written down.

Lemma 4. *Let $j \in \{0,\dots,J-1\}$ and $E^{(0)}$ be the incident wave given in* (14). *Then, the electric field E induced by $E^{(0)}$ in the presence of a susceptibility χ of the form* (12) *is given by*

$$\check{E}(\omega,x) = \check{f}(\omega)\left(\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)x_3} - \frac{n_{j+1}(\omega)-n_j(\omega)}{n_{j+1}(\omega)+n_j(\omega)}\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)z_{j+1}}\mathrm{e}^{\mathrm{i}\frac{\omega}{c}n_j(\omega)(x_3-z_{j+1})}\right)\eta$$

for $x_3 > z_{j+1}$, and by

$$\check{E}(\omega,x) = \check{f}(\omega)\frac{2n_j(\omega)}{n_{j+1}(\omega)+n_j(\omega)}\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)z_{j+1}}\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_{j+1}(\omega)(x_3-z_{j+1})}\eta$$

for $x_3 < z_{j+1}$, where the refractive indices n_j and n_{j+1} are defined by (15).

Proof. Clearly, $\check{E}$ satisfies the differential equation (4) in both regions $x_3 > z_{j+1}$ and $x_3 < z_{j+1}$. Moreover, $\check{E}^{(0)}$ is the only incoming wave in $\check{E}$. Therefore, it only remains to check that $\check{E}$ has sufficient regularity to be the weak solution along the discontinuity of the susceptibility at $x_3 = z_{j+1}$, meaning that

$$\lim_{x_3 \uparrow z_{j+1}} \check{E}(\omega,x) = \lim_{x_3 \downarrow z_{j+1}} \check{E}(\omega,x),$$
$$\lim_{x_3 \uparrow z_{j+1}} n_{j+1}(\omega)\partial_{x_3}\check{E}(\omega,x) = \lim_{x_3 \downarrow z_{j+1}} n_j(\omega)\partial_{x_3}\check{E}(\omega,x).$$

Both identities are readily verified.

For Problem 2, the situation is more complicated and we settle for an approximate solution for the electric field. For that, we assume (using the same notation as in (8)) that the susceptibility $\chi_j^{(\mathrm{p})}$ of the random particles does not differ much from the background χ_j, so that the difference between the induced field and the incident field becomes small, and we do a first order approximation in the difference $\chi_j^{(\mathrm{p})} - \chi_j$. For that purpose, we write the differential equation (4) in the form

$$\operatorname{curl}\operatorname{curl}\check{E}(\omega,x) - \frac{\omega^2}{c^2} n_j^2(\omega)(1+\bar{\phi}_{j,\sigma_j}(\omega,x))\check{E}(\omega,x) = 0,$$

where, according to (8),

$$\bar{\phi}_{j,\sigma_j}(\omega,x) = \sum_{\ell=1}^{N_j} \boldsymbol{\chi}_{B_R(\sigma_{j,\ell})}(x)\,\phi_j(\omega),$$

and we abbreviate

$$\phi_j(\omega) = \frac{\check{\chi}_j^{(\mathrm{p})}(\omega) - \check{\chi}_j(\omega)}{1+\check{\chi}_j(\omega)}. \tag{16}$$

In first order in $\bar{\phi}$, we then approximate the field by the solution $\check{E}_{N_j,\sigma_j}^{(1)}$ of the equation

$$\operatorname{curl}\operatorname{curl}\check{E}^{(1)}(\omega,x) - \frac{\omega^2}{c^2} n_j^2(\omega)\check{E}^{(1)}(\omega,x) = \frac{\omega^2}{c^2} n_j^2(\omega)\bar{\phi}_{j,\sigma_j}(\omega,x)\check{E}^{(0)}(\omega,x),$$

the so called Born approximation. Using that the fundamental solution G of the Helmholtz equation, which by definition fulfils

$$\Delta G(\kappa,x) + \kappa^2 G(\kappa,x) = -\delta(x),$$

is given by

$$G(\kappa,x) = \frac{\mathrm{e}^{\mathrm{i}\kappa|x|}}{4\pi|x|},$$

we obtain the expression

$$\begin{aligned}
&\check{E}_{N_j,\sigma_j}^{(1)}(\omega,x) = \check{E}^{(0)}(\omega,x) \\
&+ \left(\frac{\omega^2}{c^2} n_j^2(\omega) + \operatorname{grad}\operatorname{div}\right) \sum_{\ell=1}^{N_j} \int_{B_R(\sigma_{j,\ell})} G(\tfrac{\omega}{c} n_j(\omega), x-y)\phi_j(\omega)\check{E}^{(0)}(\omega,y)\mathrm{d}y
\end{aligned} \tag{17}$$

for the Born approximation of the induced field, see, for example, [6, Proposition 4].

We now want to determine the expected value of $E_{N_j,\sigma_j}^{(1)}$ in the limit where the number of particles N_j and the width L_j of the region where the particles are horizontally distributed tend to infinity, while keeping the ratio $\rho_j = \frac{N_j}{L_j^2}$ of particles per surface area constant, that is, we want to calculate the expression

$$\bar{E}^{(1)}(\omega,x) = \lim_{N_j\to\infty} \int_{\mathscr{X}_j} \check{E}^{(1)}_{N_j,\sigma_j}(\omega,x)\mathrm{d}P_{j,N_j,L_j(N_j)}(\sigma_j), \tag{18}$$

where $L_j(N_j) = \sqrt{\frac{N_j}{\rho_j}}$ and P denotes the probability measure introduced in (7).

Lemma 5. *Let $j \in \{1,\dots,J\}$ and $\rho_j > 0$ be fixed, $E^{(0)}$ be an incident field of the form* (14)*, and χ be the susceptibility specified in* (13)*.*

Then, the expected value $\bar{E}^{(1)}$ of the Born approximation of the field induced by $E^{(0)}$ in the presence of the susceptibility χ in the limit $N_j \to \infty$ with $L_j^2\rho_j = N_j$, as introduced in (18)*, is given by*

$$\begin{aligned}\bar{E}^{(1)}(\omega,x) &= \check{E}^{(0)}(\omega,x) + (2\pi)^4\rho_j\phi_j(\omega)\check{f}(\omega) \\ &\quad\times h(2R\tfrac{\omega}{c}n_j(\omega))\left(\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)Z_j} - \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\zeta_j}\right)\mathrm{e}^{\mathrm{i}\frac{\omega}{c}n_j(\omega)(x_3-\mu_j)}\eta\end{aligned} \tag{19}$$

for $x_3 > \zeta_j + R$ and by

$$\begin{aligned}\bar{E}^{(1)}(\omega,x) &= \check{E}^{(0)}(\omega,x) + \frac{(2\pi)^4}{3}\rho_j\phi_j(\omega)\check{f}(\omega) \\ &\quad\times\left(\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)Z_j} - \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\zeta_j}\right)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)(x_3-\mu_j)}\eta\end{aligned} \tag{20}$$

for $x_3 < Z_j - R$, where $\mu_j = \frac{1}{2}(\zeta_j + Z_j)$ and

$$h(\xi) = \frac{\sin(\xi) - \xi\cos(\xi)}{\xi^3}. \tag{21}$$

Proof. Inserting the expression (17) for the Born approximation of the electric field into the formula (18) for the expected value, we obtain the equation

$$\begin{aligned}\bar{E}^{(1)}(\omega,x) &= \check{E}^{(0)}(\omega,x) \\ &\quad+ \lim_{N_j\to\infty} N_j\phi_j(\omega)\check{f}(\omega)\left(\frac{\omega^2}{c^2}n_j^2(\omega) + \operatorname{grad}\operatorname{div}\right)K_{L_j(N_j)}(\omega,x)\eta,\end{aligned} \tag{22}$$

where

$$K_L(\omega,x) = \int_{U_{j,L}}\int_{B_R(\sigma_{j,1})} G(\tfrac{\omega}{c}n_j(\omega),x-y)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)x_3}\mathrm{d}y\,\mathrm{d}\sigma_{j,1}.$$

We recall that $U_{j,L} = [-\frac{1}{2}L,\frac{1}{2}L] \times [-\frac{1}{2}L,\frac{1}{2}L] \times [Z_j,\zeta_j]$ is for $L = L_j$ the region in which the particles in the j-th layer are lying. To symmetrise the expression, we set

$$\mu_j = \frac{1}{2}(\zeta_j + Z_j) \text{ and } d_j = \frac{1}{2}(\zeta_j - Z_j)$$

and shift $U_{j,L}$ to the origin, by defining $\tilde{U}_{j,L} = U_{j,L} - \mu_j e_3$ with $e_3 = (0,0,1)$.

Introducing the probability density

$$p_L(\xi) = \frac{1}{|U_{j,L}|}\boldsymbol{\chi}_{U_{j,L}}(\mu_j e_3 + \xi) = \frac{1}{2L^2 d_j}\boldsymbol{\chi}_{\tilde{U}_{j,L}}(\xi)$$

for the variable $\xi = \sigma_{j,1} - \mu_j e_3$, we rewrite K_L in the form

$$\begin{aligned}
K_L(\omega,x) = {} & \int_{\mathbb{R}^3} p_L(\xi)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)(\mu_j+\xi_3)} \\
& \times \int_{\mathbb{R}^3} \boldsymbol{\chi}_{B_R(0)}(y)G(\tfrac{\omega}{c}n_j(\omega), x-\mu_j e_3 - \xi - y)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)y_3}\mathrm{d}y\,\mathrm{d}\xi \\
= {} & (2\pi)^{\frac{3}{2}} \int_{\mathbb{R}^3} p_L(\xi)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)(\mu_j+\xi_3)} \\
& \times \mathscr{F}[y \mapsto \boldsymbol{\chi}_{B_R(0)}(y)G(\tfrac{\omega}{c}n_j(\omega), x-\mu_j e_3 - \xi - y)](\tfrac{\omega}{c}n_j(\omega)e_3)\mathrm{d}\xi \\
= {} & (2\pi)^3 \int_{\mathbb{R}^3} p_L(\xi)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)(\mu_j+\xi_3)} \\
& \times \big(\mathscr{F}[\boldsymbol{\chi}_{B_R(0)}] * \mathscr{F}[y \mapsto G(\tfrac{\omega}{c}n_j(\omega), x-\mu_j e_3 - \xi - y)]\big)(\tfrac{\omega}{c}n_j(\omega)e_3)\mathrm{d}\xi.
\end{aligned}$$

The shift in the function G now translates in Fourier space to a multiplication by a phase factor; explicitly, we find (using the symmetry $G(\kappa,y) = G(\kappa,-y)$ and the notation $\hat{G}(\kappa,k) = \mathscr{F}[y \mapsto G(\kappa,y)](k)$) that

$$\mathscr{F}[y \mapsto G(\kappa, x-\mu_j e_3 - \xi - y)](k) = \mathrm{e}^{-\mathrm{i}\langle k, x-\mu_j e_3 - \xi\rangle}\hat{G}(\kappa,k).$$

Therefore, we can write K_L (again using the shorter notation $\hat{\boldsymbol{\chi}}_{B_R(0)} = \mathscr{F}[\boldsymbol{\chi}_{B_R(0)}]$ and $\hat{p}_L = \mathscr{F}[p_L]$) as

$$\begin{aligned}
K_L(\omega,x) = {} & (2\pi)^3 \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\mu_j} \int_{\mathbb{R}^3} \hat{\boldsymbol{\chi}}_{B_R(0)}(\tfrac{\omega}{c}n_j(\omega)e_3 - k)\hat{G}(\tfrac{\omega}{c}n_j(\omega), k) \\
& \qquad \times \mathrm{e}^{-\mathrm{i}\langle k, x-\mu_j e_3\rangle} \int_{\mathbb{R}^3} p_L(\xi)\mathrm{e}^{-\mathrm{i}\langle \frac{\omega}{c}n_j(\omega)e_3 - k, \xi\rangle}\mathrm{d}\xi\,\mathrm{d}k \\
= {} & (2\pi)^{\frac{9}{2}} \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\mu_j} \int_{\mathbb{R}^3} \hat{\boldsymbol{\chi}}_{B_R(0)}(\tfrac{\omega}{c}n_j(\omega)e_3 - k) \\
& \qquad \times \hat{p}_L(\tfrac{\omega}{c}n_j(\omega)e_3 - k)\hat{G}(\tfrac{\omega}{c}n_j(\omega), k)\mathrm{e}^{-\mathrm{i}\langle k, x-\mu_j e_3\rangle}\mathrm{d}k.
\end{aligned} \tag{23}$$

Remarking that the Fourier transform of p_L is explicitly given by

$$\begin{aligned}
\hat{p}_L(k) &= \frac{1}{(2\pi)^{\frac{3}{2}}L^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{e}^{-\mathrm{i}k_1\xi_1}\mathrm{d}\xi_1 \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{e}^{-\mathrm{i}k_2\xi_2}\mathrm{d}\xi_2 \int_{\mathbb{R}} \boldsymbol{\chi}_{[-d_j,d_j]}(\xi_3)\mathrm{e}^{-\mathrm{i}k_3\xi_3}\mathrm{d}\xi_3 \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}L^2} \frac{2\sin(\frac{1}{2}Lk_1)}{k_1} \frac{2\sin(\frac{1}{2}Lk_2)}{k_2} \int_{\mathbb{R}} \boldsymbol{\chi}_{[-d_j,d_j]}(\xi_3)\mathrm{e}^{-\mathrm{i}k_3\xi_3}\mathrm{d}\xi_3,
\end{aligned}$$

we find with the abbreviation $\hat{\boldsymbol{\chi}}_{[-d_j,d_j]} = \mathscr{F}[\boldsymbol{\chi}_{[-d_j,d_j]}]$ in the limit $N_j \to \infty$ that

$$N_j\hat{p}_{L_j(N_j)}(k) \to 2\pi\rho_j\delta(k_1)\delta(k_2)\hat{\boldsymbol{\chi}}_{[-d_j,d_j]}(k_3)\ (N_j \to \infty). \tag{24}$$

Using (24) in (23), we can calculate the behaviour of K_L in this limit to be

$$\begin{aligned}\lim_{N_j\to\infty} N_jK_{L_j(N_j)}(\omega,x) &= (2\pi)^{\frac{11}{2}}\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\mu_j}\rho_j \\ &\times \int_{\mathbb{R}} \hat{\boldsymbol{\chi}}_{B_R(0)}((\tfrac{\omega}{c}n_j(\omega)-k_3)e_3)\hat{G}(\tfrac{\omega}{c}n_j(\omega),k_3e_3)\hat{\boldsymbol{\chi}}_{[-d_j,d_j]}(k_3)\mathrm{e}^{-\mathrm{i}k_3(x_3-\mu_j)}\mathrm{d}k_3.\end{aligned}$$

Using further that $\hat{G}$ can be computed by taking the Fourier transform of the Helmholtz equation, giving us

$$\hat{G}(\kappa,k) = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{|k|^2-\kappa^2},$$

and calculating the Fourier transform of the characteristic function of a sphere to be

$$\begin{aligned}\hat{\boldsymbol{\chi}}_{B_R(0)}(k) &= \frac{1}{\sqrt{2\pi}}\int_0^R\int_0^\pi r^2\sin\theta\mathrm{e}^{-\mathrm{i}r|k|\cos\theta}\mathrm{d}\theta\mathrm{d}r = \frac{1}{\sqrt{2\pi}}\int_0^R\frac{r}{\mathrm{i}|k|}(\mathrm{e}^{\mathrm{i}r|k|}-\mathrm{e}^{-\mathrm{i}r|k|})\mathrm{d}r \\ &= \frac{1}{|k|^3}\sqrt{\frac{2}{\pi}}\int_0^{R|k|}\alpha\sin(\alpha)\mathrm{d}\alpha = \frac{1}{|k|^3}\sqrt{\frac{2}{\pi}}(\sin(R|k|)-R|k|\cos(R|k|));\end{aligned}$$

we are left with

$$\begin{aligned}\lim_{N_j\to\infty} N_jK_{L_j(N_j)}(\omega,x) &= (2\pi)^4\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\mu_j}\rho_j\sqrt{\frac{2}{\pi}} \\ &\times \int_{\mathbb{R}} h(R(\tfrac{\omega}{c}n_j(\omega)-k_3))\frac{1}{k_3^2-\frac{\omega^2}{c^2}n_j^2(\omega)}\hat{\boldsymbol{\chi}}_{[-d_j,d_j]}(k_3)\mathrm{e}^{-\mathrm{i}k_3(x_3-\mu_j)}\mathrm{d}k_3,\end{aligned} \tag{25}$$

where we used the abbreviation h from (21).

Inserting finally

$$\hat{\boldsymbol{\chi}}_{[-d_j,d_j]}(k_3) = \frac{1}{\sqrt{2\pi}}\int_{-d_j}^{d_j}\mathrm{e}^{-\mathrm{i}k_3x_3}\mathrm{d}x_3 = \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}k_3}\left(\mathrm{e}^{\mathrm{i}k_3d_j}-\mathrm{e}^{-\mathrm{i}k_3d_j}\right),$$

we see that the integrand in (25) can for $x_3-\mu_j > d_j+R$ (that is, for $x_3 > \zeta_j+R$) be meromorphically extended to a function of k_3 in the lower half complex plane which decays sufficiently fast at infinity, so that the residue theorem yields

$$\begin{aligned}\lim_{N_j\to\infty} N_jK_{L_j(N_j)}(\omega,x) &= (2\pi)^4\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)\mu_j}\rho_j\frac{h(2R\frac{\omega}{c}n_j(\omega))}{\frac{\omega^2}{c^2}n_j^2(\omega)} \\ &\times\left(\mathrm{e}^{\mathrm{i}\frac{\omega}{c}n_j(\omega)d_j}-\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)d_j}\right)\mathrm{e}^{\mathrm{i}\frac{\omega}{c}n_j(\omega)(x_3-\mu_j)}.\end{aligned}$$

Putting this into (22), we obtain with $\mu_j + d_j = \zeta_j$ and $\mu_j - d_j = Z_j$ the formula (19).

Similarly, we extend the integrand for $x_3 - \mu_j < -d_j - R$ (that is, for $x_3 < Z_j - R$) meromorphically to a function of k_3 in the upper half plane and find with the residue theorem that

$$\lim_{N_j\to\infty} N_j K_{L_j(N_j)}(\omega,x) = (2\pi)^4 \rho_j \frac{h(0)}{\frac{\omega^2}{c^2} n_j^2(\omega)} \times \left(\mathrm{e}^{\mathrm{i}\frac{\omega}{c}n_j(\omega)d_j} - \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)d_j}\right)\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}n_j(\omega)x_3},$$

which gives us with (22) and with $h(0) = \frac{1}{3}$ the formula (20).

5 Recovering the susceptibility with optical coherence elastography

So far, we have presented a way to model the measurements of an optical coherence tomography setup for a layered medium of the form (6). The question we are really interested in, however, is how to reconstruct the properties of the medium from this data.

Let us first consider one of the layer stripping steps for a susceptibility χ of the form (9) with X_1 being either of the form (12) of Problem 1 or of the form (13) of Problem 2. We make the additional assumption that $\operatorname{supp}\chi_j \subset [0,T]$ and $\operatorname{supp}\chi_j^{(\mathrm{p})} \subset [0,T]$ for a sufficiently small $T > 0$. Then, we see that by choosing a sufficiently short pulse as incident wave, that is, $E^{(0)}(t,x) = f(t+\frac{x_3}{c})\eta$ (assuming for the background medium $\chi_0 = 0$) with f having a sufficiently narrow support (this ability is of course limited by the available frequencies), we can arrange it such that the field E in the presence of χ and the field E_1 in the presence of X_1 (where we are content with the averaged Born approximation of the electric field, see (18), in the case of Problem 2) are such that $E_1(t,x_0) = E(t,x_0)$ for all $t < t_0$ and $E_1(t,x_0) = 0$ for $t \ge t_0$ at the detector $x_0 \in \mathbb{R}^3$ for some time $t_0 \in \mathbb{R}$. This allows us to split the reconstruction of the electric susceptibility by a layer stripping method and reconstruct each layer separately.

We will therefore only describe the inductive steps, in which we independently consider the subproblems described in Section 4.

We want to start with measurements from an optical coherence elastography setup, that is, we have optical coherence tomography data for different elastic states of the medium. Concretely, we apply a force proportional to some parameter $\delta \in \mathbb{R}$ perpendicular to the layers of the medium, which causes under the assumption of a linear elastic medium a linear displacement of the position z_j of the layer. Correspondingly, the refractive indices in the medium, defined by (15), will change, which we assume to be linear as well. Thus, each layer at the compression state corresponding to δ will be characterised by a refractive index $\bar{n}_j$ and a vertical position $\bar{z}_j$ of the beginning of the layer of the form

$$\bar{n}_j(\omega,\delta) = n_j(\omega) + \delta n'_j(\omega), \text{ and } \bar{z}_j(\delta) = z_j + \delta z'_j,$$

for some functions $n'_j : \mathbb{R} \to \mathbb{C}$ and some slopes $z'_j \in \mathbb{R}$.

In the first reconstruction step, we have that the first layer is the background in which the medium resides, which we assume to be well described by the vacuum $n_0 = 1$ and not to be affected by the compression, that is, $n'_0 = 0$. Moreover, the distance between the detector and the medium shall be kept fixed during the compression so that $z'_1 = 0$ as well.

According to Lemma 4, the measurements at the detector $x_0 \in \mathbb{R}^3$ with $x_{0,3} > z_1$ then allow us to extract (knowing $\bar{n}_0 = 1$, the incident field $E^{(0)}$, and the vertical position $x_{0,3}$ of the detector explicitly) the information

$$m_0[n_1,n'_1,z](\omega,\delta) = \frac{\bar{n}_1(\omega,\delta)-1}{\bar{n}_1(\omega,\delta)+1} e^{-2\mathrm{i}\frac{\omega}{c}z_1}. \tag{26}$$

From this data, we can uniquely compute the functions n_1, n'_1, and z_1.

Lemma 6. *Let $I \subset \mathbb{R}$ be a set which contains at least two incommensurable points $\omega_1, \omega_2 \in I \setminus \{0\}$ (that is, $\frac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$). Assume that we have (n_1,n'_1,z_1) and $(\tilde{n}_1,\tilde{n}'_1,\tilde{z}_1)$ with $n'_1(\omega) \neq 0$, $\tilde{n}'_1(\omega) \neq 0$, and*

$$m_0[n_1,n'_1,z_1](\omega,\delta) = m_0[\tilde{n}_1,\tilde{n}'_1,\tilde{z}_1](\omega,\delta) \text{ for all } \omega \in I,\ \delta \in \mathbb{R}. \tag{27}$$

Then, we have

$$n_1(\omega) = \tilde{n}_1(\omega),\ n'_1(\omega) = \tilde{n}'_1(\omega), \text{ and } z_1 = \tilde{z}_1 \text{ for all } \omega \in I.$$

Proof. Expanding the fractions in (27), the equation reduces to the zeroes of a quadratic polynomial in δ. Comparing the coefficients of second order of δ, we find that

$$n'_1(\omega)\tilde{n}'_1(\omega)\left(e^{-2\mathrm{i}\frac{\omega}{c}z_1} - e^{-2\mathrm{i}\frac{\omega}{c}\tilde{z}_1}\right) = 0.$$

Thus, we get

$$e^{-2\mathrm{i}\frac{\omega}{c}z_1} = e^{-2\mathrm{i}\frac{\omega}{c}\tilde{z}_1} \text{ for all } \omega \in I.$$

Evaluating this at ω_1 and ω_2, we have that there exist two integers $\lambda_1, \lambda_2 \in \mathbb{Z}$ with

$$z_1 - \tilde{z}_1 = \frac{\pi c}{\omega_1}\lambda_1 = \frac{\pi c}{\omega_2}\lambda_2.$$

If $\lambda_2 \neq 0$, then we would get the contradiction $\frac{\lambda_1}{\lambda_2} = \frac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$. Therefore, $\lambda_2 = 0$, which means that $z_1 = \tilde{z}_1$.

With this, (27) evaluated at $\delta = 0$ simplifies to

$$n_1(\omega) = \tilde{n}_1(\omega) \text{ for all } \omega \in I.$$

Finally, looking at the terms of first order in δ in the expanded version of (27), we find that they have been reduced to give the equation

$$n_1'(\omega) = \tilde{n}_1'(\omega).$$

After having recovered the parameters up to the j-th layer, $j \in \{1,\dots,J\}$, we can clean our measurement data from all effects caused by the previous layers and consider the next subproblem, namely the signal originating from the region of the randomly distributed particles. Here, the unknown parameters consist of

- the radius R of the particles, which we will assume to be so small that the approximation $R = 0$ is reasonable and that the particles can also after compression be considered to have a round shape;
- the ratio $\rho_j > 0$ of particles per surface area, which we assume to be invariant under the compression;
- the refractive index $\bar{\nu}_j$ of the particles, which we assume to deform linearly according to

$$\bar{\nu}_j(\omega,\delta) = \nu_j(\omega) + \delta \nu_j'(\omega), \text{ where } \nu_j(\omega) = \sqrt{1+\check{\chi}_j^{(\mathrm{p})}(\omega)},$$

  under compression; and
- the vertical positions $\bar{\zeta}_j$ and $\bar{Z}_j$ of the beginning and the end of the random medium inside the j-th layer, which are also assumed to change linearly according to

$$\bar{\zeta}_j(\delta) = \zeta_j + \delta\zeta_j' \text{ and } \bar{Z}_j(\delta) = Z_j + \delta Z_j'.$$

We collect these unknowns in the tuple $S_j = (\rho_j, \nu_j, \nu_j', \zeta_j, \zeta_j', Z_j, Z_j')$. The (corrected) incident wave $E^{(0)}$ and the refractive index n_j and its rate n_j' of change under compression are presumed to be already calculated.

From the measurements of the electric field for this subproblem, provided that it can be well approximated by the expected value of the Born approximation as calculated in Lemma 5, we can extract the data (rewriting the expression (16) for ϕ_j in (19) in terms of the refractive indices)

$$\begin{aligned} M_j[S_j](\omega,\delta) &= \rho_j(\bar{\nu}_j^2(\omega,\delta) - \bar{n}_j^2(\omega,\delta)) \\ &\quad \times \left(\mathrm{e}^{-\mathrm{i}\frac{\omega}{2c}\bar{n}_j(\omega,\delta)(\bar{\zeta}_j(\delta)+3\bar{Z}_j(\delta))} - \mathrm{e}^{-\mathrm{i}\frac{\omega}{2c}\bar{n}_j(\omega,\delta)(3\bar{\zeta}_j(\delta)+\bar{Z}_j(\delta))}\right), \end{aligned}$$

Lemma 7. *Let $j \in \{1,\dots,J\}$ be fixed, $I \subset \mathbb{R}$ be an arbitrary subset and n_j, n_j' be given such that $n_j(\omega) \neq 0$ for every $\omega \in I$ and that there exists a value $\omega_0 \in I \setminus \{0\}$ with $\Im\mathrm{m}(n_j'(\omega_0)) > 0$. Assume that we have $S_j = (\rho_j, \nu_j, \nu_j', \zeta_j, \zeta_j', Z_j, Z_j')$ and $\tilde{S}_j = (\tilde{\rho}_j, \tilde{\nu}_j, \tilde{\nu}_j', \tilde{\zeta}_j, \tilde{\zeta}_j', \tilde{Z}_j, \tilde{Z}_j')$ with*

$$M_j[S_j](\omega,\delta) = M_j[\tilde{S}_j](\omega,\delta) \text{ for all } \omega \in I,\ \delta \in \mathbb{R}. \tag{28}$$

Additionally, we enforce the ordering $Z_j < \zeta_j$ and $\tilde{Z}_j < \tilde{\zeta}_j$ about the beginning and the end of the random layer and make the assumptions $Z_j' > \zeta_j' > 0$ and $\tilde{Z}_j' > \tilde{\zeta}_j' > 0$ that the layer shrinks when being compressed.

Moreover, we assume the existence of an element $\omega_1 \in I$ so that

$$\frac{n_j'(\omega_1)}{n_j(\omega_1)} \neq \frac{\nu_j'(\omega_1)}{\nu_j(\omega_1)}. \tag{29}$$

Then, we have

$$S_j = \tilde{S}_j.$$

Proof. Considering the different orders of decay in δ in the exponents in (28), we require that all of them match, which yields the equation system

$$\delta^2 \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(\zeta_j' + 3Z_j') = \delta^2 \frac{\omega}{c} \Im\mathrm{m}(n_j'(\omega))(\tilde{\zeta}_j' + 3\tilde{Z}_j') \text{ and}$$
$$\delta^2 \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(3\zeta_j' + Z_j') = \delta^2 \frac{\omega}{c} \Im\mathrm{m}(n_j'(\omega))(3\tilde{\zeta}_j' + \tilde{Z}_j')$$

for the exponents quadratic in δ, which implies for the frequency $\omega = \omega_0$ for which $\Im\mathrm{m}(n_j'(\omega_0)) > 0$ that $\zeta_j' = \tilde{\zeta}_j'$ and $Z_j' = \tilde{Z}_j'$, and, using this result, the equation system

$$\delta \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(3\zeta_j + Z_j) = \delta \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(3\tilde{\zeta}_j + \tilde{Z}_j) \text{ and}$$
$$\delta \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(\zeta_j + 3Z_j) = \delta \frac{\omega}{2c} \Im\mathrm{m}(n_j'(\omega))(\tilde{\zeta}_j + 3\tilde{Z}_j)$$

for the exponents linear in δ, which further implies $\zeta_j = \tilde{\zeta}_j$ and $Z_j = \tilde{Z}_j$.

At this point, (28) is reduced to

$$\rho_j \left((\nu_j + \delta \nu_j')^2 - (n_j + \delta n_j')^2 \right) = \tilde{\rho}_j \left((\tilde{\nu}_j + \delta \tilde{\nu}_j')^2 - (n_j + \delta n_j')^2 \right).$$

Comparing coefficients with respect to δ gives us the equation system

$$\rho_j \left(\nu_j'^2 - n_j'^2 \right) = \tilde{\rho}_j \left(\tilde{\nu}_j'^2 - n_j'^2 \right), \tag{30}$$
$$\rho_j \left(\nu_j \nu_j' - n_j n_j' \right) = \tilde{\rho}_j \left(\tilde{\nu}_j \tilde{\nu}_j' - n_j n_j' \right), \tag{31}$$
$$\rho_j \left(\nu_j^2 - n_j^2 \right) = \tilde{\rho}_j \left(\tilde{\nu}_j^2 - n_j^2 \right). \tag{32}$$

We use equation (32) in (30) and (31) to eliminate of the variables ρ_j and $\tilde{\rho}_j$, and interpret the result as an equation system for the variables $\tilde{\nu}_j$ and $\tilde{\nu}_j'$. Solving these equations then for $\tilde{\nu}_j'$, gives us

$$(\nu_j^2 - n_j^2)\tilde{\nu}_j'^2 = (\tilde{\nu}_j^2 - n_j^2)\nu_j'^2 + (\nu_j^2 - \tilde{\nu}_j^2)n_j'^2,$$
$$(\nu_j^2 - n_j^2)\tilde{\nu}_j \tilde{\nu}_j' = (\tilde{\nu}_j^2 - n_j^2)\nu_j \nu_j' + (\nu_j^2 - \tilde{\nu}_j^2)n_j n_j'.$$

Eliminating further $\tilde{\nu}_j'$ by multiplying the first equation with $\tilde{\nu}_j$ and subtracting the squared second equation, we find after some algebraic manipulations

$$(\tilde{\nu}_j^2 - n_j^2)(\nu_j^2 - \tilde{\nu}_j^2)(\nu_j' n_j - \nu_j n_j')^2 = 0.$$

Evaluating this at the value ω_1, we see that the last factor is by assumption (29) not zero. Thus, there are only two cases.

1. Either we have $\tilde{\nu}_j(\omega_1) = \nu_j(\omega_1) \neq n_j(\omega_1)$ and therefore by (32) that $\tilde{\rho}_j = \rho_j$; then we get with (32) and (30) that $\tilde{\nu}_j = \nu_j$ and $\tilde{\nu}_j' = \nu_j'$ holds on the whole set I, which means that we have shown $\tilde{S}_j = S_j$.
2. Or we have that $\tilde{\nu}_j(\omega_1) = n_j(\omega_1)$. Then, (32) tells us that also $\nu_j(\omega_1) = n_j(\omega_1)$ and thus, by combining (30) and (31), that $\tilde{\nu}_j'(\omega_1) = \nu_j'(\omega_1)$. Furthermore, we know from assumption (29) that in this case $\nu_j'(\omega_1) \neq n_j'(\omega_1)$, and therefore (30) implies $\tilde{\rho}_j = \rho_j$ from which we again conclude that $\tilde{S}_j = S_j$.

As last type of subproblem, we encounter then the interface between the layer j and the layer $j+1$. Similarly to the case of the initial layer, we obtain here from Lemma 4 the data

$$m_j[n_{j+1}, n_{j+1}', z_{j+1}, z_{j+1}'](\omega, \delta) = \frac{\bar{n}_{j+1}(\omega,\delta) - \bar{n}_j(\omega,\delta)}{\bar{n}_{j+1}(\omega,\delta) + \bar{n}_j(\omega,\delta)} \mathrm{e}^{-2\mathrm{i}\frac{\omega}{c}\bar{n}_j(\omega,\delta)\bar{z}_{j+1}(\delta)}.$$

Again, this data allows us to uniquely obtain the variables n_{j+1}, n_{j+1}', z_{j+1}, and z_{j+1}' from the already reconstructed values n_j and n_j'.

Lemma 8. *Let $j \in \{1,\dots,J-1\}$ be fixed, $I \subset \mathbb{R}$ be an arbitrary subset and n_j, n_j' be given such that $n_j(\omega) \neq 0$ for every $\omega \in I$ and that there exists a value $\omega_0 \in I \setminus \{0\}$ with $\Im\mathrm{m}(n_j'(\omega_0)) > 0$. Assume that we have $(n_{j+1}, n_{j+1}', z_{j+1}, z_{j+1}')$ and $(\tilde{n}_{j+1}, \tilde{n}_{j+1}', \tilde{z}_{j+1}, \tilde{z}_{j+1}')$ with*

$$m_j[n_{j+1}, n_{j+1}', z_{j+1}, z_{j+1}'](\omega, \delta) = m_j[\tilde{n}_{j+1}, \tilde{n}_{j+1}', \tilde{z}_{j+1}, \tilde{z}_{j+1}'](\omega, \delta) \tag{33}$$

for all $\omega \in I$ and $\delta \in \mathbb{R}$.

Then, we have

$$n_{j+1}(\omega) = \tilde{n}_{j+1}(\omega),\ n_{j+1}'(\omega) = \tilde{n}_{j+1}'(\omega),\ z_{j+1} = \tilde{z}_{j+1},\ \textit{and}\ z_{j+1}' = \tilde{z}_{j+1}'$$

for all $\omega \in I$.

Proof. Comparing again the different orders of decay in δ in the exponents in (33), we require that the coefficients on both sides coincide:

$$2\delta^2 \frac{\omega}{c} \Im\mathrm{m}(n_j'(\omega))(z_{j+1}' - \tilde{z}_{j+1}') = 0 \text{ and}$$
$$4\delta \frac{\omega}{c} \left(\Im\mathrm{m}(n_j(\omega))(z_{j+1}' - \tilde{z}_{j+1}') + \Im\mathrm{m}(n_j'(\omega))(z_{j+1} - \tilde{z}_{j+1})\right) = 0.$$

Because of the assumption that $\Im\mathrm{m}(n_j'(\omega_0)) > 0$, this is equivalent to

$$z_{j+1}' = \tilde{z}_{j+1}' \text{ and } z_{j+1} = \tilde{z}_{j+1}.$$

As in the proof of Lemma 6, equation (33) for $\delta = 0$ then gives us

$$2n_j(\omega)(n_{j+1}(\omega) - \tilde{n}_{j+1}(\omega)) = 0,$$

resulting in $n_{j+1}(\omega) = \tilde{n}_{j+1}(\omega)$.

Finally, dividing both sides of (33) by the exponential factors (which we already know to be the same), we get a quadratic equation for δ and equating the first order terms in δ, we obtain

$$2n_j(\omega)(n'_{j+1}(\omega) - \tilde{n}'_{j+1}(\omega)) = 0,$$

which yields $n'_{j+1}(\omega) = \tilde{n}'_{j+1}(\omega)$.

Conclusion

We have thus shown that by analysing a layered medium endued with independently uniformly distributed scatterers in each layer with optical coherence tomography, we can reduce the inverse problem of reconstructing the electric susceptibility of the medium to subproblems for each layer separately by a layer stripping argument, provided the homogeneous parts between the different regions are not too small.

Then by combining this imaging method with an elastography setup by recording measurements for different compression states (normal to the layered structure), we find out that this allows for the reconstruction of the optical parameters and leads to a unique reconstructability of all the optical parameters: the electric susceptibilities and positions of the layers, the electric susceptibilities of the randomly distributed particles, their particle density, and the locations of the regions of these particles (at every compression state). Of course, the recovered shifts of the layer boundaries for the different compression states could then be used in a next step to determine elastic parameters of the medium.

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