Abstracts

The inverse electromagnetic scattering problem in OCT for anisotropic media

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In this work we consider the inverse electromagnetic scattering problem for inhomogeneous anisotropic media placed in an Optical Coherence Tomography (OCT) system.

OCT is a non-invasive imaging technique producing high-resolution images of biological tissues. It is based on low (time) coherence interferometry and it is capable of imaging micro-structures within a few micrometers resolution. Standard OCT operates using broadband and continuous wave light in the visible and near-infrared spectrum. Images are obtained by measuring the time delay and the intensity of back-scattered light from the sample under investigation. There also exist contrast-enhanced OCT techniques like polarisation-sensitive OCT (PS-OCT) which allows for simultaneously detecting different polarisation states of the back-scattered light [2].

We model the propagation of electromagnetic waves through the inhomogeneous anisotropic medium using Maxwell's equations [1]. The sample is hereby considered as a linear dielectric non-magnetic medium. Moreover, we assume that it is weakly scattering, meaning that the electromagnetic field inside the medium is sufficiently well described by a first order Born approximation.

The optical properties of the medium are then characterised by the electric susceptibility $\chi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$, the quantity to be recovered, where the causality requires that $\chi(t,x) = 0$ for $t < 0$. The measurements M are given as a combination of the back-scattered field by the sample and the back-reflected field from a reference mirror. In an OCT system, the detector is placed in a distance much bigger than the size of the medium, therefore we consider as measurement data the far field approximation of the electromagnetic field.

Then, the direct problem is modelled as an integral operator $\mathcal{F}$ mapping the susceptibility χ to the measurement data M. The inverse problem we are interested in is to solve the operator equation

$$\mathcal{F}\chi = M$$

for χ, given measurements for different positions of the mirror and different incident polarisation vectors. Under some restrictions on the OCT setup [3] and assuming that the incoming plane wave $E^{(0)}$ propagates in the direction $-e_3$, we can formulate the inverse problem as the reconstruction of χ from the expressions

$$\mathcal{H}_j^{-1} M = p_j \left[\vartheta \times (\vartheta \times \tilde{\chi}(\omega, \tfrac{\omega}{c}(\vartheta + e_3))p)\right]_j, \quad j = 1, 2, \tag{1}$$

where $\mathcal{H}_j$ are some explicitly known, well-posed operators, $p \in \mathbb{R}^2 \times \{0\}$ is the polarisation of the initial illumination, $\omega \in \mathbb{R} \setminus \{0\}$ is the frequency and $\vartheta \in S^2_+$

is the direction from the origin (where the sample is located) to a detector point. Here $\tilde{\chi}$ denotes the Fourier transform of χ with respect to time and space.

In the special case of an isotropic medium, meaning that χ is a multiple of the unit matrix, it remains the problem to reconstruct the four dimensional susceptibility from the three dimensional measurement data. We propose an iterative scheme, assuming a certain discretisation of χ with respect to the detection points and its support, that provides us with the values of a limited angle Radon transform. For non-dispersive media, where the temporal Fourier transform $\hat{\chi}$ of χ does not depend on frequency, the equation (1) determines the spatial Fourier transform of $\hat{\chi}$ in a cone. Thus, the problem reduces to the inversion of the three-dimensional Fourier transform with limited data and there exist several algorithms for recovering the scalar susceptibility under these assumptions.

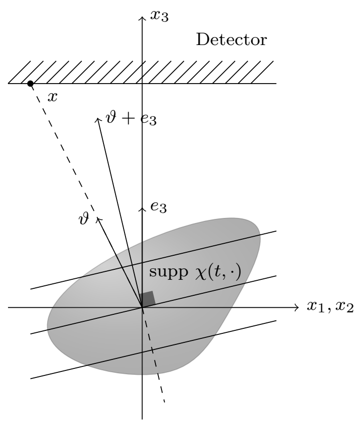

FIGURE 1

For the most general case of a matrix valued susceptibility (non-symmetric), using the discretisation of χ we show that three incident polarisation vectors uniquely determine the Radon transform of the projection of χ over planes orthogonal to the vector $\vartheta + e_3$, see Figure 1. To be able to recover the Fourier transform of χ we have to repeat the experiment for different orientations of the sample. More precisely, we have to tilt slightly the sample three times for every incident polarisation.

The above analysis concerns the study of standard OCT where the sample is illuminated by light with fixed polarisation (usually linear).

On the other hand, in PS-OCT the interferometer with the addition of polarizers and quarter-wave plates change the polarisation state of light to produce circularly polarized light incident on the sample. The output signal now is split into its horizontal and vertical components to be measured at two different photo detectors. In this setting, we consider an orthotropic non-dispersive medium where the susceptibility is now a symmetric matrix with only four unknowns and its temporal Fourier transform is frequency independent.

From [1] we know that the scattered field can be written as a linear integral operator $\mathcal{G}$ applied to the product of the temporal Fourier transforms $\hat{\chi}$ and $\hat{E}^{(0)}$ of the susceptibility χ and the incident field $E^{(0)}$, respectively. The kernel of the operator $\mathcal{G}$ is the Green tensor related to Maxwell's equations in the frequency domain. This relation is known as Lippmann–Schwinger equation. The kth order Born approximation $\hat{E}^{(k)}$ is defined by

$$\hat{E}^{(k)} = \hat{E}^{(0)} + \mathcal{G}[\hat{\chi}\hat{E}^{(k-1)}], \quad k \in \mathbb{N}.$$

We consider the second order Born approximation together with the far field approximation, that affects only the Green tensor resulting to an operator $\mathcal{G}^\infty$, which leads to

$$\hat{E}^{(2)} = \hat{E}^{(0)} + \mathcal{G}^\infty[\hat{\chi}(\hat{E}^{(0)} + \mathcal{G}[\hat{\chi}\hat{E}^{(0)}])].$$

We assume that the variations of $\hat{\chi}$ are small compared to the constant background value $\hat{\chi}_0$ (which is given), meaning $\hat{\chi} = \hat{\chi}_0 + \epsilon\hat{\chi}_1$, for some small $\epsilon > 0$. Equating the first order terms we consider only the term

$$\mathcal{G}^\infty[\hat{\chi}_1(\hat{E}^{(0)} + \mathcal{G}[\hat{\chi}_0\hat{E}^{(0)}])] + \mathcal{G}^\infty[\hat{\chi}_0\mathcal{G}[\hat{\chi}_1\hat{E}^{(0)}]]$$

as the one that has the necessary information. Similar to the derivation of (1), the inverse problem of determining $\hat{\chi}_1$ from the OCT measurements, related to the field $\hat{E}^{(2)}$, reduces to the reconstruction of $\hat{\chi}_1$ from the expressions

$$p_j^r\big[\vartheta \times (\vartheta \times ((\tilde{\chi}_1 + \hat{\chi}_0\mathcal{K}[\tilde{\chi}_1] + \mathcal{L}[\tilde{\chi}_1]\hat{\chi}_0)p^s))\big]_j, \quad j = 1, 2, \tag{2}$$

for some integral operators $\mathcal{K}$ and $\mathcal{L}$ related to $\mathcal{G}$ and polarisation vectors p^r and p^s describing the change of the polarisation state of light travelling in the reference and sample arm, respectively. These changes can be easily modelled using Jones matrices.

We show that two linear independent incident polarisations provide enough information for recovering $\hat{\chi}_1$. Expression (2), for two incident polarisations of the form $p = e_1$ and $p = e_2$ results in a system of Fredholm integral equations of the second kind for the three (out of four) unknown components of $\hat{\chi}_1$. In addition, the integral operator is compact. The fourth component is given as a solution of a Fredholm integral equation of the first kind, where the right-hand side depends on the three previously computed solutions.

As a future work we plan to consider the numerical implementation of the above results for simulated and real data. Even though we have a theoretical result that makes the reconstruction of $\tilde{\chi}$ possible, we still have to tackle the ill-posedness of the inverse problem. Recall that in a real world experiment we only have access to noisy band-limited and limited-angle measurements due to the limited spectrum of the light source and the size of the detector.

References

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