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Asymptotic Expansions for Higher Order Elliptic Equations with an Application to Quantitative Photoacoustic Tomography*

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Abstract. In this paper, we derive new asymptotic expansions for the solutions of higher order elliptic equations in the presence of small inclusions. As a byproduct, we derive a topological derivative based algorithm for the reconstruction of piecewise smooth functions. This algorithm can be used for edge detection in imaging, topological optimization, and inverse problems, such as quantitative photoacoustic tomography, for which we demonstrate the effectiveness of our asymptotic expansion method numerically.

Key words. asymptotic expansions, higher order equations, thin stripes, inverse problems, quantitative photoacoustic

AMS subject classifications. Primary, 35G15; Secondary, 35C20, 35R30

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1. Introduction.

In this paper, we propose a new algorithm based on *topological derivatives* (see, for example, [4, 5, 29, 41, 42] for a review on topological optimization) which allows for detection of discontinuities of a given function f and of its *derivatives*. The basic idea is that f can be viewed as a piecewise smooth function, and edges in f and its derivatives can be modeled accurately by a set of singularities along small line segments. More precisely, given a noisy image f, in order to find a smoothed version u of f, we consider the following functional for a fixed order $m \in \mathbb{N}$,

$$\mathcal{J}(u;v) := \frac{1}{2}\int_\Omega (u-f)^2\,dx + \frac{\alpha}{2}\int_\Omega v\,|\nabla^m u|^2\,dx, \tag{1.1}$$

where $v \in L^\infty(\Omega)$, $v > 0$, and Ω is a Lipschitz domain in $\mathbb{R}^2$ where f is defined. The case $m = 1$ has already been studied in [13]. The first term in (1.1) measures the fidelity to the given image f, while the second one is a regularization term which accounts for the presence

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of discontinuities. The minimizer $u \in H_0^m(\Omega)$ of (1.1), where $H_0^m(\Omega)$ is the classical Sobolev space of traces of derivatives which are zero on $\partial\Omega$ up to $m-1$, is also a weak solution of

$$\begin{cases} u + \alpha\, (-1)^m (\nabla \cdot)^m (v \nabla^m u) = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $(\nabla \cdot)^m$ is the divergence operator applied m times, n is the outward normal vector on $\partial\Omega$, $\frac{\partial}{\partial n}$ is the normal derivative, and, recursively, $\frac{\partial^l u}{\partial n^l} = \frac{\partial}{\partial n}(\frac{\partial^{l-1} u}{\partial n^{l-1}})$ for $l = 2, \ldots, m-1$. In order to detect discontinuities in f and its derivatives, we study the variation of the minimizer of the functional with respect to small variations of v. We denote by v_ε and u_ε the perturbation of v and u, respectively. Obviously, u_ε satisfies the same equation in (1.2), substituting v with v_ε. With small variations, we mean that v_ε differs from v by a constant times a characteristic function of small support—in our case a thin neighborhood of a line segment. Specifically, denoting with $\Omega_\varepsilon(y, \tau)$ the thin strip centered at y and along the direction τ, and with κ a real number such that $0 < \kappa < \frac{1}{2}$, we have that

$$(v_\varepsilon - v)(x) = \begin{cases} \kappa - 1, & x \in \overline{\Omega}_\varepsilon(y, \tau), \\ 0 & \text{otherwise.} \end{cases}$$

This procedure introduces discontinuities in derivatives of order m of the minimizer along line segments, and an accurate asymptotic analysis of the minimizer with respect to v_ε allows us to compare the perturbed functional, $\mathcal{J}(u_\varepsilon; v_\varepsilon)$, with the unperturbed one, $\mathcal{J}(u; v)$, and to derive a rigorous formula of the topological derivative of the functional. Precisely, we prove that

$$\mathcal{J}(u_\varepsilon; v_\varepsilon) = \mathcal{J}(u; v) + 2\varepsilon^3 \alpha(\kappa - 1)\mathbb{M}\nabla^m u(y) \cdot \nabla^m u(y) + o(\varepsilon^3),$$

where $\mathbb{M}$ is a tensor of order $2m$, which contains geometrical information on the direction of the line segment along which the discontinuity occurs.

The idea of "drilling small holes" in the domain for edge detection, when $m = 1$, has been introduced in [32] and has been successfully implemented in [23], where also a conceptual connection to Mumford–Shah minimization and the Ambrosio–Tortorelli relaxation [3] has been highlighted. Later, these results have been extended to the case where the edge set is covered with thin stripes rather than with balls in [13]. Therefore, the main novelty in this paper is to use higher order derivative discontinuities of the minimizer u to more accurately recover image discontinuities. In order to get these results we need to overcome several significant difficulties, by combining known methods and recent results on higher order elliptic equations with discontinuous coefficients (see [10]) and tensors (see [19]), and generalizing some of them, in a novel way.

In the first part of our paper we derive an asymptotic expansion for the perturbed functional $u \to \mathcal{J}(\cdot, v_\varepsilon)$ due to the presence of a small measurable set via compensated compactness, following the ideas of [17] and [18] where an asymptotic expansion of perturbations of solutions to the conductivity equation has been derived. Here, however, the minimizer of the functional is the solution to a partial differential equation of order $2m$, where $m > 1$, with discontinuous leading coefficients, for which the theoretical basis is much less advanced (see [10]). Also, to the best of our knowledge, asymptotic expansions of solutions of higher order elliptic equations have been derived only in the case of diametrically small domains [6], while the analysis in the case of arbitrary measurable small domains, and in the form of thin strips like the ones considered here, is new, not at all straightforward, and interesting on its own. In particular, we obtain a full characterization of the Polya–Szego polarization tensor in the case of thin strips, generalizing the results obtained in [12, 11] and in [15] for the conductivity equation and the linearized system of elasticity.

As explained in [29], topological gradient methods have been successfully applied to different areas of applications, such as topology optimization problems, image processing, and image reconstructions to name a few.

In the second part of the paper, we use the above results in an application to *quantitative photoacoustic tomography* (qPAT) (see [14]). This is an imaging technique that excites a specimen by electromagnetic waves and records the resulting ultrasound; see [43] for an extensive treatment of the experimental issues, and see [27, 28] for an extensive mathematical analysis. Since the electromagnetic pulses used in photoacoustic tomography are very short, the complete energy is deposited almost instantaneously compared to the travel times of the induced acoustic waves, and therefore the pressure wave p can be assumed to have been generated by an *initial pressure* $\mathcal{H}(x)$; that is, it satisfies the wave equation

$$\begin{aligned}\partial_{tt}p(x,t) - \Delta p(x,t) &= 0,\\ \partial_t p(x,0) &= 0,\\ p(x,0) &= \mathcal{H}(x),\end{aligned}$$

and $p|_{\mathcal{M}}$, where $\mathcal{M}$ denotes a *measurement surface*, can be obtained from ultrasound measurements.

By solving an *inverse problem for the wave equation* (see, e.g., [28] for a review on mathematical and numerical techniques), these ultrasound measurements can be used to estimate the *initial pressure* $\mathcal{H}(x)$. Since

$$\mathcal{H}(x) = \Gamma(x)\mathcal{E}(x) = \Gamma(x)\mu(x)u(x), \tag{1.3}$$

in the particular case where Γ is constant, $\mathcal{H}(x)$ and the *absorbed energy* $\mathcal{E}(x)$ are proportional, which is in turn proportional to the product of *optical absorption coefficient* $\mu(x)$ and the fluence $u(x)$ (the time-integrated laser power received at x). Therefore, the initial pressure visualizes contrast in μ. The coefficient $\Gamma(x)$ is called the *Grüneisen coefficient* (or *photoacoustic efficiency* since it describes the efficiency of conversion from absorbed energy to acoustic signal) [21, 20, 44].

Quantitative photoacoustic tomography (qPAT) consists in determining spatially heterogeneous Grüneisen, absorption, and diffusion coefficients, Γ, μ, and D, from photoacoustic measurements of the absorbed energy $\mathcal{H} = \Gamma\mu u$, where u satisfies

$$\begin{aligned}-\nabla\cdot(D\nabla u) + \mu u &= 0 \text{ in } \Omega\subset\mathbb{R}^2,\\ u &= g \text{ on } \partial\Omega.\end{aligned} \tag{1.4}$$

Here g is a given function which describes the illumination pattern.

Many facets of qPAT have been considered in the literature; an incomplete list is [2, 9, 20, 21, 24, 25, 31, 36, 38, 45, 46]. In general the problem of estimating Γ, μ, and D is ill-posed since it admits an infinite number of solution pairs (see [8, 9]).

Here, we consider piecewise constant material parameters μ and D and a known constant Γ as in [33, 14]. Opposed to the general setting, if μ and D are piecewise constant functions, then they can be uniquely determined from knowledge of the absorbed energy $\mathcal{E}$ if, in addition, the values of the two parameters at the boundary are known [33]. Moreover, it is a useful fact that the union of the jumps of μ and D are contained in the set of discontinuities of derivatives up to the 2nd order of the qPAT measurement absorbed energy $\mathcal{E}$, which then takes the role of the data function f in (1.1).

In order to recover its discontinuities and consequently the discontinuities of μ and D, we use the topological derivative approach described in the first part. This allows us to detect discontinuities of $\mathcal{E}$ more accurately than in [14], where a variational method based on an Ambrosio–Tortorelli relaxation of a Mumford–Shah-like functional has been considered. In fact, the numerical algorithm that we present here is sufficiently robust to identify both μ and D even in the presence of noise in the data f; see section 6 for more details and comparisons.

The outline of the paper is as follows: In section 2 we introduce the basic notation and state the general assumptions used throughout the paper. In section 3, we study the properties of the minimizers of the functional (1.1) and of its perturbed version. This is, in fact, the first step in order to introduce, in section 4, the tensor of order $2m$ which is the basic analytical tool needed to characterize the topological gradient (see section 5) and to describe the variations of the minimizers of the functionals (1.1). Finally, we complement this analytical theory with numerical examples. In fact, section 6 is devoted to the presentation of generalized versions of the numerical algorithms described in [13], which are applied to qPAT.

2. Notation and main assumptions.

In this section we recall the notation regarding tensors and functional spaces and the main assumptions used throughout this paper.

Notation 2.1. We first recall that a tensor of order m can also be represented as hypermatrices by choosing a basis, i.e., $A_{i_1,\dots,i_m}$, where $i_k = 1,\dots,d$ and d is the dimension of vector space; see, for example, [19]. In what follows $d = 2$.

Tensors and hypermatrices.

- Latin capital letters, e.g., A, E, V, indicate tensors of order m. The blackboard bold letters, e.g., $\mathbb{M}$, represent tensors of order $2m$.
- Given a vector of components $(i_1,\dots,i_m)$, with $m \geq 2$, we use the symbol $\boldsymbol{i} := (i_1,\dots,i_m)$ in order to shorten the notation of indices for tensors. For instance, we will often write $A_{\boldsymbol{i}}$ instead of $A_{i_1,\dots,i_m}$.
- Let $A, B \in \mathbb{R}^{2\times 2\times\cdots\times 2}$ be two tensors of order m; then $A\cdot B = \sum_{\boldsymbol{i},\boldsymbol{j}=1}^{2} a_{\boldsymbol{i}\boldsymbol{j}} b_{\boldsymbol{i}\boldsymbol{j}}$ denotes the usual scalar product, i.e.,

$$A\cdot B = \sum_{\substack{\boldsymbol{i}=(i_1,\dots,i_m)=1\\ \boldsymbol{j}=(j_1,\dots,j_m)=1}}^{2} a_{i_1,\dots,i_m,j_1,\dots,j_m} b_{i_1,\dots,i_m,j_1,\dots,j_m}.$$

- Let $A, B \in \mathbb{R}^{2\times 2\times \cdots \times 2}$ be two tensors of order m; then $\mathbb{M}_{\boldsymbol{ij}} := A \otimes B = \left[a_{\boldsymbol{i}} b_{\boldsymbol{j}}\right]$ is a tensor of order $2m$.
- Let $A \in \mathbb{R}^{2\times 2\times \cdots \times 2}$ be a tensor of order m; then $|A| := \left(\sum_{\boldsymbol{i}=(i_1,\dots,i_m)=1}^{2} a_{\boldsymbol{i}}^2\right)^{\frac{1}{2}}$ denotes the Frobenius norm of A.

Function spaces.

- $H^j(\Omega)$ denotes the Sobolev space, where all derivatives up to order $j \in \mathbb{N}$ are square integrable (see, for instance, [1]).
- $H^j_0(\Omega) := \overline{C_0^\infty(\Omega)}^{H^j(\Omega)}$ is the subspace of functions which satisfy the homogeneous boundary condition. See again [1].
- ∇^m u represents the tensor of the derivatives of order m of the function u, i.e.,

$$\nabla^m u = (\nabla^m u)_{i_1,\dots,i_m} = \frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}},$$

  where $i_k = 1, 2$ for $k = 1, \dots, m$.
- $(\nabla\cdot)$ is the divergence operator.
- Let $A : \Omega \to \mathbb{R}^{2\times 2\times \cdots \times 2}$ be a tensor of order m; then we define

$$((\nabla\cdot)A)_{i_1,\dots,i_{m-1}} := \sum_{i_m=1}^{2} \partial_{x_{i_m}} A_{i_1\cdots i_m}.$$

- We define the divergence operator, $(\nabla\cdot)^m A$, applied m times inductively, i.e., $(\nabla\cdot)^m = (\nabla\cdot)((\nabla^{m-1}\cdot)A)$.
- The polyharmonic operator is defined inductively by $\Delta^m u = \Delta(\Delta^{m-1}u)$, with $m \geq 2$.
- Denoting with n the outward normal vector on $\partial\Omega$ and with $\frac{\partial}{\partial n}$ the normal derivative, the symbol $\frac{\partial^m u}{\partial n^m}$ is defined recursively by $\frac{\partial^m u}{\partial n^m} = \frac{\partial}{\partial n}(\frac{\partial^{m-1} u}{\partial n^{m-1}})$.
- ζ and f will denote bounded functions such that $\zeta \in L^\infty_+(\Omega) := \{\zeta \in L^\infty(\Omega) : \zeta > 0 \text{ a.e. in } \Omega\}$ and $f \in L^\infty(\Omega)$.

Parameters.

- $\alpha > 0$ is fixed during the whole paper and has the role of a regularization parameter.

Let us now state the main assumptions.

Assumption 2.2 (Domains).

1. Ω denotes an open, connected, and bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.
2. $\mathcal{B}_\rho(y)$ denotes a two-dimensional ball of radius ρ and center $y \in \mathbb{R}^2$.
3. K denotes a closed subset of Ω with positive 2D-Hausdorff measure.
4. $\delta_0 > 0$, and $L_0 \subseteq \Omega \backslash K$ is an open domain with smooth boundary satisfying

$$\operatorname{dist}(\overline{L}_0, \partial\Omega \cup K) \geq \delta_0 > 0. \tag{2.1}$$

5. $0 < \varepsilon < 1$, $y \in L_0$ and $\tau \in \mathbb{S}^1$ are fixed, such that

$$\Omega_\varepsilon(y,\tau) := \left\{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Sigma_\varepsilon(y,\tau)) < \varepsilon^2\right\}, \tag{2.2}$$

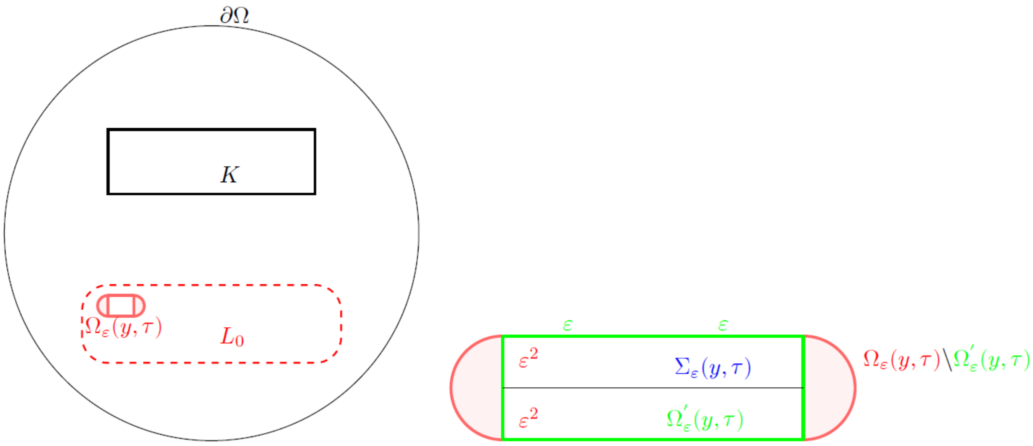

Figure 1. *L_0 does not touch $\partial\Omega$ and K and can contain the stripe $\Omega_\varepsilon(y,\tau)$. The scaling of the thin stripe: $\Omega_\varepsilon(y,\tau) = \Omega_\varepsilon^{'}(y,\tau) \cup \Omega_\varepsilon(y,\tau)\backslash\Omega_\varepsilon^{'}(y,\tau)$.*

with

$$\Sigma_\varepsilon(y,\tau) := \left\{x \in \mathbb{R}^2 : x = y + \rho\tau,\ -\varepsilon \le \rho \le \varepsilon\right\}, \tag{2.3}$$

contained in L_0. In particular, $\Omega_\varepsilon(y,\tau)$ does not contain K. Moreover, we denote by $\Omega_\varepsilon^{'}$ the rectangular box around $\Sigma_\varepsilon(y,\tau)$ and the two caps by $\Omega_\varepsilon\backslash\Omega_\varepsilon^{'}$ (see Figure 1). Since $0 < \varepsilon < 1$, $y \in L_0$, and $\tau \in \mathbb{S}^1$ are fixed throughout the paper, we omit the dependencies of $\Sigma_\varepsilon(y,\tau)$ and $\Omega_\varepsilon(y,\tau)$ on y and τ.

6. n_ε and n denote the outward normal vectors to $\partial\Omega_\varepsilon$ and $\partial\Omega$, respectively.

Remark 2.3. We use the terminology of morphological image analysis (see, for instance, [40]). Let $B \subseteq \mathbb{R}^2$ be a *structuring element* and $A \subseteq \mathbb{R}^2$ an arbitrary set; then the *dilation* of A with respect to the structuring element B is defined as follows:

$$A \oplus B := \{x + y : x \in A \text{ and } y \in B\}.$$

In particular

$$\Sigma_\varepsilon(y,\tau) = \{y\} \oplus \Sigma_\varepsilon(0,\tau) \text{ and } \Omega_\varepsilon(y,\tau) = \Sigma_\varepsilon(y,\tau) \oplus \mathcal{B}_\rho(0).$$

Assumption 2.4 (Functions). Let $0 < \kappa < \frac{1}{2}$. We define the following functions:

1.
$$v = \kappa\chi_K + 1\chi_{\Omega\backslash K}. \tag{2.4}$$

2. For given (y,τ) such that $\Omega_\varepsilon(y,\tau) \subseteq L_0$ we define $v_\varepsilon : \Omega \to \mathbb{R}$ by

$$v_\varepsilon(x) = \begin{cases} \kappa, & x \in K \cup \overline{\Omega}_\varepsilon(y,\tau), \\ 1 & \text{otherwise}. \end{cases} \tag{2.5}$$

Therefore, we have

$$(v_\varepsilon - v)(x) = \begin{cases} \kappa - 1, & x \in \overline{\Omega}_\varepsilon(y,\tau), \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

Remark 2.5. We note that

$$|\Omega_\varepsilon| = \varepsilon^3(4+\pi\varepsilon) = \mathcal{O}(\varepsilon^3), \left|\Omega_\varepsilon^{'}\right| = 4\varepsilon^3 = \mathcal{O}(\varepsilon^3), \text{ and } \left|\Omega_\varepsilon \backslash \Omega_\varepsilon^{'}\right| = \pi\varepsilon^4 = \mathcal{O}(\varepsilon^4). \tag{2.7}$$

We finally stress that throughout this paper C denotes a generic constant, which can depend on $\kappa, \delta_0, y, \tau, K, \Omega, L_0, \alpha,$ and f but does not necessarily need to depend on all of them. That is,

$$C := C(\kappa, \Omega, K, \delta_0, L_0, \|f\|_{L^\infty(\Omega)}, \alpha) > 0. \tag{2.8}$$

3. Asymptotic analysis.

Let $\zeta \in L^\infty_+(\Omega)$, $f \in L^\infty(\Omega)$, $\alpha > 0$, and $m \geq 2$ be fixed. In this section we are analyzing the following functional defined on $H_0^m(\Omega)$:

$$\mathcal{J}(u;\zeta) := \frac{1}{2}\int_\Omega (u-f)^2\, dx + \frac{\alpha}{2}\int_\Omega \zeta\, |\nabla^m u|^2\, dx. \tag{3.1}$$

In the following we characterize the minimizer of $\mathcal{J}(\cdot;\zeta)$ as the solution of a $2m$-order partial differential equation.

Lemma 3.1. *Let v be defined as in (2.4); then there exists a unique minimum $u \in H_0^m(\Omega)$ of $\mathcal{J}(u;v)$, and u is also a weak solution of*

$$\begin{cases} u + \alpha\ (-1)^m(\nabla\cdot)^m(v\nabla^m u) = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1}u}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

In addition, let v_ε be defined as in (2.5); then there exists a unique minimum $u_\varepsilon \in H_0^m(\Omega)$ of $\mathcal{J}(u_\varepsilon;v_\varepsilon)$, and u_ε is also a weak solution of

$$\begin{cases} u_\varepsilon + \alpha\ (-1)^m(\nabla\cdot)^m(v_\varepsilon\nabla^m u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = \frac{\partial u_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m-1}u_\varepsilon}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Moreover, the following energy estimate holds:

$$\max\left\{\|u\|_{H^m(\Omega)}, \|u_\varepsilon\|_{H^m(\Omega)}\right\} \leq C\,\|f\|_{L^\infty(\Omega)}.$$

See Appendix A for the proof of this lemma.

We define the function $w_\varepsilon := u_\varepsilon - u$. As a consequence of Theorem 3.1, we have the following lemma.

Lemma 3.2. *From (2.6), (3.2), and (3.3), function $w_\varepsilon \in H_0^m(\Omega)$ is the weak solution to*

$$\begin{cases} w_\varepsilon + \alpha\ (-1)^m(\nabla\cdot)^m(v\nabla^m w_\varepsilon) = \alpha\ (-1)^m(\nabla\cdot)^m((1-\kappa)\chi_{\Omega_\varepsilon}\nabla^m u_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon = \frac{\partial w_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m-1}w_\varepsilon}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

and

$$\begin{cases} w_\varepsilon + \alpha\ (-1)^m (\nabla\cdot)^m (v_\varepsilon \nabla^m w_\varepsilon) = \alpha\ (-1)^m (\nabla\cdot)^m ((1-\kappa)\chi_{\Omega_\varepsilon} \nabla^m u) & \text{in } \Omega, \\ w_\varepsilon = \frac{\partial w_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m-1} w_\varepsilon}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Proof. The proof follows by subtracting (3.2) from (3.3) and then by adding and substracting $v\nabla^m u_\varepsilon$ and $v_\varepsilon \nabla^m u$ properly. In fact, it is straightforward to find that the weak formulations of the two problems (3.4) and (3.5), are, for all $\varphi \in H_0^m(\Omega)$,

$$\int_\Omega w_\varepsilon \varphi\, dx + \alpha \int_\Omega v \nabla^m w_\varepsilon \cdot \nabla^m \varphi\, dx = \alpha(1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m \varphi\, dx \tag{3.6}$$

and

$$\int_\Omega w_\varepsilon \varphi\, dx + \alpha \int_\Omega v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m \varphi\, dx = \alpha(1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m \varphi\, dx, \tag{3.7}$$

respectively. ■

3.1. Asymptotics of w_ε. We need the following estimates for u, which is a consequence of the local regularity results for polyharmonic equations which can be found in [22].

Lemma 3.3. *Let u be the solution of (3.2). Then $u \in H^{2m}(L_0)$ and there exists a constant C such that*

$$\|\nabla^m u\|_{L^\infty(L_0)} \le C \|f\|_{L^\infty(\Omega)}. \tag{3.8}$$

Proof. Since $v = 1$ in L_0, the equation in (3.2) is equal to

$$u + \alpha(-\Delta)^m u = f \qquad \text{in } L_0;$$

hence, by interior regularity results for polyharmonic operators with smooth coefficients (see, for instance, [22, Thm. 2.20]), it follows that $u \in H^{2m}(L_0)$ and

$$\|u\|_{H^{2m}(L_0)} \le C(\|f\|_{L^\infty(\Omega)} + \|u\|_{H^m(\Omega)}) \le C \|f\|_{L^\infty(\Omega)}.$$

Then, by using Sobolev's embedding theorem, [1, Thm. 6.2], we find that

$$H^{2m}(L_0) \subset C^{m,\gamma}(L_0), \qquad \text{with } \ \gamma \in (0,1);$$

hence

$$\|\nabla^m u\|_{L^\infty(L_0)} \le C\|u\|_{H^{2m}(L_0)} \le C \|f\|_{L^\infty(\Omega)}.$$ ■

In the following lemma we get some asymptotic behavior on the function w_ε.

Lemma 3.4. *Let $p, q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, with $q \in (1,2)$ and $p \in (2,+\infty)$, and $\eta_{m,q,k} := \frac{k}{m}(\frac{1}{q} - \frac{1}{2})$ for $k = 1, \dots, m-1$, where $\eta_{m,q,k} > 0$ for every $k = 1, \dots, m-1$. Then there exists some positive constant C independent of ε such that w_ε satisfies*

$$\|w_\varepsilon\|_{H^m(\Omega)} \le C\, |\Omega_\varepsilon|^{\frac{1}{2}} \ \ \text{and} \ \ \|w_\varepsilon\|_{H^{m-k}(\Omega)} \le C\, |\Omega_\varepsilon|^{\frac{1}{2}+\eta_{m,q,k}} \tag{3.9}$$

for every $k = 1, \dots, m$.

Proof. From (3.7), using the test function $\varphi = w_\varepsilon$ (which is an element of $H_0^m(\Omega)$ according to Theorem 3.1), we obtain

$$\int_\Omega {w_\varepsilon}^2 \, dx + \alpha \int_\Omega v_\varepsilon \left|\nabla^m w_\varepsilon\right|^2 \, dx = \alpha(1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m w_\varepsilon \, dx.$$

Application of the Cauchy–Schwarz inequality and the use of (3.8), (2.5), and the fact that $\Omega_\varepsilon \subset\subset L_0$ (see Figure 1) then give

$$\begin{aligned}\kappa \left\|\nabla^m w_\varepsilon\right\|^2_{L^2(\Omega)} &\leq (1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m w_\varepsilon \, dx \leq (1-\kappa) \left\|\nabla^m u\right\|_{L^\infty(\Omega_\varepsilon)} \int_{\Omega_\varepsilon} \left|\nabla^m w_\varepsilon\right| \, dx \\ &\leq C \left\|\nabla^m w_\varepsilon\right\|_{L^2(\Omega)} \left|\Omega_\varepsilon\right|^{\frac{1}{2}}.\end{aligned}$$

Thus, there exists a positive constant C such that

$$\left\|\nabla^m w_\varepsilon\right\|_{L^2(\Omega)} \leq C \left|\Omega_\varepsilon\right|^{\frac{1}{2}},$$

and thus the first estimate in (3.9) follows by means of the Poincaré inequality; see Theorem A.1.

To prove the second inequality in (3.9), we consider an auxiliary function $\overline{W}_\varepsilon$ which satisfies

$$\begin{cases} \overline{W}_\varepsilon + \alpha \, (-1)^m (\nabla\cdot)^m (v \nabla^m \overline{W}_\varepsilon) = w_\varepsilon & \text{in } \Omega, \\ \overline{W}_\varepsilon = \frac{\partial \overline{W}_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m-1} \overline{W}_\varepsilon}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

for which the weak solution is given by

$$\int_\Omega \overline{W}_\varepsilon \varphi + \alpha \int_\Omega v \nabla^m \overline{W}_\varepsilon \cdot \nabla^m \varphi = \int_\Omega w_\varepsilon \varphi$$

for all $\varphi \in H_0^m(\Omega)$. Inserting $\varphi = w_\varepsilon$ into the last equation and $\varphi = \overline{W}_\varepsilon$ into (3.6) and then subtracting the resulting equations, we find

$$\int_\Omega w_\varepsilon^2 \, dx = \alpha(1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m \overline{W}_\varepsilon \, dx.$$

To estimate the left-hand side of the previous equation, we apply the Hölder inequality on the term on the right-hand side. With this aim, we first observe that $\overline{W}_\varepsilon$ is more regular in Ω_ε because it is the solution of a polyharmonic operator with constant coefficients. We carefully explain this fact later, in (3.15). By hypothesis, we choose p and q such that

$$p \in (2, +\infty), \; q \in (1, 2), \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1; \tag{3.11}$$

hence

$$\begin{aligned}\left\|w_\varepsilon\right\|^2_{L^2(\Omega)} &\leq \alpha(1-\kappa) \left\|\nabla^m u_\varepsilon\right\|_{L^q(\Omega_\varepsilon)} \left\|\nabla^m \overline{W}_\varepsilon\right\|_{L^p(\Omega_\varepsilon)} \\ &\leq \alpha(1-\kappa) \left(\left\|\nabla^m w_\varepsilon\right\|_{L^q(\Omega_\varepsilon)} + \left\|\nabla^m u\right\|_{L^q(\Omega_\varepsilon)}\right) \left\|\nabla^m \overline{W}_\varepsilon\right\|_{L^p(\Omega_\varepsilon)}.\end{aligned} \tag{3.12}$$

Now, we estimate the terms on the right-hand side of (3.12).

Estimate of $\|\nabla^m u\|_{L^q(\Omega_\varepsilon)}$. We apply Theorem 3.3; in fact since $\Omega_\varepsilon \subset\subset L_0$ (cf. Figure 1) it follows that there exists some constant C such that

(3.13)
$$\|\nabla^m u\|^q_{L^q(\Omega_\varepsilon)} \le \|\nabla^m u\|^q_{L^\infty(L_0)} \left(\int_{\Omega_\varepsilon} dx\right) \le C\,|\Omega_\varepsilon|, \qquad \text{that is,} \qquad \|\nabla^m u\|_{L^q(\Omega_\varepsilon)} \le C\,|\Omega_\varepsilon|^{\frac{1}{q}}.$$

Estimate of $\|\nabla^m w_\varepsilon\|_{L^q(\Omega_\varepsilon)}$. Using again Hölder's inequality with $r = \frac{2}{q} \in (1,2)$ and $s = \frac{2}{2-q}$ it follows that

$$\|\nabla^m w_\varepsilon\|^q_{L^q(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} |\nabla^m w_\varepsilon|^q \; dx \le \left(\int_{\Omega_\varepsilon} |\nabla^m w_\varepsilon|^2 \; dx\right)^{\frac{q}{2}} |\Omega_\varepsilon|^{\frac{1}{s}} = \|\nabla^m w_\varepsilon\|^q_{L^2(\Omega_\varepsilon)} |\Omega_\varepsilon|^{1-\frac{q}{2}},$$

which then together with the first, already proven, inequality in (3.9) gives

$$\|\nabla^m w_\varepsilon\|_{L^q(\Omega_\varepsilon)} \le \|\nabla^m w_\varepsilon\|_{L^2(\Omega_\varepsilon)} |\Omega_\varepsilon|^{\frac{1}{q}-\frac{1}{2}} \le C\,|\Omega_\varepsilon|^{\frac{1}{q}}. \tag{3.14}$$

Estimate of $\left\|\nabla^m \overline{W}_\varepsilon\right\|_{L^p(\Omega_\varepsilon)}$. As $\Omega_\varepsilon \subset\subset L_0$ and $v = 1$ in L_0, for the estimate of this term we can apply the local regularity results for polyharmonic equations with constant coefficients; see [22]. Indeed, since $\overline{W}_\varepsilon$ is the weak solution of (3.10) and $w_\varepsilon \in L^2(\Omega)$, we have that $\overline{W}_\varepsilon \in H^{2m}(L_0)$; hence

$$\left\|\overline{W}_\varepsilon\right\|_{H^{2m}(L_0)} \le C\,\|w_\varepsilon\|_{L^2(\Omega)}.$$

Finally, applying the Sobolev embedding theorem (see [1, 22]), we find that

$$\left\|\nabla^m \overline{W}_\varepsilon\right\|_{L^p(L_0)} \le \|\overline{W}_\varepsilon\|_{W^{m,p}(L_0)} \le C\|\overline{W}_\varepsilon\|_{H^{2m}(L_0)} \le C\,\|w_\varepsilon\|_{L^2(\Omega)}. \tag{3.15}$$

Therefore, inserting (3.13), (3.14), and (3.15) into (3.12), we get that there exists a positive constant C such that

$$\|w_\varepsilon\|^2_{L^2(\Omega)} \le C\,|\Omega_\varepsilon|^{\frac{1}{q}}\,\|w_\varepsilon\|_{L^2(\Omega)}, \tag{3.16}$$

which finally implies that

$$\|w_\varepsilon\|_{L^2(\Omega)} \le C\,|\Omega_\varepsilon|^{\frac{1}{q}}. \tag{3.17}$$

So far we have found the estimate of w_ε in $H^m(\Omega)$ and $L^2(\Omega)$. Finally, the assertion of the theorem, i.e., the second inequality of (3.9), follows by the application of classical interpolation inequalities in Sobolev spaces (see [30]), with the results (3.17) and the first inequality in (3.9). Indeed, for every $k = 1, \ldots, m-1$, we have that

$$\|w_\varepsilon\|_{H^{m-k}} \le C\|w_\varepsilon\|^{1-\frac{k}{m}}_{H^m(\Omega)}\|w_\varepsilon\|^{\frac{k}{m}}_{L^2(\Omega)} \le C|\Omega_\varepsilon|^{\frac{1}{2}+\frac{k}{m}(\frac{1}{q}-\frac{1}{2})}, \qquad k = 1, \ldots, m-1,$$

which gives, together with (3.17), the assertion of the theorem. ■

Remark 3.5. For q and m fixed, it is straightforward to observe that the following relations between $\eta_{m,q,k}$ hold: $\eta_{m,q,1} \leq \eta_{m,q,2} \leq \cdots \leq \eta_{m,q,m-1}$.

We define some auxiliary functions.

Definition 3.6. *In what follows we use the following notation:* $\boldsymbol{i} = (i_1, i_2, \ldots, i_m)$*, where* $i_k = 1, 2$ *for* $k = 1, \ldots, m$ *and* $x_{\boldsymbol{i}} = x_{i_1} \ldots x_{i_m}$*. We denote the polynomial of degree* m *and in the variables* x_1 *and* x_2 *by*

$$V^{i_1 \cdots i_m}(x) := \frac{1}{m!} x_{i_1} \cdots x_{i_m} = \frac{1}{m!} x_{\boldsymbol{i}} \quad \text{for} \quad x \in \Omega. \tag{3.18}$$

To shorten the notation we define $V^{\boldsymbol{i}} := V^{i_1 \cdots i_m}$*.*

These functions satisfy

$$(\nabla \cdot)^m (\nabla^m V^{\boldsymbol{i}}) = 0 \text{ in } \Omega. \tag{3.19}$$

Moreover, we denote by $V_\varepsilon^{\boldsymbol{i}}$ *the solution of*

$$\begin{cases} (\nabla \cdot)^m (\gamma_\varepsilon \nabla^m V_\varepsilon^{\boldsymbol{i}})) = 0 & \text{in } \Omega, \\ V_\varepsilon^{\boldsymbol{i}} = V^{\boldsymbol{i}} & \text{on } \partial\Omega, \\ \frac{\partial V_\varepsilon^{\boldsymbol{i}}}{\partial n} = \frac{\partial V^{\boldsymbol{i}}}{\partial n}, \ldots, \frac{\partial^{m-1} V_\varepsilon^{\boldsymbol{i}}}{\partial n^{m-1}} = \frac{\partial^{m-1} V^{\boldsymbol{i}}}{\partial n^{m-1}} & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

where

$$\gamma_\varepsilon(x) = \begin{cases} \kappa, & x \in \Omega_\varepsilon(y, \tau), \\ 1 & \text{otherwise.} \end{cases} \tag{3.21}$$

Consistently we denote

$$\gamma_0 = 1 \text{ in } \Omega. \tag{3.22}$$

In the following, we provide estimates for the functions $V_\varepsilon^{\boldsymbol{i}}$ and $V^{\boldsymbol{i}}$.

Lemma 3.7. *Under the assumptions of Lemma 3.4, there exists some constant* C *independent of* ε *such that*

$$\left\| V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}} \right\|_{H^m(\Omega)} \leq C \left| \Omega_\varepsilon \right|^{\frac{1}{2}} \text{ and } \left\| V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}} \right\|_{H^{m-k}(\Omega)} \leq C \left| \Omega_\varepsilon \right|^{\frac{1}{2} + \eta_{m,q,k}} \tag{3.23}$$

for any $k = 1, \ldots, m$*.*

The proof is similar to the proof of Theorem 3.4.

Lemma 3.8. *Let* $\eta_{m,q,1}$ *be defined as in Lemma 3.4. Then, for all test functions* $\phi \in C_0^m(L_0)$ *the following identity holds:*

$$\left| \int_{\Omega_\varepsilon} (\nabla^m u_\varepsilon \cdot \nabla^m V^{\boldsymbol{i}}) \phi \, dx - \int_{\Omega_\varepsilon} (\nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m u) \phi \, dx \right| = \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}). \tag{3.24}$$

Proof. We first observe that $\gamma_\varepsilon = v_\varepsilon$ in L_0. We use the weak formulations (3.6) and (3.7) of w_ε, where we choose $\varphi = V^{\boldsymbol{i}}\phi$ and $\varphi = V_\varepsilon^{\boldsymbol{i}}\phi$, respectively, i.e.,

$$\int_\Omega w_\varepsilon V^{\boldsymbol{i}}\phi\, dx + \alpha \int_\Omega v\nabla^m w_\varepsilon \cdot \nabla^m (V^{\boldsymbol{i}}\phi)\, dx = \alpha(1-\kappa)\int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m (V^{\boldsymbol{i}}\phi)\, dx, \tag{3.25}$$

and

$$\int_\Omega w_\varepsilon V_\varepsilon^{\boldsymbol{i}}\phi\, dx + \alpha \int_\Omega v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m (V_\varepsilon^{\boldsymbol{i}}\phi)\, dx = \alpha(1-\kappa)\int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m (V_\varepsilon^{\boldsymbol{i}}\phi)\, dx. \tag{3.26}$$

Then, subtracting (3.25) from (3.26), and since ϕ has compact support in L_0, we find

$$\begin{aligned}\int_{L_0} w_\varepsilon (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}})\phi\, dx + \alpha \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m (V_\varepsilon^{\boldsymbol{i}}\phi)\, dx - \alpha \int_{L_0} \nabla^m w_\varepsilon \cdot \nabla^m (V^{\boldsymbol{i}}\phi)\, dx \\ = \alpha(1-\kappa)\int_{\Omega_\varepsilon} \left[\nabla^m u \cdot \nabla^m (V_\varepsilon^{\boldsymbol{i}}\phi) - \nabla^m u_\varepsilon \cdot \nabla^m (V^{\boldsymbol{i}}\phi)\right] dx.\end{aligned}$$

In the last expression we specify the terms containing the derivative of maximum order of all the functions, i.e.,

$$\begin{aligned}
&\int_{L_0} w_\varepsilon (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}})\phi\, dx + \alpha \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m V_\varepsilon^{\boldsymbol{i}}\phi\, dx + \alpha \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m \phi V_\varepsilon^{\boldsymbol{i}}\, dx \\
&- \alpha \int_{L_0} \nabla^m w_\varepsilon \cdot \nabla^m V^{\boldsymbol{i}}\phi\, dx + \alpha \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n V_\varepsilon^{\boldsymbol{i}} \otimes \nabla\phi + \nabla^n \phi \otimes \nabla V_\varepsilon^{\boldsymbol{i}}\right)\right] dx \\
&- \alpha \int_{L_0} \nabla^m w_\varepsilon \cdot \nabla^m \phi V^{\boldsymbol{i}}\, dx - \alpha \int_{L_0} \nabla^m w_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n V^{\boldsymbol{i}} \otimes \nabla\phi + \nabla^n \phi \otimes \nabla V^{\boldsymbol{i}}\right)\right] dx \\
&= \alpha(1-\kappa)\Bigg\{ \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m V_\varepsilon^{\boldsymbol{i}}\phi\, dx + \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m \phi V_\varepsilon^{\boldsymbol{i}}\, dx - \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m V^{\boldsymbol{i}}\phi\, dx \\
&- \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m \phi V^{\boldsymbol{i}}\, dx + \int_{\Omega_\varepsilon} \nabla^m u \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n V_\varepsilon^{\boldsymbol{i}} \otimes \nabla\phi + \nabla^n \phi \otimes \nabla V_\varepsilon^{\boldsymbol{i}}\right)\right] dx \\
&- \int_{L_0} \nabla^m u_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n V^{\boldsymbol{i}} \otimes \nabla\phi + \nabla^n \phi \otimes \nabla V^{\boldsymbol{i}}\right)\right] dx \Bigg\}.
\end{aligned} \tag{3.27}$$

We next use the weak formulation of the equations satisfied by $V^{\boldsymbol{i}}$ and $V_\varepsilon^{\boldsymbol{i}}$ (see (3.19) and (3.20)) in L_0. The idea is to find an expression of the terms in w_ε with the maximum order of derivation, which are in the left-hand side of the previous formula, in terms of the derivatives of lower order with respect to w_ε. Specifically, for all test functions $w_\varepsilon\phi$, we have

$$\int_{L_0} v_\varepsilon \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m (w_\varepsilon \phi)\, dx = 0, \qquad \text{and} \qquad \int_{L_0} \nabla^m V^{\boldsymbol{i}} \cdot \nabla^m (w_\varepsilon \phi)\, dx = 0;$$

hence, specifying the terms $\nabla^m(w_\varepsilon\phi)$, we get for the first equation

(3.28)
$$\int_{L_0} v_\varepsilon \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m w_\varepsilon \phi \, dx = -\int_{L_0} v_\varepsilon \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m \phi w_\varepsilon \, dx - \int_{L_0} v_\varepsilon \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n w_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla w_\varepsilon \right) \right] dx$$

and, analogously for the second equation,

(3.29)
$$\int_{L_0} \nabla^m V^{\boldsymbol{i}} \cdot \nabla^m w_\varepsilon \phi \, dx = -\int_{L_0} \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m \phi w_\varepsilon \, dx - \int_{L_0} \nabla^m V^{\boldsymbol{i}} \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n w_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla w_\varepsilon \right) \right] dx.$$

We use (3.28) and (3.29) in (3.27). Then, in the resulting equation, we add and subtract $V^{\boldsymbol{i}}$ in each term containing $V_\varepsilon^{\boldsymbol{i}}$ (but not in the terms already containing the difference $V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}$ and not in the first integral in the left-hand side of the equality (3.27)). After some calculations and simplifications, we find

$$\begin{aligned}
&\alpha(1-\kappa)\int_{\Omega_\varepsilon} (\nabla^m u \cdot \nabla^m V_\varepsilon^{\boldsymbol{i}} - \nabla^m u_\varepsilon \cdot \nabla^m V^{\boldsymbol{i}})\phi \, dx \\
&= \int_{L_0} w_\varepsilon (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}})\phi \, dx \\
&- \alpha \Bigg\{ \int_{L_0} v_\varepsilon \nabla^m (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \cdot \nabla^m \phi w_\varepsilon \, dx - \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \nabla^m \phi (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \, dx \\
&+ \int_{L_0} v_\varepsilon \nabla^m (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n w_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla w_\varepsilon \right) \right] dx \\
&- \int_{L_0} v_\varepsilon \nabla^m w_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \otimes \nabla\phi + \nabla^n \phi \otimes \nabla (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \right) \right] dx \\
&+ (\kappa - 1) \Bigg[\int_{\Omega_\varepsilon} \nabla^m V^{\boldsymbol{i}} \cdot \left(\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n w_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla w_\varepsilon \right) \right) dx \\
&+ \int_{\Omega_\varepsilon} \nabla^m V^{\boldsymbol{i}} \cdot \nabla^m \phi w_\varepsilon \, dx \Bigg] \Bigg\} \\
&- \alpha(1-\kappa) \Bigg\{ \int_{\Omega_\varepsilon} \nabla^m u \cdot \nabla^m \phi (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \, dx \\
&+ \int_{\Omega_\varepsilon} \nabla^m u \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n} \left(\nabla^n (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \otimes \nabla\phi + \nabla^n \phi \otimes \nabla (V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}) \right) \right] dx \Bigg\}.
\end{aligned}$$

Getting the assertion of the theorem, i.e., the estimate of the integral on the left-hand side of the previous equality, in terms of $|\Omega_\varepsilon|^{1+\eta_{m,q,1}}$, is a lengthy task. It is based on the application, on the terms in the right-hand side, of the Cauchy–Schwarz inequality, the fact that $\|\nabla^n \phi\|_{L^\infty(L_0)} \le C$ for all $n:\ 0 \le n \le m-1$ and $|v_\varepsilon| \le 1$ in L_0, and the estimates (3.9) and (3.23). We do not give the calculations in detail here; we only note that all these integrals can be estimated in terms of the product $|\Omega_\varepsilon|^{\frac{1}{2}}$ and $|\Omega_\varepsilon|^{\frac{1}{2}+\eta_{m,q,1}}$. ■

Remark 3.9. The identity (3.24) holds also with u_ε replaced by $V_\varepsilon^{\boldsymbol{h}}$ and u replaced by $V^{\boldsymbol{h}}$, respectively, since in L_0 the pairs satisfy the same equations.

Remark 3.10. From Theorem 3.4 it follows that

(3.30) $$\|w_\varepsilon\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{\frac{3}{2}}) \quad \text{and} \quad \|w_\varepsilon\|_{H^{m-k}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{3}{2}+3\eta_{m,q,k}}) \qquad \forall k = 1, \dots, m-1.$$

Moreover, from (3.23), it follows that

(3.31) $$\left\|V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}\right\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{\frac{3}{2}}) \text{ and } \left\|V_\varepsilon^{\boldsymbol{i}} - V^{\boldsymbol{i}}\right\|_{H^{m-k}(\Omega)} = \mathcal{O}(\varepsilon^{\frac{3}{2}+3\eta_{m,q,k}}) \qquad \forall k = 1, \dots, m-1.$$

4. Definition and properties of the $2m$-order tensor.

We start this section with the definition of the tensor $\mathbb{M}$. We note that the family of functions $(\frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. Hence, by the Riesz representation theorem, we have that the family of measures

(4.1) $$d\mu_\varepsilon := \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\, dx$$

is bounded in $C^0(\overline{\Omega})^*$, and hence by the Banach–Alaoglu theorem (see, for instance, [39]), possibly up to the extraction of a subsequence, we have that

(4.2) $$d\mu_\varepsilon := \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\, dx \to^* \ d\mu \quad \text{for } \varepsilon \to 0,$$

where $\to^*$ denotes the weak*-convergence of $C^0(\overline{\Omega})^*$, that is,

$$\int_\Omega \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\widetilde{\phi}\, dx \to \int_\Omega \widetilde{\phi}\, d\mu \text{ for } \varepsilon \to 0, \qquad \text{for all } \ \widetilde{\phi} \in C^0(\overline{\Omega}).$$

It is also immediate to see that, due to the form of Ω_ε, the measure μ is concentrated at the point y, i.e., $\mu = \delta_y$. Analogously, using the energy estimates (3.23), it follows that the family of functions $(\frac{1}{|\Omega_\varepsilon|}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\chi_{\Omega_\varepsilon})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$, and therefore the family of measures

$$d\nu_\varepsilon = \frac{1}{|\Omega_\varepsilon|}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\chi_{\Omega_\varepsilon}\, dx$$

converges, possibly up to subsequences, to a Borel measure

(4.3) $$d\nu_\varepsilon = \frac{1}{|\Omega_\varepsilon|}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\chi_{\Omega_\varepsilon}\, dx \to^* \mathbb{M}_{i_1\cdots i_m j_1\cdots j_m}\delta_y \ \text{ for } \varepsilon \to 0,$$

which is equivalent to saying that

$$\frac{1}{|\Omega_\varepsilon|}\int_\Omega \frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\chi_{\Omega_\varepsilon}\widetilde{\phi}\,dx \to \mathbb{M}_{i_1\cdots i_m j_1\cdots j_m}\widetilde{\phi}(y) \quad \text{for all} \quad \widetilde{\phi}\in C^0(\overline{\Omega}). \tag{4.4}$$

To shorten the notation we define $\mathbb{M}_{\boldsymbol{ij}} := \mathbb{M}_{i_1\cdots i_m j_1\cdots j_m}$.

Before putting together the results in Theorem 3.8 and (4.4), we observe a local regularity result on u_ε and $V_\varepsilon^{\boldsymbol{i}}$.

Remark 4.1. We note that the function u_ε is $C^m(\Omega_\varepsilon)$ since it satisfies, in the open set Ω_ε, a nonhomogeneous polyharmonic equation with a forcing term in L^∞ (by assumption). Indeed, this result follows from local regularity theorems (see [22]), for which $u_\varepsilon \in H^{2m}(\Omega_\varepsilon)$, and consequently from the application of the embeddings theorems.

This result also holds for $V_\varepsilon^{\boldsymbol{i}}$ since these functions satisfy a homogeneous polyharmonic equation.

From this regularity result, it follows that both $\nabla^m u_\varepsilon$ and $\nabla^m V_\varepsilon^{\boldsymbol{i}}$ are symmetric tensors in Ω_ε.

Therefore, from the symmetries of $\nabla^m u_\varepsilon$ in Ω_ε, for all $\phi\in C_0^m(L_0)$, by Theorem 3.8 and using (4.4) (where we choose $\widetilde{\phi} = \nabla^m u\phi$), we also have that

$$\begin{aligned}&\int_\Omega \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\nabla^m u_\varepsilon\cdot\nabla^m V^{\boldsymbol{i}}\phi\,dx\\ &\qquad = \int_\Omega \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\frac{\partial^m u_\varepsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\phi\,dx \to \mathbb{M}\nabla^m u(y)\phi(y) \quad \text{for} \quad \varepsilon\to 0.\end{aligned} \tag{4.5}$$

Remark 4.2. We notice that $(\frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\nabla^m u_\varepsilon\cdot\nabla^m V^{\boldsymbol{i}})_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ due to (3.18) and the use of energy estimates (3.9) after summing and subtracting u in $\nabla^m u_\varepsilon$. Therefore, this sequence converges in the weak* topology of $C(\overline{\Omega})$, up to a subsequence, to a Borel measure, i.e.,

$$\frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\nabla^m u_\varepsilon\cdot\nabla^m V^{\boldsymbol{i}}\,dx \to^* d\mu_{\boldsymbol{i}}. \tag{4.6}$$

Defining a bounded domain L_1, with smooth boundary, such that $\Omega_\varepsilon\subset L_1\subset L_0$, and exploiting (4.6) and (4.5), we deduce that

$$\int_\Omega \phi\,d\mu_{\boldsymbol{i}} = \mathbb{M}\nabla^m u(y)\phi(y) \qquad \forall\phi\in C^m(\overline{L_1}), \tag{4.7}$$

and by the density of $C^m(\overline{L_1})$ in $C^0(\overline{L_1})$, we have that

$$d\mu_{\boldsymbol{i}} = \mathbb{M}_{\boldsymbol{i}j_1\cdots j_m}\frac{\partial u}{\partial x_{j_1}\cdots\partial x_{j_m}}\delta_y.$$

Let us now state some key properties of the tensor $\mathbb{M}$.

Proposition 4.3. *We denote with $\sigma(\boldsymbol{i})$ any permutation of the indices $i_1, \dots, i_m$. The $2m$-order tensor $\mathbb{M}$ has the full symmetries, i.e.,*

$$\mathbb{M}_{\boldsymbol{ij}} = \mathbb{M}_{\sigma(\boldsymbol{i})\boldsymbol{j}} = \mathbb{M}_{\boldsymbol{i}\sigma(\boldsymbol{j})} = \mathbb{M}_{\boldsymbol{ji}}. \tag{4.8}$$

Proof. The first equality in (4.8) follows immediately from the fact that $V^{\boldsymbol{i}} = V^{\sigma(\boldsymbol{i})}$, which implies $V_\varepsilon^{\boldsymbol{i}} = V_\varepsilon^{\sigma(\boldsymbol{i})}$ (from the uniqueness result for (3.20)). The second identity follows from the regularity results for $V_\varepsilon^{\boldsymbol{i}}$ (see Remark 4.1), which implies that we can interchange the order of differentiation in (4.4).

The identity $\mathbb{M}_{\boldsymbol{ij}} = \mathbb{M}_{\boldsymbol{ji}}$ follows by substituting u and u_ε with $V^{\boldsymbol{j}}$ and $V_\varepsilon^{\boldsymbol{j}}$, respectively, in Theorem 3.8 (see Remark 3.9) and using the symmetry of the tensor $\nabla^m V_\varepsilon^{\boldsymbol{i}}$ (see Remark 4.1). In fact,

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m V^{\boldsymbol{j}} \phi \, dx = \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \nabla^m V_\varepsilon^{\boldsymbol{j}} \cdot \nabla^m V^{\boldsymbol{i}} \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{\eta_{m,q,1}}) \tag{4.9}$$

for all test-functions $\phi \in C_0^m(L_0)$.

Therefore, using (4.3) and Remark 4.1, we get for the left-hand side

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \nabla^m V_\varepsilon^{\boldsymbol{i}} \cdot \nabla^m V^{\boldsymbol{j}} \phi \, dx = \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1} \cdots \partial x_{j_m}} \phi \, dx \to \mathbb{M}_{\boldsymbol{ij}} \phi(y).$$

On the other hand, the first term in the right-hand side of (4.9) gives

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \nabla^m V_\varepsilon^{\boldsymbol{j}} \cdot \nabla^m V^{\boldsymbol{i}} \phi \, dx = \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \frac{\partial^m V_\varepsilon^{\boldsymbol{j}}}{\partial x_{i_1} \cdots \partial x_{i_m}} \phi \, dx \to \mathbb{M}_{\boldsymbol{ji}} \phi(y); \tag{4.10}$$

that is the assertion. ∎

By the symmetry properties of $\mathbb{M}$ we can consider, without loss of generality, the space of symmetric tensor of order m, i.e.,

$$S^m(\mathbb{R}^2) := \left\{ A \text{ such that } \; A_{\boldsymbol{i}} = A_{\sigma(\boldsymbol{i})} \right\}.$$

Therefore, we have that $\mathbb{M} : S^m(\mathbb{R}^2) \to S^m(\mathbb{R}^2)$. For a review of some results on symmetric tensors of generic order, their properties, their decomposition, and their relations with hypermatrices, see [19], [26], and references therein.

We introduce some auxiliary functions which are needed to get some bounds on the tensor of order $2m$.

Definition 4.4. *Given $E \in S^m(\mathbb{R}^2)$, we define the auxialiary functions*

$$V := \sum_{\boldsymbol{i}=(i_1,\cdots,i_m)=1}^{2} E_{\boldsymbol{i}} V^{\boldsymbol{i}} \quad \text{and} \quad V_\varepsilon := \sum_{\boldsymbol{i}=(i_1,\cdots,i_m)=1}^{2} E_{\boldsymbol{i}} V_\varepsilon^{\boldsymbol{i}}. \tag{4.11}$$

From (3.19) and (3.20), we find that by construction V and V_ε are solutions of

$$(4.12)\qquad \begin{cases} (\nabla\cdot)^m(\nabla^m V) = 0 & \text{in } \Omega,\\ V = \sum_{\boldsymbol{i}=1}^{2} E_{\boldsymbol{i}} V^{\boldsymbol{i}} & \text{on } \partial\Omega,\\ \frac{\partial V}{\partial n} = \frac{\partial}{\partial n}\left(\sum_{\boldsymbol{i}=1}^{2} E_{\boldsymbol{i}} V^{\boldsymbol{i}}\right), \dots, \frac{\partial^{m-1} V}{\partial n^{m-1}} = \frac{\partial^{m-1}}{\partial n^{m-1}}\left(\sum_{\boldsymbol{i}=1}^{2} E_{\boldsymbol{i}} V^{\boldsymbol{i}}\right) & \text{on } \partial\Omega \end{cases}$$

and

$$(4.13)\qquad \begin{cases} (\nabla\cdot)^m(\gamma_\varepsilon\nabla^m V_\varepsilon) = 0 & \text{in } \Omega,\\ V_\varepsilon = V & \text{on } \partial\Omega,\\ \frac{\partial V_\varepsilon}{\partial n} = \frac{\partial V}{\partial n}, \dots, \frac{\partial^{m-1} V_\varepsilon}{\partial n^{m-1}} = \frac{\partial^{m-1} V}{\partial n^{m-1}} & \text{on } \partial\Omega, \end{cases}$$

respectively.

Remark 4.5. We observe that from the definition of $V^{\boldsymbol{i}}$ (see (3.18)) it follows that

$$(4.14)\qquad \nabla^m V = \sum_{\boldsymbol{i}=1}^{2} E_{\boldsymbol{i}} \nabla^m V^{\boldsymbol{i}} = E.$$

In fact, from (3.18), since $x_{i_l} = x_1$ or $x_{i_l} = x_2$ for $l = 1, \dots, m$, we have that

$$x_{i_1} x_{i_2} \cdots x_{i_m} = x_1^{m-h} x_2^h,$$

where h is the number of x_{i_l} for $l = 1, \dots, m$ equal to x_1. In this way, we immediately get that

$$(4.15)\qquad \frac{\partial^m V^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}} = \frac{1}{m!}\frac{\partial^m(x_{i_1}x_{i_2}\cdots x_{i_m})}{\partial x_{j_1}\cdots\partial x_{j_m}} = \frac{1}{m!}\frac{\partial^m(x_1^{m-h}x_2^h)}{\partial x_1^{m-k}\partial x_2^k} = \begin{cases} \frac{(m-k)!k!}{m!} & \text{if } h = k,\\ 0 & \text{if } h \neq k. \end{cases}$$

Then, since for hypothesis $E_{i_1 i_2 \cdots i_m}$ is a symmetric tensor, it follows that the number of tensors $E_{i_1\cdots i_m} = E_{\sigma(i_1\cdots i_m)}$, in the sum of (4.14), which have $m-k$ components equal to 1 and k components equal to 2 is given by $\binom{m}{k}$. This consideration and (4.15) give the identity in (4.14).

Definition 4.6. *Given V and V_ε as in (4.11), we define*

$$W_\varepsilon := V_\varepsilon - V.$$

By means of the difference of (4.12) and (4.13), and then adding and subtracting $(\nabla\cdot)^m(\gamma_\varepsilon\nabla^m V)$, according to (2.5), we find that W_ε satisfies

$$(4.16)\qquad \begin{cases} (\nabla\cdot)^m(\gamma_\varepsilon\nabla^m W_\varepsilon) = (\nabla\cdot)^m((1-\kappa)\chi_{\Omega_\varepsilon}\nabla^m V) & \text{in } \Omega,\\ W_\varepsilon = 0 & \text{on } \partial\Omega,\\ \frac{\partial W_\varepsilon}{\partial n} = \cdots = \frac{\partial^{m-1} W_\varepsilon}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Using these auxiliary functions, in the following we determine some sharp bounds on $\mathbb{M}$.

Proposition 4.7. *The* $2m$*-order tensor* $\mathbb{M}$ *is positive definite and satisfies*

$$|E|^2 \leq \mathbb{M}E \cdot E \leq \frac{1}{\kappa}|E|^2 \quad \textit{for all } E \in S^m(\mathbb{R}^2).$$

Proof. The starting point to prove this proposition is the definition of the tensor $\mathbb{M}_{\boldsymbol{ij}}$ (see (4.4)), which we specialize for the case where the test function $\phi \in C^m(\overline{\Omega})$. This choice of regularity will be clear in the second part of the proof, where it is needed in order to get uniform estimates. By (4.4), we have that

$$\mathbb{M}_{\boldsymbol{ij}}\phi(y) = \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\phi\,dx \qquad \text{for all } \phi \in C^m(\overline{\Omega}). \tag{4.17}$$

Applying this identity with the test function $E_{\boldsymbol{i}}E_{\boldsymbol{j}}\phi$, with $\phi \in C^m(\overline{\Omega})$, it follows from (4.17) that

$$\mathbb{M}E\cdot E\phi(y) = \sum_{\boldsymbol{i},\boldsymbol{j}=1}^{2} E_{\boldsymbol{i}}\mathbb{M}_{\boldsymbol{ij}}E_{\boldsymbol{j}}\phi(y) = \sum_{\boldsymbol{i},\boldsymbol{j}=1}^{2}\left\{\lim_{\varepsilon\to 0}E_{\boldsymbol{i}}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}E_{j_1\cdots j_m}\phi\,dx\right\}. \tag{4.18}$$

Using (4.14) in (4.18), we can rewrite $E_{j_1\cdots j_m} = \frac{\partial^m V}{\partial x_{j_1}\cdots\partial x_{j_m}}$; that is, it follows that

$$\begin{aligned}\mathbb{M}E\cdot E\phi(y) &= \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\sum_{\boldsymbol{i},\boldsymbol{j}=1}^{2}E_{\boldsymbol{i}}\frac{\partial^m V_\varepsilon^{\boldsymbol{i}}}{\partial x_{j_1}\cdots\partial x_{j_m}}\frac{\partial^m V}{\partial x_{j_1}\cdots\partial x_{j_m}}\phi\,dx \\ &= \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\nabla^m W_\varepsilon\cdot\nabla^m V\phi\,dx + \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|\nabla^m V|^2\phi\,dx,\end{aligned} \tag{4.19}$$

where in the right-hand side of the second equality we have added and subtracted $V^{\boldsymbol{i}}$ in the term containing $V_\varepsilon^{\boldsymbol{i}}$ and then we have used (4.11), i.e., the fact that $\sum_{\boldsymbol{i}=2}^{2}E_{\boldsymbol{i}}W_\varepsilon^{\boldsymbol{i}} = W_\varepsilon$. Now, we derive an expansion for the first limit in the right-hand side of the second equality of (4.19); i.e., to be more precise, we prove that

$$\int_{\Omega_\varepsilon}\nabla^m W_\varepsilon\cdot\nabla^m V\phi\,dx = \frac{1}{1-\kappa}\int_{\Omega}\gamma_\varepsilon|\nabla^m W_\varepsilon|^2\phi\,dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}). \tag{4.20}$$

With this aim, we use the weak formulation of the problem (4.16); i.e., for all $\varphi \in H_0^m(\Omega)$

$$\int_{\Omega}\gamma_\varepsilon\nabla^m W_\varepsilon\cdot\nabla^m\varphi\,dx = (1-\kappa)\int_{\Omega_\varepsilon}\nabla^m V\cdot\nabla^m\varphi\,dx. \tag{4.21}$$

In this last equation, we choose $\varphi = W_\varepsilon\phi$, where $\phi \in C^m(\overline{\Omega})$; hence

$$\int_{\Omega}\gamma_\varepsilon\nabla^m W_\varepsilon\cdot\nabla^m(W_\varepsilon\phi)\,dx = (1-\kappa)\int_{\Omega_\varepsilon}\nabla^m V\cdot\nabla^m(W_\varepsilon\phi)\,dx.$$

Specifying the term of higher order derivative with respect to W_ε, we get

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \nabla^m W_\varepsilon \cdot \nabla^m W_\varepsilon \phi\, dx + \int_\Omega \gamma_\varepsilon \nabla^m W_\varepsilon \cdot \nabla^m \phi W_\varepsilon\, dx \\ &\qquad + \int_\Omega \gamma_\varepsilon \nabla^m W_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n W_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla W_\varepsilon\right)\right] dx \\ &= (1-\kappa)\Bigg\{ \int_{\Omega_\varepsilon} \nabla^m V \cdot \nabla^m W_\varepsilon \phi\, dx + \int_{\Omega_\varepsilon} \nabla^m V \cdot \nabla^m \phi W_\varepsilon\, dx \\ &\qquad + \int_{\Omega_\varepsilon} \nabla^m V \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n W_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla W_\varepsilon\right)\right] dx \Bigg\}.\end{aligned}$$

Therefore, we can rewrite the previous equality as

$$\begin{aligned}&(1-\kappa)\int_{\Omega_\varepsilon} \nabla^m V \cdot \nabla^m W_\varepsilon \phi\, dx - \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi\, dx \\ &= \int_\Omega \gamma_\varepsilon \nabla^m W_\varepsilon \cdot \nabla^m \phi W_\varepsilon\, dx + \int_\Omega \gamma_\varepsilon \nabla^m W_\varepsilon \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n W_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla W_\varepsilon\right)\right] dx \\ &- (1-\kappa)\Bigg\{ \int_{\Omega_\varepsilon} \nabla^m V \cdot \left[\sum_{n=1}^{m-1} \nabla^{m-1-n}\left(\nabla^n W_\varepsilon \otimes \nabla\phi + \nabla^n \phi \otimes \nabla W_\varepsilon\right)\right] dx \\ &\qquad + \int_{\Omega_\varepsilon} \nabla^m V \cdot \nabla^m \phi W_\varepsilon\, dx \Bigg\} =: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{4.22}$$

As was already done in the proof of Lemma 3.8, we can estimate each integral, in the right-hand side of the previous formula, in terms of the product of $|\Omega_\varepsilon|^{\frac{1}{2}}$ and $|\Omega_\varepsilon|^{\frac{1}{2}+\eta_{m,q,1}}$ by using the Cauchy–Schwarz inequality, the fact that $\phi \in C^m(\overline{\Omega})$, and the estimates in Lemma 3.7.

Estimate of I_1.

$$|I_1| \le \max_{\overline{\Omega}} |\nabla^m \phi|\; \|\nabla^m W_\varepsilon\|_{L^2(\Omega)} \|W_\varepsilon\|_{L^2(\Omega)} \le C|\Omega_\varepsilon|^{\frac{1}{2}} |\Omega_\varepsilon|^{\frac{1}{2}+\eta_{m,q,1}} = C|\Omega_\varepsilon|^{1+\eta_{m,q,1}}. \tag{4.23}$$

Estimate of I_2. We note that in the sum only the derivatives of order equal to or less than $m-1$ appear. Therefore,

$$|I_2| \le c\sum_{n=1}^{m-1} \|\nabla^m W_\varepsilon\|_{L^2(\Omega)} \|\nabla^n W_\varepsilon\|_{L^2(\Omega)} \le C\|W_\varepsilon\|_{H^m(\Omega)} \|W_\varepsilon\|_{H^{m-1}(\Omega)} = C|\Omega_\varepsilon|^{1+\eta_{m,q,1}}. \tag{4.24}$$

Estimate of I_3. In the sum only the derivatives of order equal or less than $m-1$ appear; hence

$$|I_4| \le C|\nabla^m V| \sum_{n=1}^{m-1} \|\nabla^n W_\varepsilon\|_{L^2(\Omega_\varepsilon)} |\Omega_\varepsilon|^{\frac{1}{2}} \le C\; |\Omega_\varepsilon|^{\frac{1}{2}} \|W_\varepsilon\|_{H^{m-1}(\Omega_\varepsilon)} \le C|\Omega_\varepsilon|^{1+\eta_{m,q,1}}. \tag{4.25}$$

Estimate of I_4.

$$|I_3| \le |\nabla^m V| \max_{\overline{\Omega}_\varepsilon} |\nabla^m \phi| \, \|W_\varepsilon\|_{L^2(\Omega_\varepsilon)} \, |\Omega_\varepsilon|^{\frac{1}{2}} \le C|\Omega_\varepsilon|^{1+\eta_{m,q,1}}. \tag{4.26}$$

Inserting (4.23), (4.24), (4.25), and (4.26) into (4.22), we get (4.20), i.e.,

$$\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot \nabla^m V \phi \, dx = \frac{1}{1-\kappa} \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}).$$

We are now in position to prove the two estimates in the statement of the proposition.

Inequality $\mathbb{M}E \cdot E \ge |E|^2$. Using (4.20) in (4.19), we find

$$\begin{aligned} \phi(y)\mathbb{M}E \cdot E = {} & \frac{1}{1-\kappa} \lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|} \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx \\ & + \lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|} \left[\int_{\Omega_\varepsilon} |\nabla^m V|^2 \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}) \right]. \end{aligned} \tag{4.27}$$

In this last equation, we choose nonnegative test functions $\phi \in C^m(\overline{\Omega})$, i.e., $\phi \ge 0$, and recalling that $\gamma_\varepsilon > 0$ and $1-\kappa > 0$ (see Assumptions 2.4 and (3.21)), we get

$$\phi(y)\mathbb{M}E \cdot E \ge \lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|} \left[\int_{\Omega_\varepsilon} |\nabla^m V|^2 \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}) \right].$$

By (4.14), we find

$$\phi(y)\mathbb{M}E \cdot E \ge |E|^2 \lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \phi \, dx + \lim_{\varepsilon\to 0} \mathcal{O}(|\Omega_\varepsilon|^{\eta_{m,q,1}}) = |E|^2 \phi(y),$$

from which it follows that

$$\mathbb{M}E \cdot E \ge |E|^2.$$

Inequality $\mathbb{M}E \cdot E \le \frac{1}{\kappa}|E|^2$. Taking (4.20) and assuming again that $\phi \ge 0$, we find

$$\begin{aligned} \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx &= \sqrt{\kappa} \int_{\Omega_\varepsilon} \frac{1-\kappa}{\sqrt{\kappa}} \sqrt{\phi} \nabla^m W_\varepsilon \cdot \nabla^m V \sqrt{\phi} \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}) \\ &\le \left(\int_{\Omega_\varepsilon} \kappa |\nabla^m W_\varepsilon|^2 \phi \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_\varepsilon} \frac{(1-\kappa)^2}{\kappa} |\nabla^m V|^2 \phi \, dx \right)^{\frac{1}{2}} + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}) \\ &\le \frac{1}{2} \int_{\Omega_\varepsilon} \kappa |\nabla^m W_\varepsilon|^2 \phi \, dx + \frac{1}{2}(1-\kappa)^2 \int_{\Omega_\varepsilon} \frac{1}{\kappa} |\nabla^m V|^2 \phi + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}), \end{aligned} \tag{4.28}$$

where in the first inequality we have applied the Cauchy–Schwarz inequality and in the second one the Young inequality. Then, since all the terms in the first integral on the right-hand side of the last inequality are positive, and $\gamma_\varepsilon \ge \kappa$ in Ω (see (3.21)), we find that

$$\frac{1}{2} \int_{\Omega_\varepsilon} \kappa |\nabla^m W_\varepsilon|^2 \phi \, dx = \frac{1}{2} \int_{\Omega_\varepsilon} \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx \le \frac{1}{2} \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx;$$

hence, using this estimate in (4.28), and then summing up the resulting terms appropriately, we get

$$\int_{\Omega} \gamma_\varepsilon |\nabla^m W_\varepsilon|^2 \phi \, dx \leq (1-\kappa)^2 \int_{\Omega_\varepsilon} \frac{1}{\kappa} |\nabla^m V|^2 \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}).$$

Inserting this inequality into (4.27), we get

$$\phi(y)\mathbb{M}E \cdot E \leq \lim_{\varepsilon \to 0} \frac{1-\kappa}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \frac{1}{\kappa} |\nabla^m V|^2 \phi \, dx + \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \left[\int_{\Omega_\varepsilon} |\nabla^m V|^2 \phi \, dx + \mathcal{O}(|\Omega_\varepsilon|^{1+\eta_{m,q,1}}) \right],$$

where, using again (4.14), it follows that

$$\phi(y)\mathbb{M}E \cdot E \leq \frac{1}{\kappa} |E|^2 \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \phi \, dx + \lim_{\varepsilon \to 0} \mathcal{O}(|\Omega_\varepsilon|^{\eta_{m,q,1}}) = \frac{1}{\kappa} |E|^2 \phi(y),$$

which implies that

$$\mathbb{M}E \cdot E \leq \frac{1}{\kappa} |E|^2. \qquad \blacksquare$$

Spectral decomposition of $\mathbb{M}$. In the following we derive the spectral decomposition of the tensor $\mathbb{M}$. To simplify the notation, we assume without loss of generality that $\Omega_\varepsilon := \Omega_\varepsilon(0, e_1)$, where $e_1 = (1,0)$ and $e_2 = (0,1)$. The more general setting can be always obtained by rotation of Ω_ε. From [19] we know that the dimension of the space of symmetric tensors of order m is equal to $\dim(S^m(\mathbb{R}^2)) = m+1$. In the following, we denote with σ_m any permutation of the elements of $e_{i_1} \otimes \cdots \otimes e_{i_m}$, where $i_k = 1,2$ for $k = 1, \dots, m$ with the clause that the resulting permuted object is not repeated if it coincides with one of the already existing outcome. We define with E^h, for $h = 1, \dots, m+1$, the orthonormal canonical basis of $S^m(\mathbb{R}^2)$, i.e.,

$$\begin{aligned} E^1 &= \underbrace{e_1 \otimes \cdots \otimes e_1}_{m-elements}, \\ E^h &= \frac{1}{\sqrt{\binom{m}{h-1}}} \sum_{\sigma_m} \underbrace{e_1 \otimes \cdots \otimes e_1}_{(m-h+1)-elements} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{(h-1)-elements} \qquad \text{for} \quad h = 2, \dots, m, \\ E^{m+1} &= \underbrace{e_2 \otimes \cdots \otimes e_2}_{m-elements}. \end{aligned}$$

Remark 4.8. By the definition of the permutation σ_m that we are adopting, it is straightforward to observe that each E^h, for $h = 1, \dots, m+1$, is the sum of only $\binom{m}{h-1}$ tensors of order m which have $m-h+1$ elements equal to e_1 and $h-1$ elements equal to e_2. For this reason, the quantity $1/\sqrt{\binom{m}{h-1}}$ is only a normalization coefficient; i.e., it is such that $|E^h|^2 = 1$ for $h = 2, \dots, m$.

In order to derive the desired spectral value properties of $\mathbb{M}$, we use regularity properties of solutions of higher order equations with discontinuous coefficients recently derived by Barton in [10].

Spectral values along E^h, $h = 1, \ldots, m$. We will start by showing that the quadratic form associated to $\mathbb{M}$, i.e., $\mathbb{M}E \cdot E$, attains the value 1 along $E = E^h$ for $h = 1, \ldots, m$.

Proposition 4.9. *Under the notational simplification that $\Omega_\varepsilon = \Omega_\varepsilon(0, e_1)$, for all $h = 1, \ldots, m$, it holds that*

$$\mathbb{M}E^h \cdot E^h = 1.$$

Proof. The proof of this proposition is essentially based on the use of (4.19), where we choose $\phi = 1$, i.e.,

$$\mathbb{M}E \cdot E = |E|^2 + \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E \, dx, \tag{4.29}$$

where we used the fact that $\nabla^m V = E$ see (4.14). Next, we use the equation satisfied by W_ε (see (4.16)) and the regularity estimates proved in [10] to get an estimate of the integral term in the previous formula, when we choose $E = E^h$ for $h = 1, \ldots, m$.

We first note that $W_\varepsilon \in H_0^m(\Omega)$ and in (4.16) the source term $((1-\kappa)\chi_{\Omega_\varepsilon} \nabla^m V) \in L^2(\Omega)$. Then, for all $p' \in (2, 3)$ and $p \in (\frac{3}{2}, 2)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, by Theorem 24 in [10], it follows that there exists C, independent of ε and E, such that

$$\sup_{\Omega_\varepsilon} |\nabla^{m-1} W_\varepsilon| \le C \|\nabla^{m-1} W_\varepsilon\|_{L^p(\Omega)} + \|(1-\kappa)\chi_{\Omega_\varepsilon} E\|_{L^{p'}(\Omega)}. \tag{4.30}$$

In addition, since $p \in (\frac{3}{2}, 2)$, we also have that

$$\|\nabla^{m-1} W_\varepsilon\|_{L^p(\Omega)} \le C \|\nabla^{m-1} W_\varepsilon\|_{L^2(\Omega)} \le C \|W_\varepsilon\|_{H^{m-1}(\Omega)} \le C \|W_\varepsilon\|_{H^m(\Omega)}.$$

The last inequality and (4.30) imply that

$$\sup_{\Omega_\varepsilon} |\nabla^{m-1} W_\varepsilon| \le C \|W_\varepsilon\|_{H^m(\Omega)} + C'|E||\Omega_\varepsilon|^{\frac{1}{p'}} \le C|\Omega_\varepsilon|^{\min(\frac{1}{2}, \frac{1}{p'})} \le C|\Omega_\varepsilon|^{\frac{1}{p'}}. \tag{4.31}$$

Spectral value along E^1. Choosing $E = E^1$ in (4.29), we get

$$\mathbb{M}E^1 \cdot E^1 = 1 + \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m} \, dx, \tag{4.32}$$

and moreover

$$\int_{\Omega_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m} \, dx = \int_{\Omega_\varepsilon \setminus \Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m} \, dx + \int_{\Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m} \, dx =: I_1^1 + I_2^1.$$

We estimate I_1^1 and I_2^1:

$$\begin{aligned} |I_1^1| = \left| \int_{\Omega_\varepsilon \setminus \Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m} \, dx \right| &\le \left\| \frac{\partial^m W_\varepsilon}{\partial x_1^m} \right\|_{L^2(\Omega_\varepsilon \setminus \Omega'_\varepsilon)} |\Omega_\varepsilon \setminus \Omega'_\varepsilon|^{\frac{1}{2}} \\ &\le \|\nabla^m W_\varepsilon\|_{L^2(\Omega)} |\Omega_\varepsilon \setminus \Omega'_\varepsilon|^{\frac{1}{2}} = \mathcal{O}(|\Omega_\varepsilon|^{1+\frac{1}{6}}), \end{aligned} \tag{4.33}$$

where in the last equality we have used the results in (2.7). For the integral I_2^1 we have

$$I_2^1 = \int_{\Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m}\, dx = \int_{-\varepsilon^2}^{\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^m}\, dx_1\, dx_2$$
$$= \int_{-\varepsilon^2}^{\varepsilon^2} \left(\frac{\partial^{m-1} W_\varepsilon(\varepsilon, x_2)}{\partial x_1^{m-1}} - \frac{\partial^{m-1} W_\varepsilon(-\varepsilon, x_2)}{\partial x_1^{m-1}} \right) dx_2.$$

Then, using the result in (4.31), we get that

$$|I_2^1| = \left| \int_{-\varepsilon^2}^{\varepsilon^2} \left(\frac{\partial^{m-1} W_\varepsilon(\varepsilon, x_2)}{\partial x_1^{m-1}} - \frac{\partial^{m-1} W_\varepsilon(-\varepsilon, x_2)}{\partial x_1^{m-1}} \right) dx_2 \right| \le C \sup_{x_2 \in [-\varepsilon^2, \varepsilon^2]} \left| \frac{\partial^{m-1} W_\varepsilon}{\partial x_1^{m-1}} \right| \varepsilon^2$$
$$\le C \|\nabla^{m-1} W_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\, \varepsilon^2 = \mathcal{O}(|\Omega_\varepsilon|^{\frac{1}{p'} + \frac{2}{3}}), \tag{4.34}$$

where, since for hypothesis $p' \in (2, 3)$, we have that $\frac{1}{p'} + \frac{2}{3} = 1 + \delta$, $\delta > 0$. Therefore, inserting the results in (4.33) and (4.34) into (4.32), we finally find the equation

$$\mathbb{M} E^1 \cdot E^1 = 1.$$

Spectral value along E^h for $h = 2, \ldots, m$. Choosing $E = E^h$ for $h = 2, \ldots, m$ in (4.29), we get

$$\mathbb{M} E^h \cdot E^h = 1 + \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E^h\, dx, \tag{4.35}$$

where, using the symmetries of $\nabla^m W_\varepsilon$ (which come from the regularity property of the polyharmonic function W_ε in Ω_ε; see (4.16) and Remark 4.1) and Remark 4.8, we get

$$\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E^h\, dx = \sqrt{\binom{m}{h-1}} \int_{\Omega_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}}\, dx_1\, dx_2,$$

and moreover

$$\int_{\Omega_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}}\, dx_1\, dx_2 = \int_{\Omega_\varepsilon \setminus \Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}}\, dx_1\, dx_2 + \int_{\Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}}\, dx_1\, dx_2$$
$$=: I_1^h + I_2^h.$$

As already done for the terms I_1^1 and I_2^1, we find that

$$|I_1^h| = \left| \int_{\Omega_\varepsilon \setminus \Omega'_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}}\, dx_1\, dx_2 \right| \le \|\nabla^m W_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega'_\varepsilon)} |\Omega_\varepsilon \setminus \Omega'_\varepsilon|^{\frac{1}{2}} = \mathcal{O}(|\Omega_\varepsilon|^{1+\frac{1}{6}}), \tag{4.36}$$

where in the last equality we have used the results in (2.7). For the integral I_2^h we get

$$|I_2^h| \le \left| \int_{\Omega_\varepsilon} \frac{\partial^m W_\varepsilon}{\partial x_1^{m-h+1} \partial x_2^{h-1}} \right| = \left| \int_{-\varepsilon^2}^{\varepsilon^2} \frac{\partial^{m-1} W_\varepsilon(\varepsilon, x_2)}{\partial x_1^{m-h} \partial x_2^{h-1}} - \frac{\partial^{m-1} W_\varepsilon(-\varepsilon, x_2)}{\partial x_1^{m-h} \partial x_2^{h-1}}\, dx_2 \right|$$
$$\le C \|\nabla^{m-1} W_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\, \varepsilon^2 = \mathcal{O}(|\Omega_\varepsilon|^{1+\delta}), \tag{4.37}$$

where $\delta > 0$. Inserting (4.36) and (4.37) into (4.35), we get the assertion. ∎

Spectral values along E^{m+1}. We will now show that the quadratic form associated to $\mathbb{M}$, $\mathbb{M}E \cdot E$, attains value $\frac{1}{\kappa}$ along $E = E^{m+1}$. To prove this proposition we will need to show two preliminary lemmas. The first one is an adaptation of [18, Lemma 3].

Lemma 4.10. *For all $E \in S^m(\mathbb{R}^2)$ we have*

$$(\kappa-1)\mathbb{M}E \cdot E = \frac{\kappa-1}{\kappa}|E|^2 + \lim_{\varepsilon\to 0}\left(\frac{1}{|\Omega_\varepsilon|}\min_{\widetilde{W}\in H_0^m(\Omega)}\int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2\right). \tag{4.38}$$

Proof. Since $\gamma_\epsilon = \kappa$ in Ω_ε (see (2.5)), it follows by just squaring the quadratic term that

$$\begin{aligned}&\int_\Omega \gamma_\varepsilon \left|\nabla^m W_\varepsilon + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2 dx\\ &= \int_{\Omega_\varepsilon} \frac{(\kappa-1)^2}{\kappa}|E|^2\, dx + \int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2\, dx + 2(\kappa-1)\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E\, dx.\end{aligned} \tag{4.39}$$

Then, using the weak formulation (4.21) of W_ε, in which we choose $\varphi = W_\varepsilon$ as a test function, we can rewrite the second integral in the right-hand side of the previous formula in the following way:

$$\int_\Omega \gamma_\varepsilon |\nabla^m W_\varepsilon|^2\, dx = (1-\kappa)\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot \nabla^m V\, dx = (1-\kappa)\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E\, dx, \tag{4.40}$$

where in the last equality we have used (4.14). Inserting (4.40) into (4.39), we find

$$\int_\Omega \gamma_\varepsilon \left|\nabla^m W_\varepsilon + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2 dx = |\Omega_\varepsilon|\frac{(\kappa-1)^2}{\kappa}|E|^2 + (\kappa-1)\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E\, dx;$$

hence

$$(\kappa-1)\int_{\Omega_\varepsilon} \nabla^m W_\varepsilon \cdot E\, dx = -|\Omega_\varepsilon|\frac{(\kappa-1)^2}{\kappa}|E|^2 + \int_\Omega \gamma_\varepsilon \left|\nabla^m W_\varepsilon + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2.$$

Then, inserting this expression into (4.29), we get

$$(\kappa-1)\mathbb{M}E\cdot E = (\kappa-1)|E|^2 - \frac{(\kappa-1)^2}{\kappa}|E|^2 + \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_\Omega \gamma_\varepsilon \left|\nabla^m W_\varepsilon + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2 dx.$$

Finally, noting that W_ε is the minimizer of the functional

$$\widetilde{W} \in H_0^m(\Omega) \to \int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2,$$

the assertion follows. ∎

Next, we have the following lemma.

Lemma 4.11. *There exists a function* $\overline{w}_\varepsilon \in H_0^m(\Omega)$ *such that*

$$\int_\Omega |\nabla^m \overline{w}_\varepsilon - \chi_{\Omega_\varepsilon} \nabla^m x_2^m|^2 \, dx = o(|\Omega_\varepsilon|) \qquad \text{for } \varepsilon \to 0.$$

To prove this lemma, we first need to introduce some parameters and three functions, defined on the real axis, which are involved in the definition of the function $\overline{w}_\varepsilon$:

1. Let

$$a \in \left(\frac{m-1}{2m-1}, \frac{4m-3}{2(2m-1)} \right), \tag{4.41}$$

and let $\psi_\varepsilon \in C_0^\infty(\mathbb{R})$ (see [34]) be the function which satisfies for some constant $C_1 > 0$
   (a) $\psi_\varepsilon(x_2) = 1 \;\; \forall x_2 \in [-\varepsilon^2, \varepsilon^2]$;
   (b) $\operatorname{supp}(\psi_\varepsilon) \subset (-\varepsilon^2 - \varepsilon^{2a}, \varepsilon^2 + \varepsilon^{2a})$;
   (c) $0 \le \psi_\varepsilon \le 1$;
   (d) $\left|\frac{d\psi_\varepsilon}{dx_2}\right| \le \frac{C_1}{\varepsilon^{2a}}, \; \left|\frac{d^2\psi_\varepsilon}{dx_2^2}\right| \le \frac{C_1}{\varepsilon^{4a}}, \dots, \left|\frac{d^m\psi_\varepsilon}{dx_2^m}\right| \le \frac{C_1}{\varepsilon^{2ma}}$.
2. Moreover, let

$$\beta \in \left(\frac{1}{2}, \frac{1}{2(2m-1)} + a \right), \tag{4.42}$$

and let $\varphi_\varepsilon \in C_0^\infty(\mathbb{R})$ (see [34]) be the function which satisfies for some constant $C_2 > 0$
   (a) $\varphi_\varepsilon(x_1) = 1 \;\; \forall x_1 \in [-\varepsilon, \varepsilon]$;
   (b) $\operatorname{supp}(\varphi_\varepsilon) \subset (-\varepsilon - \varepsilon^{2\beta}, \varepsilon + \varepsilon^{2\beta})$;
   (c) $0 \le \varphi_\varepsilon \le 1$;
   (d) $\left|\frac{d\varphi_\varepsilon}{dx_1}\right| \le \frac{C_2}{\varepsilon^{2\beta}}, \; \left|\frac{d^2\varphi_\varepsilon}{dx_1^2}\right| \le \frac{C_2}{\varepsilon^{4\beta}}, \dots, \left|\frac{d^m\varphi_\varepsilon}{dx_1^m}\right| \le \frac{C_2}{\varepsilon^{2m\beta}}$.

   Remark 4.12. Parameters a and β vary in the triangular orange region in Figure 3.
3. Finally, let $\overline{v}_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$\overline{v}_\varepsilon(x_2) := \begin{cases} \sum_{h=1}^m \frac{m!(-1)^{h-1}\varepsilon^{2h} x_2^{m-h}}{h!(m-h)!} & \text{if } x_2 > \varepsilon^2, \\ x_2^m & \text{if } x_2 \in [-\varepsilon^2, \varepsilon^2], \\ \sum_{h=1}^m \frac{m!(-1)^{2h-1}\varepsilon^{2h} x_2^{m-h}}{h!(m-h)!} & \text{if } x_2 < -\varepsilon^2. \end{cases} \tag{4.43}$$

Remark 4.13. Function $\overline{v}_\varepsilon$ is an element of H^m on every compact interval of $\mathbb{R}$. To show that, it is sufficient to prove that $\overline{v}_\varepsilon$ and its derivatives $\frac{d^l \overline{v}_\varepsilon}{dx_2^l}$, for $l = 1, \dots, m-1$, are continuous polynomial functions across $x_2 = \varepsilon^2$ and $x_2 = -\varepsilon^2$. With this aim, we first notice that in $\{x_2 > \varepsilon^2\}$

$$\overline{v}_\varepsilon(x_2) = \sum_{h=1}^m \frac{m!(-1)^{h-1}\varepsilon^{2h} x_2^{m-h}}{h!(m-h)!} = \sum_{h=1}^m (-1)^{h-1} \binom{m}{h} \varepsilon^{2h} x_2^{m-h} \qquad \text{for } x_2 > \varepsilon^2, \tag{4.44}$$

and, analogously, the derivatives of $\overline{v}_\varepsilon$ in $\{x_2 > \varepsilon^2\}$ satisfy

$$\begin{aligned}\frac{d^l\overline{v}_\varepsilon}{dx_2^l}(x_2) &= \sum_{h=1}^{m-l} \frac{m!(-1)^{h-1}(m-h)(m-h-1)\cdots(m-h-l+1)\varepsilon^{2h}x_2^{m-h-l}}{h!(m-h)!}\\ &= m!\sum_{h=1}^{m-l}\frac{(-1)^{h-1}\varepsilon^{2h}x_2^{m-h-l}}{h!(m-h-l)!} \qquad \text{for } x_2 > \varepsilon^2.\end{aligned}$$

Multiplying and dividing for $(m-l)!$ in the last expression, we get

(4.45)
$$\begin{aligned}\frac{d^l\overline{v}_\varepsilon}{dx_2^l}(x_2) &= \frac{m!}{(m-l)!}\sum_{h=1}^{m-l}(-1)^{h-1}\frac{(m-l)!}{h!(m-l-h)!}\varepsilon^{2h}x_2^{m-h-l}\\ &= m(m-1)\cdots(m-l+1)\sum_{h=1}^{m-l}(-1)^{h-1}\binom{m-l}{h}\varepsilon^{2h}x_2^{m-h-l} \qquad \text{for } 1\le l\le m-1.\end{aligned}$$

From (4.44) and (4.45) it is straightforward to check that both $\overline{v}_\varepsilon$ and its derivatives $\frac{d^l\overline{v}_\varepsilon}{dx_2^l}$ are continuous functions across $x_2 = \varepsilon^2$. In fact,

$$\overline{v}_\varepsilon(\varepsilon^2) = -\varepsilon^{2m}\sum_{h=1}^{m}(-1)^h\binom{m}{h} = -\varepsilon^{2m}\left[-1+\sum_{h=0}^{m}(-1)^h\binom{m}{h}\right] = \varepsilon^{2m}, \tag{4.46}$$

where in the last equality we used the well-known fact that $\sum_{h=0}^{m}(-1)^h\binom{m}{h} = 0$. Analogously, for all $1 \le l \le m-1$, we have

$$\begin{aligned}\frac{d^l\overline{v}_\varepsilon}{dx_2^l}(\varepsilon^2) &= m(m-1)\cdots(m-l+1)\varepsilon^{2(m-l)}\sum_{h=1}^{m-l}(-1)^{h-1}\binom{m-l}{h}\\ &= -m(m-1)\cdots(m-l+1)\varepsilon^{2(m-l)}\sum_{h=1}^{m-l}(-1)^{h}\binom{m-l}{h}\\ &= -m(m-1)\cdots(m-l+1)\varepsilon^{2(m-l)}\left[-1+\sum_{h=0}^{m-l}(-1)^{h}\binom{m-l}{h}\right]\\ &= m(m-1)\cdots(m-l+1)\varepsilon^{2(m-l)},\end{aligned} \tag{4.47}$$

where in the last equality we used again the property $\sum_{h=0}^{m-l}(-1)^h\binom{m-l}{h} = 0$. Values in (4.46) and (4.47) coincide with those assumed by x_2^m and its derivatives $\frac{d^lx_2^m}{dx_2^l}$, for $l = 1,\dots,m-1$, across $x_2 = \varepsilon^2$. The same argument can be applied for $\overline{v}_\varepsilon(x_2)$ in $\{x_2 < -\varepsilon^2\}$.

Proof of Lemma 4.11. In the following we denote by

$$R_\varepsilon = \left\{(x_1,x_2) : -\varepsilon-\varepsilon^{2\beta} \le x_1 \le \varepsilon+\varepsilon^{2\beta}, -\varepsilon^2-\varepsilon^{2a} \le x_2 \le \varepsilon^2+\varepsilon^{2a}\right\} \tag{4.48}$$

with a and β chosen as in (4.41) and (4.42), respectively. Note that for ε sufficiently small $R_\varepsilon \subset \Omega$.

We define a test-function $\overline{w}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\overline{w}_\varepsilon(x_1, x_2) = \overline{v}_\varepsilon(x_2)\psi_\varepsilon(x_2)\varphi_\varepsilon(x_1). \tag{4.49}$$

Function $\overline{w}_\varepsilon$ satisfies the following properties:

1. $\overline{w}_\varepsilon \in H_0^m(\Omega)$,
2. $\overline{w}_\varepsilon(x_1, x_2) = \begin{cases} x_2^m & \text{in } \Omega_\varepsilon^{'}, \\ x_2^m \varphi_\varepsilon(x_1) & \text{in } \Omega_\varepsilon \backslash \Omega_\varepsilon^{'}. \end{cases}$

We can now prove the assertion of the lemma. From the definition of $\overline{w}_\varepsilon$ and property 2, it follows that

(4.50)
$$\begin{aligned}\int_\Omega |\nabla^m \overline{w}_\varepsilon - \chi_{\Omega_\varepsilon} \nabla^m x_2^m|^2 \, dx &= \int_{\Omega_\varepsilon} |\nabla^m \overline{w}_\varepsilon - \nabla^m x_2^m|^2 \, dx + \int_{\Omega\backslash\Omega_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2 \, dx \\ &= \int_{\Omega_\varepsilon\backslash\Omega_\varepsilon^{'}} |\nabla^m \overline{w}_\varepsilon - \nabla^m x_2^m|^2 \, dx + \int_{\Omega\backslash\Omega_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2 \, dx =: J_1 + J_2.\end{aligned}$$

Now we estimate J_1 and J_2.

Estimate of J_1. For the integral J_1 we first use the property 2 of $\overline{w}_\varepsilon$; hence we find

$$\int_{\Omega_\varepsilon\backslash\Omega_\varepsilon^{'}} |\nabla^m \overline{w}_\varepsilon - \nabla^m x_2^m|^2 \, dx = \int_{\Omega_\varepsilon\backslash\Omega_\varepsilon^{'}} |\nabla^m (x_2^m(\varphi_\varepsilon - 1))|^2 \, dx. \tag{4.51}$$

By properties (2a)–(2d) of φ_ε and the fact that $|x_2| \le \varepsilon^2$, we observe that

$$|\nabla^m (x_2^m(\varphi_\varepsilon - 1))|^2 \le C \left(\sum_{n=0}^m |x_2|^n \left| \frac{d^n \varphi_\varepsilon}{dx_1^n} \right| \right)^2 \le C \sum_{n=0}^m \varepsilon^{4n-4n\beta};$$

hence, using this last inequality in (4.51) and the fact that $|\Omega_\varepsilon \setminus \Omega_\varepsilon^{'}| \le C\varepsilon^4$, we get

$$\int_{\Omega_\varepsilon\backslash\Omega_\varepsilon^{'}} |\nabla^m (x_2^m(\varphi_\varepsilon - 1))|^2 \, dx \le C|\Omega_\varepsilon \setminus \Omega_\varepsilon^{'}| \sum_{n=0}^m \varepsilon^{4n-4n\beta} \le C \sum_{n=0}^m \varepsilon^{4+4n-4n\beta} = o(|\Omega_\varepsilon|),$$

where the last equality derives from the fact that, for the range of a chosen, we have that $\beta < 1$ and hence $4 + 4n - 4n\beta > 4$. Then it is sufficient to recall that $\varepsilon^3 = \mathcal{O}(|\Omega_\varepsilon|)$; see (2.7).

Estimate of J_2. Taking into account the definitions of Ω_ε and $\Omega_\varepsilon^{'}$ (see Figure 1), it follows from the properties of $\overline{w}_\varepsilon$ and the definition of R_ε (see (4.48)) that

$$\int_{\Omega\backslash\Omega_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2 \, dx \le \int_{\Omega\backslash\Omega_\varepsilon^{'}} |\nabla^m \overline{w}_\varepsilon|^2 \, dx = \int_{R_\varepsilon\backslash\Omega_\varepsilon^{'}} |\nabla^m \overline{w}_\varepsilon|^2 \, dx.$$

We divide R_ε into eight parts, consisting of

$$\begin{aligned} R_\varepsilon^1 &:= \left\{(x_1, x_2) : -\varepsilon \le x_1 \le \varepsilon, \varepsilon^2 \le x_2 \le \varepsilon^2 + \varepsilon^{2a}\right\} \subseteq \Omega\backslash\Omega_\varepsilon, \\ R_\varepsilon^2 &:= \left\{(x_1, x_2) : \varepsilon \le x_1 \le \varepsilon + \varepsilon^{2\beta}, \varepsilon^2 \le x_2 \le \varepsilon^2 + \varepsilon^{2a}\right\} \subseteq \Omega\backslash\Omega_\varepsilon, \\ R_\varepsilon^3 &:= \left\{(x_1, x_2) : \varepsilon \le x_1 \le \varepsilon + \varepsilon^{2\beta}, -\varepsilon^2 \le x_2 \le \varepsilon^2\right\} \subseteq \Omega\backslash\Omega_\varepsilon^{'}, \end{aligned}$$

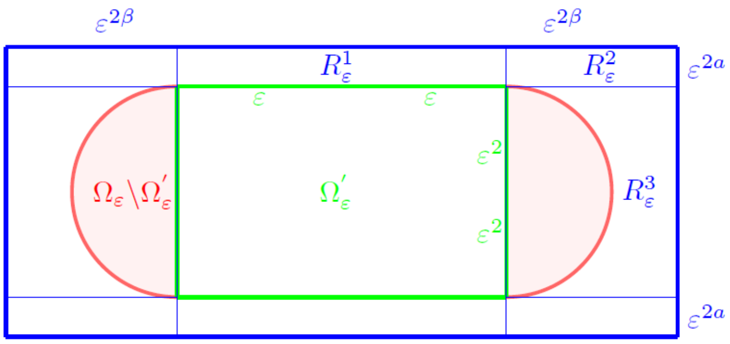

Figure 2. *R_ε is divided into 9 parts. The central object is Ω'_ε.*

and its symmetric counterparts (see Figure 2).

Then we estimate the integrals over these subdomains separately; however, because of symmetry of $\overline{w}_\varepsilon$ it suffices to only estimate the integrals over R^i_ε, $i = 1, 2, 3$.

1. Estimate of $\int_{R^1_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2 \, dx$. Using the fact that $\varphi_\varepsilon = 1$ in $R^1_\varepsilon \subseteq \Omega\backslash\Omega_\varepsilon$ and the definition of $\overline{v}_\varepsilon$ when $x_2 > \varepsilon^2$, i.e., $\overline{v}_\varepsilon = \sum_{h=1}^m \frac{m!(-1)^{h-1}\varepsilon^{2h} x_2^{m-h}}{h!(m-h)!}$, it follows that every term in $\nabla^m \overline{w}_\varepsilon$ containing at least one derivative with respect to x_1 is zero. Therefore, the only nonzero term is given by

$$\frac{\partial^m \overline{w}_\varepsilon}{\partial x_2^m} = \sum_{n=1}^{m-1} \left(\frac{d^n \overline{v}_\varepsilon}{dx_2^n} \frac{d^{m-n}\psi_\varepsilon}{dx_2^{m-n}} \right) + \overline{v}_\varepsilon \frac{d^m \psi_\varepsilon}{dx_2^m}, \tag{4.52}$$

where we have used the fact that $\frac{d^m \overline{v}_\varepsilon}{dx_2^m}\psi_\varepsilon = 0$, since the polynomial $\overline{v}_\varepsilon$ has degree $m-1$. Then, from (4.52), we get

$$\left| \frac{\partial^m \overline{w}_\varepsilon}{\partial x_2^m} \right| \leq C \sum_{n=1}^{m-1} \left| \frac{d^n \overline{v}_\varepsilon}{dx_2^n} \right| \left| \frac{d^{m-n}\psi_\varepsilon}{dx_2^{m-n}} \right| + |\overline{v}_\varepsilon| \left| \frac{d^m \psi_\varepsilon}{dx_2^m} \right|. \tag{4.53}$$

We first observe that, for $n = 0, \dots, m-1$, we get

$$\left| \frac{d^n \overline{v}_\varepsilon}{dx_2^n} \right| \leq C \sum_{h=1}^{m-n} \varepsilon^{2h} \, |x_2|^{m-h-n}. \tag{4.54}$$

In addition, from the fact that $\varepsilon^2 < x_2 < \varepsilon^2 + \varepsilon^{2a}$,

$$|x_2|^{m-h-n} < (\varepsilon^2 + \varepsilon^{2a})^{m-h-n} = \sum_{q=0}^{m-h-n} \varepsilon^{2q} \varepsilon^{2a(m-h-n-q)}.$$

In the last sum, since $a < 1$ for hypothesis (see (4.41)), we take the power with minimum exponent which corresponds to the term with $q = 0$, i.e., $\varepsilon^{2a(m-h-n)}$, for

$n = 0, \dots, m-1$ and $h = 1, \dots, m-n$, since all the other terms contain, at least, powers of ε^2; hence

$$|x_2|^{m-h-n} < (\varepsilon^2 + \varepsilon^{2a})^{m-h-n} \leq C\varepsilon^{2a(m-h-n)}. \tag{4.55}$$

Therefore, inserting (4.55) into (4.54), we find

$$\left| \frac{d^n \overline{v}_\varepsilon}{dx_2^n} \right| \leq C \sum_{h=1}^{m-n} \varepsilon^{2h} \varepsilon^{2a(m-h-n)},$$

where, again, the maximum term corresponds to that one with minimum exponent, i.e., the index $h = 1$; hence

$$\left| \frac{d^n \overline{v}_\varepsilon}{dx_2^n} \right| \leq C\varepsilon^{2+2(m-1-n)a}, \qquad \text{where } 0 \leq n \leq m-1. \tag{4.56}$$

Using this last estimate, (4.56), in (4.53) together with (1d), we find that

$$\begin{aligned} \left| \frac{d^n \overline{v}_\varepsilon}{dx_2^n} \right| \left| \frac{d^{m-n} \psi_\varepsilon}{dx_2^{m-n}} \right| &\leq C\varepsilon^{2-2a}, \\ |\overline{v}_\varepsilon| \left| \frac{d^m \psi_\varepsilon}{dx_2^m} \right| &\leq C\varepsilon^{2-2a}. \end{aligned} \tag{4.57}$$

Therefore, from (4.57) and (4.53), we get

$$\int_{R_\varepsilon^1} |\nabla^m \overline{w}_\varepsilon|^2 \, dx \leq C|R_\varepsilon^1| \, \varepsilon^{4-4a} = C\varepsilon^{5-2a}, \tag{4.58}$$

where we have used the fact that $|R_\varepsilon^1| = 2\varepsilon^{1+2a}$. Observe that, since $a < 1$ for hypothesis, we have that $5 - 2a > 3$; hence (4.58) gives

$$\int_{R_\varepsilon^1} |\nabla^m \overline{w}_\varepsilon|^2 \, dx = o(|\Omega_\varepsilon|). \tag{4.59}$$

2. Estimate of $\int_{R_\varepsilon^2} |\nabla^m \overline{w}_\varepsilon|^2 \, dx$. In this case $\overline{w}_\varepsilon(x_1, x_2) = \overline{v}_\varepsilon(x_2)\psi_\varepsilon(x_2)\varphi_\varepsilon(x_1)$, where $\overline{v}_\varepsilon$ is the polynomial of degree $m-1$ in $x_2 > \varepsilon^2$, i.e., $\overline{v}_\varepsilon = \sum_{h=1}^m \frac{m!(-1)^{h-1}\varepsilon^{2h} x_2^{m-h}}{h!(m-h)!}$. Then

$$|\nabla^m \overline{w}_\varepsilon| \leq C \left| \sum_{n=0}^m \frac{d^n(\overline{v}_\varepsilon \psi_\varepsilon)}{dx_2^n} \, \frac{d^{m-n} \varphi_\varepsilon}{dx_1^{m-n}} \right|. \tag{4.60}$$

In the previous inequality, recalling that $\frac{d^m \overline{v}_\varepsilon}{dx_2^m} = 0$, we split the term related to the maximum order of derivative, i.e., $\frac{d^m(\overline{v}_\varepsilon \psi_\varepsilon)}{dx_2^m}$, from the terms with derivative of lower orders; hence we have that

$$\begin{aligned}|\nabla^m \overline{w}_\varepsilon| &\le C\left|\sum_{n=0}^{m} \frac{d^n(\overline{v}_\varepsilon \psi_\varepsilon)}{dx_2^n} \frac{d^{m-n}\varphi_\varepsilon}{dx_1^{m-n}}\right| \\ &= \left|\varphi_\varepsilon \sum_{r=0}^{m-1}\binom{m}{r}\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\frac{d^{m-r}\psi_\varepsilon}{dx_2^{m-r}} + \sum_{n=0}^{m-1}\sum_{r=0}^{n}\binom{n}{r}\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\frac{d^{n-r}\psi_\varepsilon}{dx_2^{n-r}}\frac{d^{m-n}\varphi_\varepsilon}{dx_1^{m-n}}\right| \\ &\le C\left[|\varphi_\varepsilon|\sum_{r=0}^{m-1}\left|\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\right|\left|\frac{d^{m-r}\psi_\varepsilon}{dx_2^{m-r}}\right| + \sum_{n=0}^{m-1}\sum_{r=0}^{n}\left|\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\right|\left|\frac{d^{n-r}\psi_\varepsilon}{dx_2^{n-r}}\right|\left|\frac{d^{m-n}\varphi_\varepsilon}{dx_1^{m-n}}\right|\right] \\ &\le C\left[\sum_{r=0}^{m-1}\left|\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\right|\left|\frac{d^{m-r}\psi_\varepsilon}{dx_2^{m-r}}\right| + \sum_{n=0}^{m-1}\sum_{r=0}^{n}\left|\frac{d^r\overline{v}_\varepsilon}{dx_2^r}\right|\left|\frac{d^{n-r}\psi_\varepsilon}{dx_2^{n-r}}\right|\left|\frac{d^{m-n}\varphi_\varepsilon}{dx_1^{m-n}}\right|\right] \\ &=: S_1 + S_2.\end{aligned}$$

Using (1d) and (4.56), we find that

$$S_1 \le C\varepsilon^{2-2a}.$$

Analogously, by means of (1d), (2d), and (4.56), we get that

$$S_2 \le C\sum_{n=0}^{m-1}\varepsilon^{2+2a(m-1)-2an-2\beta(m-n)}.$$

Then, using the fact that $|R_\varepsilon^2| = \varepsilon^{2a+2\beta}$, we find

$$\begin{aligned}\int_{R_\varepsilon^2} |\nabla^m \overline{w}_\varepsilon|^2\, dx &\le C\Big(\varepsilon^{4-2a+2\beta} + \sum_{n=0}^{m-1}\varepsilon^{4+4a(m-1)-4an-4\beta(m-n)+2a+2\beta}\Big) \\ &= C\Big(\varepsilon^{4-2a+2\beta} + \sum_{n=0}^{m-1}\varepsilon^{4+2(a-\beta)(2m-2n-1)}\Big).\end{aligned} \tag{4.61}$$

Therefore, from the hypothesis made for a and β (see (4.41) and (4.42)), we find that all the exponents in the previous formula are greater than 3, i.e., $4-2a+2\beta > 3$ and $4+2(a-\beta)(2m-2n-1) > 3$. Indeed, $4-2a+2\beta > 3$ is equivalent to $\beta > -\frac{1}{2}+a$; hence we immediately observe that, in Figure 3, the orange region, where a and β vary, satisfies $\beta > -\frac{1}{2}+a$. On the other hand, condition $4+2(a-\beta)(2m-2n-1) > 3$ is equivalent to $\beta \le \frac{1}{2(2m-2n-1)} + a$ for all $n = 0,\dots,m-1$. We observe that the function $h(n) := \frac{1}{2(2m-2n-1)} + a$ is an increasing function with respect to n and hence the minimum value is $h(0) = \frac{1}{4m-2} + a$; see the red line in Figure 3, which corresponds to the upper bound for β in (4.42). Even in this case, the orange region satisfies the required condition $\beta \le \frac{1}{2(2m-2n-1)} + a$ for all $n = 0,\dots,m-1$.
Therefore, for the choices of a and β in (4.41) and (4.42), we have that

$$\int_{R_\varepsilon^2} |\nabla^m \overline{w}_\varepsilon|^2\, dx = o(|\Omega_\varepsilon|).$$

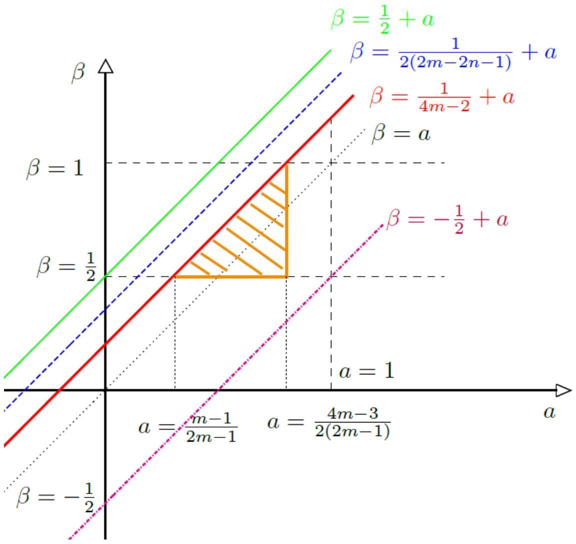

Figure 3. *Parameters a and β (see (4.41) and (4.42)) vary in the orange region. Lines show that, in (4.61), $4-2a-2\beta > 3$ (related to the purple line) and $4+2(a-\beta)(2m-2n-1) > 3$ (related to the blue line) for $n = 0, \dots, m-1$. The green line is the case $n = m-1$, and the red line is the case $n = 0$.*

3. Estimate of $\int_{R^3_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2\, dx$. In the set R^3_ε, function $\psi_\varepsilon(x_2) = 1$ and $\overline{v}_\varepsilon = x_2^m$, and hence $\overline{w}_\varepsilon = x_2^m \varphi_\varepsilon(x_1)$, which implies that

$$|\nabla^m \overline{w}_\varepsilon| \le C \sum_{n=0}^{m} |x_2^n| \left| \frac{d^n \varphi_\varepsilon}{dx_1^n} \right| \le C \sum_{n=0}^{m} \varepsilon^{2n-2n\beta}.$$

We notice that, since for hypothesis $\beta < 1$, $2n(1-\beta) \ge 0$ for all $n = 0, \dots, m$, and hence, using the fact that $|R^3_\varepsilon| = 2\varepsilon^{2+2\beta}$, we get

$$\int_{R^3_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2\, dx \le C |R^3_\varepsilon| \sum_{n=0}^{m} \varepsilon^{4n(1-\beta)} \le C\varepsilon^{2+2\beta},$$

where in the last inequality we have used the fact that $4n(1-\beta) + 2 + 2\beta \ge 2 + 2\beta$ for all $n = 0, \dots, m$. For the choice made for β, i.e., $\beta > \frac{1}{2}$, we have that $2 + 2\beta > 3$, which means that

$$\int_{R^3_\varepsilon} |\nabla^m \overline{w}_\varepsilon|^2\, dx = o(|\Omega_\varepsilon|).$$

■

In the following we show that the maximal eigenvalue of $\mathbb{M}$ (as defined in (4.3)) is $\frac{1}{\kappa}$.

Proposition 4.14. *Under the notational simplification that $\Omega_\varepsilon = \Omega_\varepsilon(0, e_1)$ it follows that*

$$\mathbb{M} E^{m+1} \cdot E^{m+1} = \frac{1}{\kappa}. \tag{4.62}$$

Proof. We first observe that from the fact $\widehat{W} \in H_0^m(\Omega)$ it follows that $\widetilde{W} := -\frac{\kappa-1}{\kappa}\widehat{W} \in H_0^m(\Omega)$. Therefore, from (4.38) it follows that

$$\min_{\widetilde{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2 dx = \min_{\widehat{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\frac{1-\kappa}{\kappa}\nabla^m \widehat{W} - \frac{1-\kappa}{\kappa}\chi_{\Omega_\varepsilon} E\right|^2 dx$$
$$= \frac{(1-\kappa)^2}{\kappa^2} \min_{\widehat{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\nabla^m \widehat{W} - \chi_{\Omega_\varepsilon} E\right|^2 dx.$$

Now, in the previous equation, we choose $E = m!E^{m+1}$, and hence

(4.63)
$$\min_{\widetilde{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} m!E^{m+1}\right|^2 dx \leq \frac{(1-\kappa)^2}{\kappa^2}\int_\Omega \gamma_\varepsilon \left|\nabla^m \overline{w}_\varepsilon - \chi_{\Omega_\varepsilon} m!E^{m+1}\right|^2 dx$$
$$= \frac{(1-\kappa)^2}{\kappa^2}\int_\Omega \gamma_\varepsilon \left|\nabla^m \overline{w}_\varepsilon - \chi_{\Omega_\varepsilon} \nabla^m x_2^m\right|^2 dx.$$

Applying Theorem 4.11, we get

$$\min_{\widetilde{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} m!E^{m+1}\right|^2 dx = o(|\Omega_\varepsilon|) \tag{4.64}$$

as $\varepsilon \to 0$. Finally, substituting (4.64) into (4.38), where we choose $E = m!E^{m+1}$, we get

$$\begin{aligned} &(m!)^2(\kappa-1)\mathbb{M}E^{m+1}\cdot E^{m+1} \\ &= (m!)^2\frac{\kappa-1}{\kappa} + \lim_{\varepsilon\to 0}\left(\frac{1}{|\Omega_\varepsilon|}\min_{\widetilde{W}\in H_0^m(\Omega)} \int_\Omega \gamma_\varepsilon \left|\nabla^m \widetilde{W} + \frac{\kappa-1}{\kappa}\chi_{\Omega_\varepsilon} m!E^{m+1}\right|^2\right) dx, \end{aligned} \tag{4.65}$$

which gives the assertion. ∎

Main result on spectral decomposition of $\mathbb{M}$. From the results of the previous section, we are now ready to prove the following spectral decomposition.

Theorem 4.15. *Under the geometrical simplification that $\Omega_\varepsilon = \Omega_\varepsilon(0, e_1)$, the tensor $\mathbb{M}$ has the following spectral decomposition:*

$$\mathbb{M} = \sum_{n=1}^{m} E^n \otimes E^n + \frac{1}{\kappa}E^{m+1} \otimes E^{m+1}. \tag{4.66}$$

Proof. By Propositions 4.3 and 4.7, the tensor $\mathbb{M}$ is symmetric and positive definite. Hence, its eigenvalues are positive and real, and by (4.7) they lie between 1 and $\frac{1}{\kappa}$. Furthermore, by Proposition 4.9, 1 is an eigenvalue with multiplicity m and corresponding eigenvectors $E^1, \ldots, E^m$. By Proposition 4.14, $\frac{1}{\kappa}$ is an eigenvalue with corresponding eigenvector E^{m+1}. Hence, for all $E \in S^m(\mathbb{R}^2)$ which, using the basis, we can represent as

$$E = \sum_{n=1}^{m+1}(E\cdot E^n)E^n,$$

we find, by applying the tensor $\mathbb{M}$, that

$$\begin{aligned}\mathbb{M}E = \sum_{n=1}^{m+1}(E\cdot E^n)\mathbb{M}E^n &= \sum_{n=1}^{m}(E\cdot E^n)E^n + \frac{1}{\kappa}(E\cdot E^{m+1})E^{m+1}\\ &= \sum_{n=1}^{m}(E^n\otimes E^n)E + \frac{1}{\kappa}(E^{m+1}\otimes E^{m+1})E,\end{aligned} \tag{4.67}$$

which implies (4.66). ■

Remark 4.16. In the general setting of $\Omega_\varepsilon(y,\tau)$, (4.66) reads as follows:

$$\mathbb{M} = \mathbb{M}(\tau) = \sum_{n=1}^{m} E^n\otimes E^n + \frac{1}{\kappa}E^{m+1}\otimes E^{m+1}, \tag{4.68}$$

where

$$\begin{aligned} E^1 &= \underbrace{\tau\otimes\cdots\otimes\tau}_{m-elements},\\ E^h &= \frac{1}{\sqrt{\binom{m}{h-1}}}\sum_{\sigma_m}\underbrace{\tau\otimes\cdots\otimes\tau}_{(m-h+1)-elements}\otimes\underbrace{\tau^\perp\otimes\cdots\otimes\tau^\perp}_{(h-1)-elements} \qquad \text{for}\quad h=2,\dots,m,\\ E^{m+1} &= \underbrace{\tau^\perp\otimes\cdots\otimes\tau^\perp}_{m-elements},\end{aligned} \tag{4.69}$$

where $\tau^\perp$ is the unit normal vector to the line segment $\sigma_\varepsilon(y,\tau)$.

Remark 4.17. Note that the proof to derive the spectral decomposition of the tensor of order $2m$ is more involved than the one in [18] since we have to deal with higher order differential equations with discontinuous coefficients.

Remark 4.18. We want to emphasize that our asymptotic analysis can be generalized in a straightforward way to the case of measurable set Ω_ε which tends to zero when $\varepsilon\to 0$. In this case,

$$\int_\Omega \frac{1}{|\Omega_\varepsilon|}\chi_{\Omega_\varepsilon}\frac{\partial^m u_\varepsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\phi\,dx \to \int_\Omega \mathbb{M}\nabla^m u\,\phi\,d\widetilde{\mu} \qquad \text{as}\quad \varepsilon\to 0,$$

where $d\widetilde{\mu}$ is a Borel measure supported in Ω. In particular, if Ω_ε is a neighborhood of a segment σ of fixed length, then $\widetilde{\mu}=\delta_\sigma$, and the decomposition of the polarization tensor (4.66) still holds.

5. Topological gradient. We are now ready to derive the topological gradient of the functional $\mathcal{J}(\cdot;\cdot)$ as defined in (3.1).

Theorem 5.1. *Let u, u_ε be the solutions of* (3.2) *and* (3.3)*, respectively; then, for $\varepsilon\to 0$, we have*

$$\boxed{\mathcal{J}(u_\varepsilon;v_\varepsilon) = \mathcal{J}(u;v) + 2\varepsilon^3\alpha(\kappa-1)\mathbb{M}\nabla^m u(y)\cdot\nabla^m u(y) + o(\varepsilon^3).} \tag{5.1}$$

Proof. Recalling the definition of the functional $\mathcal{J}(\cdot;\cdot)$ (see (3.1)), we first prove that

$$\mathcal{J}(u_\varepsilon; v_\varepsilon) - \mathcal{J}(u; v) = \frac{\alpha(\kappa-1)}{2} \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m u \, dx. \tag{5.2}$$

It follows from the first order optimality condition of $\mathcal{J}(\cdot;\cdot)$ with respect to the first component, for fixed ε and v, that

$$\int_\Omega (u_\varepsilon - f)\phi \, dx + \alpha \int_\Omega v_\varepsilon \nabla^m u_\varepsilon \cdot \nabla^m \phi \, dx = 0 \quad \text{for all } \phi \in H_0^m(\Omega) \tag{5.3}$$

and

$$\int_\Omega (u - f)\phi \, dx + \alpha \int_\Omega v \nabla^m u \cdot \nabla^m \phi \, dx = 0 \quad \text{for all } \phi \in H_0^m(\Omega). \tag{5.4}$$

Choosing $\phi = u$ in (5.3) and $\phi = u_\varepsilon$ in (5.4) and then subtracting (5.4) from (5.3), we get

$$\int_\Omega (u_\varepsilon - u) f \, dx = \alpha(1-\kappa) \int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m u \, dx. \tag{5.5}$$

On the other hand, inserting $\phi = u_\varepsilon$ into (5.3) and $\phi = u$ into (5.4), we obtain, respectively,

$$\int_\Omega (u_\varepsilon - f) u_\varepsilon \, dx + \alpha \int_\Omega v_\varepsilon \left|\nabla^m u_\varepsilon\right|^2 \, dx = 0 \tag{5.6}$$

and

$$\int_\Omega (u - f) u \, dx + \alpha \int_\Omega v \left|\nabla^m u\right|^2 \, dx = 0. \tag{5.7}$$

Now, from (3.1), we find

$$\begin{aligned}\mathcal{J}(u_\varepsilon; v_\varepsilon) - \mathcal{J}(u; v) = {} & \frac{1}{2}\int_\Omega (u_\varepsilon - f)^2 \, dx + \frac{\alpha}{2}\int_\Omega v_\varepsilon \left|\nabla^m u_\varepsilon\right|^2 \, dx - \frac{1}{2}\int_\Omega (u-f)^2 \, dx \\ & - \frac{\alpha}{2}\int_\Omega v \left|\nabla^m u\right|^2 \, dx;\end{aligned}$$

hence, by (5.6) and (5.7) we have that

$$\mathcal{J}(u_\varepsilon; v_\varepsilon) - \mathcal{J}(u; v) = -\frac{1}{2}\int_\Omega (u_\varepsilon - u) f \, dx,$$

and using (5.5), we get (5.2). Now, we estimate the right-hand side of (5.2), first observing that

$$\int_{\Omega_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m u \, dx = \int_{\Omega'_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m u \, dx + \int_{\Omega_\varepsilon \setminus \Omega'_\varepsilon} \nabla^m u_\varepsilon \cdot \nabla^m u \, dx. \tag{5.8}$$

In the second integral in the right-hand side of the previous formula, we first add and subtract $\nabla^m u$ and then we apply the Schwarz inequality, the regularity estimates (3.8), and the energy estimates (3.9); that is,

$$\begin{aligned}\left|\int_{\Omega_\varepsilon\setminus\Omega'_\varepsilon} \nabla^m u_\varepsilon\cdot\nabla^m u\,dx\right| &\le \|u\|_{L^2(\Omega)}\|u_\varepsilon-u\|_{H^m(\Omega)} + \|u\|^2_{L^2(\Omega)}\\ &\le C(\|u_\varepsilon-u\|_{H^m(\Omega)}\left|\Omega_\varepsilon\setminus\Omega'_\varepsilon\right|^{1/2} + \left|\Omega_\varepsilon\setminus\Omega'_\varepsilon\right|\\ &= o(\varepsilon^3).\end{aligned} \tag{5.9}$$

Therefore, inserting (5.9) into (5.8) and then the resulting equation into (5.2), it follows that

$$\mathcal{J}(u_\varepsilon;v_\varepsilon)-\mathcal{J}(u;v)=\frac{\alpha(\kappa-1)}{2}\int_{\Omega'_\varepsilon}\nabla^m u_\varepsilon\cdot\nabla^m u\,dx+o(\varepsilon^3).$$

Next, choosing a bounded set L_1 such that $\Omega_\varepsilon\subset L_1\subset L_0$, we use the result in Remark 4.2; hence

$$\frac{1}{|\Omega'_\varepsilon|}\int_{\Omega'_\varepsilon}\nabla^m u_\varepsilon\cdot\nabla^m u\,dx\to\mathbb{M}\nabla^m u(y)\cdot\nabla^m u(y)\text{ as }\varepsilon\to 0.$$

Recalling that $|\Omega'_\varepsilon|=4\varepsilon^3$ (see (2.7)), we finally derive

$$\mathcal{J}(u_\varepsilon;v_\varepsilon)-\mathcal{J}(u;v)=2\varepsilon^3\alpha(\kappa-1)\mathbb{M}\nabla^m u(y)\cdot\nabla^m u(y)+o(\varepsilon^3),$$

which concludes the proof. ∎

6. Numerical simulations.

We consider the problem of quantitative photoacoustic tomography(qPAT) with piecewise constant parameters μ and D (absorption and diffusion coefficients, respectively), and a constant Grüneisen parameter Γ (see (1.3) and (1.4)), as outlined in the introduction. Parameters μ and D can be detected from the set of discontinuities of derivatives up to the 2-order of the qPAT measurement data $\mathcal{H}$, which is proportional to $\mathcal{E}$ under the assumption that Γ is constant (see (1.3)). Figure 4 shows a typical example of qPAT data derived from piecewise constant material parameters.

In this section, we extend the topological based algorithms for edge detection in image data (see [13, Algorithms 1 and 2]) by using elliptic differential equations of order $2m$ (see (3.3)), with $f=\mathcal{E}$, and the topological gradient provided in (5.1). Note that in [13, Algorithms 1 and 2] the discontinuities of f have been detected by using the discontinuity of the gradient of the solution of a second order elliptic equation along line segments. We emphasize that according to [33] if the diffusion coefficient D is jumping across an interface but the absorption μ is constant, one observes a jump in the derivative of the gradient of $\mathcal{E}$.

The goal of this section is to show that using the topological gradient derived in (5.1), we are able to detect both the absorption and diffusion coefficients better than in the results obtained in [13] and the results obtained in [14], where a variational method based on an Ambrosio–Tortorelli approximation of a Mumford–Shah-like functional is used. Toward this

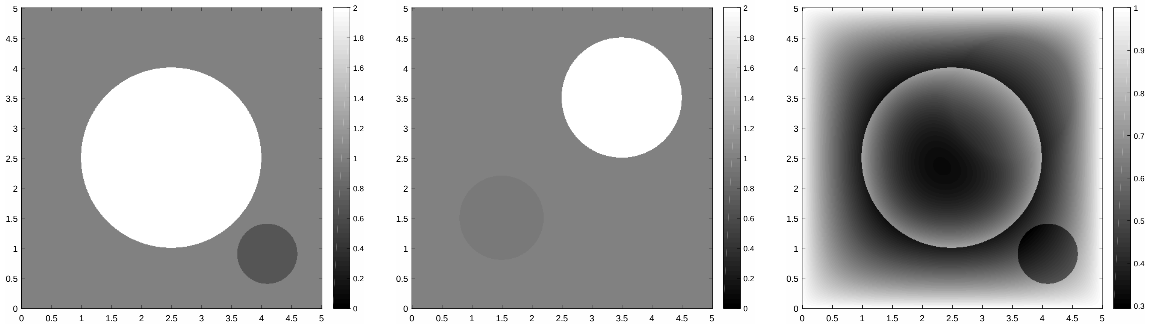

Figure 4. *From the left: The piecewise constant absorption coefficient μ, the piecewise constant diffusion coefficient D, and the simulated qPAT data $f = \mathcal{E}$. The parameters are from* (1.4)*. The test data is analogous to that in* [14].

aim, we use the asymptotic expansion provided in the previous section, specialized only to the case $m = 2, 3$, comparing the results with the case $m = 1$, which was studied in [13], and with the numerical outcomes of [33].

Using the topological asymptotic expansion (5.1) with $v = \kappa\chi_K + 1\chi_{\Omega\setminus K}$, $v_\varepsilon = \kappa\chi_{K\cup\overline{\Omega_\varepsilon}(y,\tau)} + 1\chi_{\Omega\setminus(K\cup\overline{\Omega_\varepsilon}(y,\tau))}$ (see (2.4)), where the according set K is as introduced in Assumption 2.2, we get

$$\mathcal{J}(u_\varepsilon; v_\varepsilon) - \mathcal{J}(u; v) \sim 2\varepsilon^3\alpha(\kappa - 1)\mathbb{M}\nabla^m u(y) \cdot \nabla^m u(y).$$

To develop a stable algorithm, we follow the approach proposed in [13] for the case $m = 1$. We recall here the main idea: For every $v \in L^2(\Omega)$, we define

$$m_\varepsilon(v) := \inf\left\{|S| : S \subset \mathbb{R}^2 \times \mathbb{S}^1,\ v = v_K \ \text{ with } K = \bigcup_{(y,\tau)\in S} \Omega_\varepsilon(y,\tau)\right\},$$

where v_K is the value of v in K, and we set $m_\varepsilon(v) := +\infty$ if $v \neq v_K$ for every finite subset $S \subset \mathbb{R}^2 \times \mathbb{S}^1$ with $K = \bigcup_{(y,\tau)\in S} \Omega_\varepsilon(y,\tau)$. The idea behind the algorithm is to introduce a slight modification of the functional $\mathcal{J}$ defined in (3.1) in order to take into account a constraint on the perimeter of K, i.e.,

$$\mathcal{J}_\varepsilon(u, v) := \frac{1}{2}\int_\Omega (u - f)^2\, dx + \frac{\alpha}{2}\int_\Omega v|\nabla^m u|^2\, dx + 2\beta\varepsilon m_\varepsilon(v),$$

where β is a positive parameter, for all $u \in H_0^m(\Omega)$ and $v \in L^\infty(\Omega)$. It is shown in [13] that for general $\Omega_\varepsilon(y,\tau) \cap K = \emptyset$, it yields

$$\mathcal{J}_\varepsilon(u, v_\varepsilon) - \mathcal{J}_\varepsilon(\hat{u}, v) = \mathcal{J}(u, v_\varepsilon) - \mathcal{J}(\hat{u}, v) + 2\beta\varepsilon,$$

where $\mathcal{J}$ is exactly the functional defined in (3.1) for all $u, \hat{u} \in H_0^m(\Omega)$. Therefore, by (5.1), we have

$$\mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon) - \mathcal{J}_\varepsilon(u, v) = \mathcal{J}(u_\varepsilon; v_\varepsilon) - \mathcal{J}(u; v) + 2\beta\varepsilon \sim 2\varepsilon^3\alpha(\kappa - 1)\mathbb{M}\nabla^m u(y)\cdot\nabla^m u(y) + 2\beta\varepsilon. \tag{6.1}$$

We observe that, from Proposition 4.7, we have that

$$\mathbb{M}(\tau)\nabla^m u(y) \cdot \nabla^m u(y) \leq \frac{|\nabla^m u(y)|^2}{\kappa}. \tag{6.2}$$

Therefore, substituting $\mathbb{M}(\tau)\nabla^m u(y) \cdot \nabla^m u(y) \sim \frac{|\nabla^m u(y)|^2}{\kappa}$ into (6.1), which implies that the direction of the line segment has to be chosen parallel to the eigenvector associated to the eigenvalue of $\nabla^m u(y)$ with maximum absolute value, we find

$$\mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon) - \mathcal{J}_\varepsilon(u, v) \sim -2\varepsilon^3\alpha\frac{(1-\kappa)}{\kappa}|\nabla^m u(y)|^2 + 2\beta\varepsilon;$$

hence we expect a decrease of the functional $\mathcal{J}_\varepsilon$ in case

$$|\nabla^m u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}. \tag{6.3}$$

In the particular case $m = 2$, we can be more precise, finding the maximum value of $\mathbb{M}(\tau)\nabla^m u(y) \cdot \nabla^m u(y)$ and the direction of τ where it occurs. In fact, we are able to provide the explicit expression of the tensor $\mathbb{M}$, which is optimal with respect to $\nabla^2 u(y)$, i.e., which maximizes the form $\mathbb{M}\nabla^2 u(y) \cdot \nabla^2 u(y)$.

Lemma 6.1 ($m = 2$). *Let $\lambda_1 := \lambda_1(y)$ and $\lambda_2 := \lambda_2(y)$ denote the eigenvalues of $\nabla^2 u(y)$, and the according eigenvectors are denoted by $\hat{\tau} := \hat{\tau}(y)$ and $\hat{\tau}^\perp := \hat{\tau}^\perp(y)$, respectively. Moreover, we assume that $|\lambda_1| \leq |\lambda_2|$. Then, for $\mathbb{M}$ as defined in (4.68), with $m = 2$, we have*

$$\mathbb{M}(\hat{\tau})\nabla^2 u(y) \cdot \nabla^2 u(y) = \lambda_1^2 + \frac{\lambda_2^2}{\kappa} = \max_{\tau \in \mathbb{S}^1} \mathbb{M}(\tau)\nabla^2 u(y) \cdot \nabla^2 u(y).$$

Proof. First, we note that, due to the symmetry of $\nabla^2 u(y)$, we can consider an orthogonal decomposition of the matrix $\nabla^2 u(y)$, i.e.,

$$\nabla^2 u(y) = U \Lambda U^T, \tag{6.4}$$

where U is an orthogonal matrix, and the matrices are given by

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad U = \begin{bmatrix} \hat{\tau} & \hat{\tau}^\perp \end{bmatrix} = \begin{bmatrix} \hat{\tau}_1 & -\hat{\tau}_2 \\ \hat{\tau}_2 & \hat{\tau}_1 \end{bmatrix}. \tag{6.5}$$

Then, from (4.68) (cf. (4.67)), (6.4), and (6.5), we get

$$\begin{aligned} \mathbb{M}(\tau)\nabla^2 u(y) \cdot \nabla^2 u(y) &= \left[(U\ \Lambda\ U^T) \cdot (\tau \otimes \tau)\right]^2 \\ &\quad + \frac{1}{2}\left[(U\ \Lambda\ U^T) \cdot (\tau \otimes \tau^\perp + \tau^\perp \otimes \tau)\right]^2 + \frac{1}{\kappa}\left[(U\ \Lambda\ U^T) \cdot (\tau^\perp \otimes \tau^\perp)\right]^2 \\ &= \left[(\tau^T U)\ \Lambda\ (\tau^T U)^T\right]^2 + \frac{1}{\kappa}\left[((\tau^\perp)^T U)\ \Lambda\ ((\tau^\perp)^T U)^T\right]^2 \\ &\quad + \frac{1}{2}\left[(\tau^T U)\ \Lambda\ ((\tau^\perp)^T U)^T + ((\tau^\perp)^T U)\ \Lambda\ (\tau^T U)^T\right]^2. \end{aligned} \tag{6.6}$$

Now, we denote $\chi = \tau^T U$ and $\nu = (\tau^\perp)^T U$, and then we have $\nu = \chi^\perp$ because U is orthogonal; in fact,

$$\chi \cdot \nu = \nu^T \chi = U^T \tau^\perp \tau^T U = 0.$$

Because $\|\nu\| = \|\chi\| = 1$, it follows from (6.6) and the explicit expression of Λ (see (6.5)) that

$$\begin{aligned} \mathbb{M}(\tau)\nabla^2 u(y) \cdot \nabla^2 u(y) &= \left[\chi\ \Lambda\ \chi^T\right]^2 + \frac{1}{2}\left[\chi\ \Lambda\ (\chi^\perp)^T + \chi^\perp\ \Lambda\ \chi^T\right]^2 + \frac{1}{\kappa}\left[\chi^\perp\ \Lambda\ (\chi^\perp)^T\right]^2 \\ &= (\lambda_1\chi_1^2 + \lambda_2\chi_2^2)^2 + \frac{1}{\kappa}(\lambda_1\chi_2^2 + \lambda_2\chi_1^2)^2 + 2(\lambda_2 - \lambda_1)^2\chi_1^2\chi_2^2 \\ &= \lambda_1^2\chi_1^4 + \lambda_2^2\chi_2^4 + 2\lambda_1\lambda_2\chi_1^2\chi_2^2 \\ &\quad + \frac{1}{\kappa}(\lambda_1^2\chi_2^4 + \lambda_2^2\chi_1^4 + 2\lambda_1\lambda_2\chi_1^2\chi_2^2) + 2(\lambda_2 - \lambda_1)^2\chi_1^2\chi_2^2 =: \mathcal{T}. \end{aligned} \tag{6.7}$$

Defining $\rho := \lambda_1^2 + \frac{1}{\kappa}\lambda_2^2$ and $\bar\rho := \lambda_2^2 + \frac{1}{\kappa}\lambda_1^2$, we see that the last term on the right-hand side of (6.7) equals

$$\mathcal{T} = \left(\rho(\chi_1^4 + 2\chi_1^2\chi_2^2 + \chi_2^4) - 2\chi_1^2\chi_2^2\rho - \chi_2^4\rho\right) + \chi_2^4\bar\rho + 2\chi_1^2\chi_2^2\lambda_1\lambda_2\left(1 + \frac{1}{\kappa}\right) + 2(\lambda_2 - \lambda_1)^2\chi_1^2\chi_2^2.$$

Now, we take into account that χ is a vector of norm 1, and thus

$$\begin{aligned} \chi_1^4 + 2\chi_1^2\chi_2^2 + \chi_2^4 &= 1, \\ (-\rho + \bar\rho) &= (\lambda_2^2 - \lambda_1^2)\left(1 - \frac{1}{\kappa}\right), \\ \lambda_1\lambda_2\left(1 + \frac{1}{\kappa}\right) + (\lambda_2 - \lambda_1)^2 - \rho &= (\lambda_1\lambda_2 - \lambda_2^2)\left(\frac{1}{\kappa} - 1\right), \end{aligned}$$

and thus

$$\mathcal{T} = \rho + \chi_2^4(\lambda_2^2 - \lambda_1^2)\left(1 - \frac{1}{\kappa}\right) - 2\chi_1^2\chi_2^2(\lambda_1\lambda_2 - \lambda_2^2)\left(1 - \frac{1}{\kappa}\right).$$

By the assumptions on the eigenvalues, i.e., $|\lambda_2| > |\lambda_1|$, the sum of the last two terms is always negative; hence the maximum value of $\mathcal{T}$ is given by ρ. Then, we note that this maximum occurs when $\chi_2 = 0$; that is, in terms of τ this means that

$$0 = \chi_2 = (\tau^T U)_2 = -\tau_1\hat\tau_2 + \tau_2\hat\tau_1,$$

which, equivalently, means that $\tau \perp \hat\tau^\perp$, or in other words $\tau = \pm\hat\tau$. ∎

Remark 6.2. Since $\kappa - 1 \le 0$, as a consequence of Lemma 6.1, we get that

$$\mathbb{M}(\tau)\nabla^2 u(y) \cdot \nabla^2 u(y) \le \mathbb{M}(\hat\tau)\nabla^2 u(y) \cdot \nabla^2 u(y) = \lambda_1^2(y) + \frac{\lambda_2^2(y)}{\kappa} \quad \forall \tau \in \mathbb{S}^1.$$

Therefore, by considering the approximation

$$\mathbb{M}(\tau)\nabla^2 u(y)\cdot\nabla^2 u(y) \sim \lambda_1^2(y) + \frac{\lambda_2^2(y)}{\kappa},$$

(6.1) becomes

$$\mathcal{J}_\varepsilon(u_\varepsilon, v_\varepsilon) - \mathcal{J}_\varepsilon(u,v) \sim -2\varepsilon^3\alpha(1-\kappa)\left(\lambda_1^2(y) + \frac{\lambda_2^2(y)}{\kappa}\right) + 2\beta\varepsilon. \tag{6.8}$$

Therefore, we expect a decrease of the functional $\mathcal{J}_\varepsilon$ if

$$\left(\lambda_1^2(y) + \frac{\lambda_2^2(y)}{\kappa}\right) \geq \frac{\beta}{\alpha\varepsilon^2(1-\kappa)}. \tag{6.9}$$

Remark 6.3. Getting a similar result of Lemma 6.1 for the case $m \geq 3$ is more involved, due to the fact that a decomposition of $\nabla^m u(y)$ in terms of its eigenvalues is not known a priori; see [19, section 8.2]. In fact, in the real field, the number of eigenvalues of an m-order tensor could be different from the dimensional space (in our case 2).

Using the above facts, we implement, in the cases $m = 2$ and $m = 3$, a two-step algorithm for detecting line segments; this algorithm first detects the edge position and second determines its direction according to the following rules:

1. We detect only significant edges by selecting the point $y \in L$ satisfying the stabilizing criterion (6.3).
2. At this point, we create a line segment in the same direction of the eigenvector associated to the eigenvalue of $\nabla^m u(y)$ of maximum absolute value.

Before presenting the numerical results, we make some other remarks on this algorithm.

Remark 6.4. In the case $m = 3$, to identify the eigenvalue of greatest absolute value of $\nabla^3 u(y)$ and, in particular, its corresponding eigenvector, we utilize the results in [37, Theorem 7.3] regarding the L-eigenvectors of a third order tensor. We recall here the main step: given $\nabla^3 u(y)$ we define the *kernel tensor* $U_{ij} = \sum_{k,h=1}^2 (\nabla^3 u(y))_{ikh}(\nabla^3 u(y))_{khj}$. Then, the eigenvector associated to the greatest eigenvalue of $\nabla^3 u(y)$ is equal to the eigenvector associated to the greatest eigenvalue of maximum module of the kernel matrix U; see [37] for more details.

Remark 6.5. For the case $m = 2$, we will also show the results given by steps 1 and 2, where we replace the stabilizing criterion with (6.9) and choose as the direction of the line segment the one given in Lemma 6.1 (see Algorithm 6.2 below).

We are now in position to develop and show Algorithms, 6.1, 6.2, and 6.3, generalizing those contained in [13]. These algorithms will be applied to the qPAT test data from [33], represented in Figure 4. Since we need to numerically solve higher order elliptic equations with finite element methods, the qPAT image is down-sampled to (130×130) in order to save computational time. The numerical results can be applied to more general problems which aim to detect discontinuities in an image f through the discontinuity of higher order derivatives of a smoothed version of f.

Algorithm 6.1. Our first algorithm computes a smoothed version of the input image f, where the smoothed image is the solution of a $2m$-order elliptic equations with $v = 1$; see (3.2). By only using this regularization function, we identify a sequence of thin stripes $K^{(j)}$, where $K^{(j+1)}$ is formed by including with $K^{(j)}$ a thin stripe $\Omega_\varepsilon(y^{(j)}, \tau^{(j)})$ in the position $y^{(j)}$, for which $|\nabla^m u(y)|^2$ is maximal, and along the direction $\tau^{(j)}$ until $|\nabla^m u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$. We note that the direction $\tau^{(j)}$ coincides with that of the eigenvector associated to the greatest absolute eigenvalue, when $m = 2$ or $m = 3$, and is chosen orthogonal to the gradient when $m = 1$ (see [13] for this last case). See the scheme in Algorithm 6.1.

Algorithm 6.1 Implementation without updates of v.

Data: input data f (for instance, qPAT data $\mathcal{E}$), $m = 1$ or $m = 2$ or $m = 3$, parameters $0 < \kappa \ll 1$, α, $\beta > 0$, ε, $\delta_0 > 0$, $\varrho_0 > 0$;
Result: the set of line segments S and the set of thin stripes K;
Initialization: set $S = \emptyset$, $K = \emptyset$, and $L = \Omega \setminus (\partial\Omega \oplus \mathcal{B}_{\delta_0}(0))$
compute the solution u of

$$\begin{cases} u + \alpha(-1)^m(\nabla\cdot)^m(\nabla^m u) = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases}$$

with a finite element method.
while $\max_{y\in L} |\nabla^m u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$ **do**
  find $y \in L$ such that $|\nabla^m u(y)|^2$ is maximal;
  compute the line segment $\Sigma_\varepsilon(y, \tau)$ and the thin stripe $\Omega_\varepsilon(y, \tau)$;
  set $S \leftarrow S \cup \Sigma_\varepsilon(y, \tau)$ and $K \leftarrow K \cup \Omega_\varepsilon(y, \tau)$;
  set $L \leftarrow L \setminus (\Omega_\varepsilon(y, \tau) \oplus \mathcal{B}_{\varrho_0}(0))$;
end while

Remark 6.6. When $m = 2$, we use Hsieh–Clough–Tocher C^1 finite elements to solve the fourth order differential equations. We call $\mathcal{T}_h^\Delta$ the submesh of $\mathcal{T}$ where all the triangles are split into three subtriangles at their barycenter; then

$$HCT_h = \{\eta \in C^1(\Omega) : \quad \forall T \in \mathcal{T}_h^\Delta \quad \eta_{|_T} \in P^3\},$$

where P^3 is the set of polynomials of $\mathbb{R}^2$ of degree less than or equal to 3. The degrees of freedom are the value and derivatives at vertices and the normal derivative at the middle edge point of initial meshes; see [35, 16] and references therein.

In the case $m = 3$, we use a splitting method in order to solve the corresponding sixth order equation with a finite element method [16]. In particular, we solve the system given by $v := \Delta u$ and $u - \alpha\Delta^2 v = f$. From the viewpoint of the implementation, this means that we need only to properly modify the code obtained for the fourth order equation. Certainly, the numerical results for this case can be improved using more sophisticated methods which are able to directly solve the sixth order equation with the proper boundary conditions. In fact, in our implementation only two of the three prescribed boundary conditions are satisfied, namely $u = 0$ and $\frac{\partial u}{\partial n} = 0$.

Algorithm 6.2. In this algorithm we focus our attention only on the case $m = 2$, following the same ideas contained in Algorithm 6.1, but we use, as stopping criterion and as identifier of the points where to insert a line segment, the relations in Remark 6.2. See Algorithm 6.2 for all the details.

Algorithm 6.2 Implementation without updates of v and using eigenvalues of the matrix of second order derivatives.

Data: input data f (for instance, qPAT data $\mathcal{E}$), parameters $0 < \kappa \ll 1$, α, $\beta > 0$, ε, $\delta_0 > 0$, $\varrho_0 > 0$;
Result: the set of line segments S and the set of thin stripes K;
Initialization: set $S = \emptyset$, $K = \emptyset$, and $L = \Omega \setminus (\partial\Omega \oplus \mathcal{B}_{\delta_0}(0))$
compute the solution u of

$$\begin{cases} u + \alpha(\nabla\cdot)^2(\nabla^2 u) = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

with a finite element method.
while $\max_{y\in L}(\lambda_1^2(y) + \frac{\lambda_2^2(y)}{k}) \geq \frac{\beta}{\alpha\varepsilon^2(1-\kappa)}$ **do**
  find $y \in L$ such that $\lambda_1^2(y) + \frac{\lambda_2^2(y)}{k}$ is maximal;
  compute the line segment $\Sigma_\varepsilon(y, \tau)$ and the thin stripe $\Omega_\varepsilon(y, \tau)$;
  set $S \leftarrow S \cup \Sigma_\varepsilon(y, \tau)$ and $K \leftarrow K \cup \Omega_\varepsilon(y, \tau)$;
  set $L \leftarrow L \setminus (\Omega_\varepsilon(y, \tau) \oplus \mathcal{B}_{\varrho_0}(0))$;
end while

Algorithm 6.3. We combine updates of the piecewise constant v with updates of the function u. We start with $v \equiv 1$. After adding a fixed number s of thin stripes $\Omega_\varepsilon(y, \tau)$ to the set K, using the same scheme as in Algorithm 6.1, we update the piecewise constant function v by setting

$$v = \kappa\chi_K + 1\chi_{\Omega\setminus K},$$

and then we compute a corresponding function u, solution of

$$\begin{cases} u + \alpha(-1)^m(\nabla\cdot)^m(v\nabla^m u) = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.10}$$

where $m = 1$ or $m = 2$, with a finite element method, which is then used for computing $|\nabla^m u(y)|^2$ and selecting the next at most s thin stripes $\Omega_\varepsilon(y, \tau)$ for including in the set K. The process of alternating between the addition of stripes and updates of the smoothed function u is repeated until no more admissible points $y \in L$ exist, i.e., when the inequality $|\nabla^m u(y)|^2 < \frac{\beta\kappa}{\varepsilon^2\alpha(1-\kappa)}$ holds.

Due to the complexity in solving a sixth order equation with discontinuous coefficients, we do not implement here Algorithm 6.3 when $m = 3$.

The reason behind the update of v lies in the fact that the asymptotic expansion derived above becomes increasingly inaccurate as the number of the stripes becomes larger. This means that at some point one has to update v in order to get better reconstructions. The main drawback of this method lies in the fact that we cannot update v at every step because this would involve solving an elliptic equation, which is a lengthy and costly procedure. Thus we choose the number s sufficiently large in such a way that approximately less than 8 computations of the $2m$-order elliptic equation are needed.

Algorithm 6.3 Implementation with updates of v.

Data: input data f, $m = 1$ or $m = 2$, parameters $0 < \kappa \ll 1$, α, $\beta > 0$, ε, $\delta_0 > 0$, $\varrho_0 > 0$, $s \in \mathbb{N}$;
Result: the set of line segments S and the set of thin stripes K;
Initialization: set $S = \emptyset$, $K = \emptyset$, and $L = \Omega \setminus (\partial\Omega \oplus \mathcal{B}_{\delta_0}(0))$;
compute the solution u of

$$\begin{cases} u + \alpha(-1)^m (\nabla\cdot)^m (\nabla^m u) &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with a finite element method.
while $\max_{y \in L} |\nabla^m u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$ **do**
  set $k = 1$;
  while $k < s$ and $|\nabla^m u(y)|^2 \geq \frac{\beta\kappa}{\alpha\varepsilon^2(1-\kappa)}$ **do**
    find $y \in L$ such that $|\nabla^m u(y)|^2$ is maximal;
    compute the line segment $\Sigma_\varepsilon(y,\tau)$ and the thin stripe $\Omega_\varepsilon(y,\tau)$;
    set $S \leftarrow S \cup \Sigma_\varepsilon(y,\tau)$ and $K \leftarrow K \cup \Omega_\varepsilon(y,\tau)$;
    set $L \leftarrow L \setminus (\Omega_\varepsilon(y,\tau) \oplus \mathcal{B}_{\varrho_0}(0))$; set $k \leftarrow k+1$;
  end while
  set $v = \kappa\chi_K + 1\chi_{\Omega\setminus K}$;
  compute the solution u of

$$\begin{cases} u + \alpha(-1)^m (\nabla\cdot)^m (v\nabla^m u) &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

  with a finite element method.
end while

All of the algorithms described above have been implemented in MATLAB.

Results of numerical experiments. To test the proposed algorithms we have performed five experiments; see Tests 1, 2, 3, 4, and 5 below. In Tests 1, 2, and 3, the source term f is given by the simulated qPAT data presented in Figure 4, down-sampled to an image of size 130×130 in order to save computational time. These tests are the results of the application of Algorithms 6.1, 6.2, and 6.3, respectively.

Tests 4 and 5 are performed with the same qPAT data but corrupted by a small amount of noise. Specifically, in Tests 4 and 5 we add to the image a Gaussian noise with standard deviation of 0.1% and 2%, respectively, of the average signal of qPAT data. Test 4 serves for a direct comparison with the numerical outcomes in [14].

For all the experiments, the parameter ε, corresponding to the length of a line segment and the thickness of a stripe, is set to be $\varepsilon = h$, where h is the pixel size. The size of the stripe's neighborhood, i.e., ϱ_0, is equal to $\varrho_0 = h$. Moreover, in all tests, we set $\delta_0 = 12h$; i.e., starting from an image of size 130×130, the restricted set L of Ω, defined in Algorithms 6.1, 6.2, and 6.3, has dimensions 106×106, which is indeed the size of the images in Tests 1–5.

The parameters used in the five tests are summarized in Table 1.

Table 1

Parameters used in Tests 1, 2, 3, 4, and 5. In the first column, we show the number of the test. In the second column, we indicate whether there is noise in the qPAT data, i.e., whether f is corrupted by Gaussian noise. In the third and fourth columns, we specify the algorithm we implement and the order of the derivatives we consider, respectively. The last three columns give the principal parameters which are used to implement the various algorithms.

Test	Noise	Algorithm	m (order)	α	κ	β
Test 1	no	1	1	10^{-1}	10^{-2}	0.0072
			2			$1.1029 \cdot 10^{-4}$
			3			$6.4453 \cdot 10^{-5}$
Test 2	no	2	2			9.5803×10^{-5}
Test 3	no	3	1			0.0072
			2			$1.1029 \cdot 10^{-4}$
Test 4	yes (0.1%)	1	2	1		$9.288 \cdot 10^{-4}$
Test 5	yes (2%)		1			0.0724
			2			0.0020
			3			0.0015

Test 1. We apply Algorithm 6.1. The numerical results are given in Figure 5. In the first column we have $|\nabla^m u|$, where $m = 1, 2, 3$, respectively. On the right column, we give the line segments given by the application of Algorithm 6.1. Despite the coefficients v remaining constant in all the iterations, the numerical outcomes of higher order elliptic equations (see the cases $m = 2$ and $m = 3$) lead us to identify both the absorption (μ) and the diffusion (D) coefficients. This confirms analytical results in [33, 14] which state that the union of jumps in coefficients μ and D is contained in derivatives of f up to the second order.

Test 2. We apply Algorithm 6.2. The numerical results are given in Figure 6. We observe that the numerical outcomes, by using the results in Remark 6.2, are perceptibly better than the case with the stopping rule $|\nabla^m u(y)| < \frac{\beta\kappa}{\varepsilon\alpha(1-\kappa)}$.

Test 3. We apply Algorithm 6.3. The numerical results are given in Figure 7. In this case, we set the parameter s to be $s = 40$ for $m = 1$ and $s = 50$ for $m = 2$. Comparing the numerical outcomes of this test with those of Test 1, we can appreciate how the updates of the coefficient v lead to better reconstructions.

Test 4. We apply Algorithm 6.1, with $m = 2$ to an image which is corrupted by a small Gaussian noise with standard deviation of 0.1% of the average signal value of the image. The numerical results are given in Figure 8. Due to the presence of noise, we use a greater value of α with respect to the previous case, which is set to be $\alpha = 1$. The numerical results are more stable than those in [14, Figure 3]; in fact, we are able to detect all four circles in the image.

Test 5. We apply Algorithm 6.1 to an image which is corrupted by a Gaussian noise with standard deviation of 2% of the average signal value of the image. The numerical results are given in Figure 9. Comparing our results with those in [14, Figure 3], we observe that they are rather good. We emphasize that, for the case $m = 3$, we expect better results in the case where more sophisticated finite element methods to find the solution of the sixth order equation are applied. Certainly, the splitting method, used in this paper, introduces some errors in the reconstructions because, numerically, only two of the three prescribed boundary conditions are satisfied, namely $u = 0$ and $\frac{\partial u}{\partial n} = 0$.

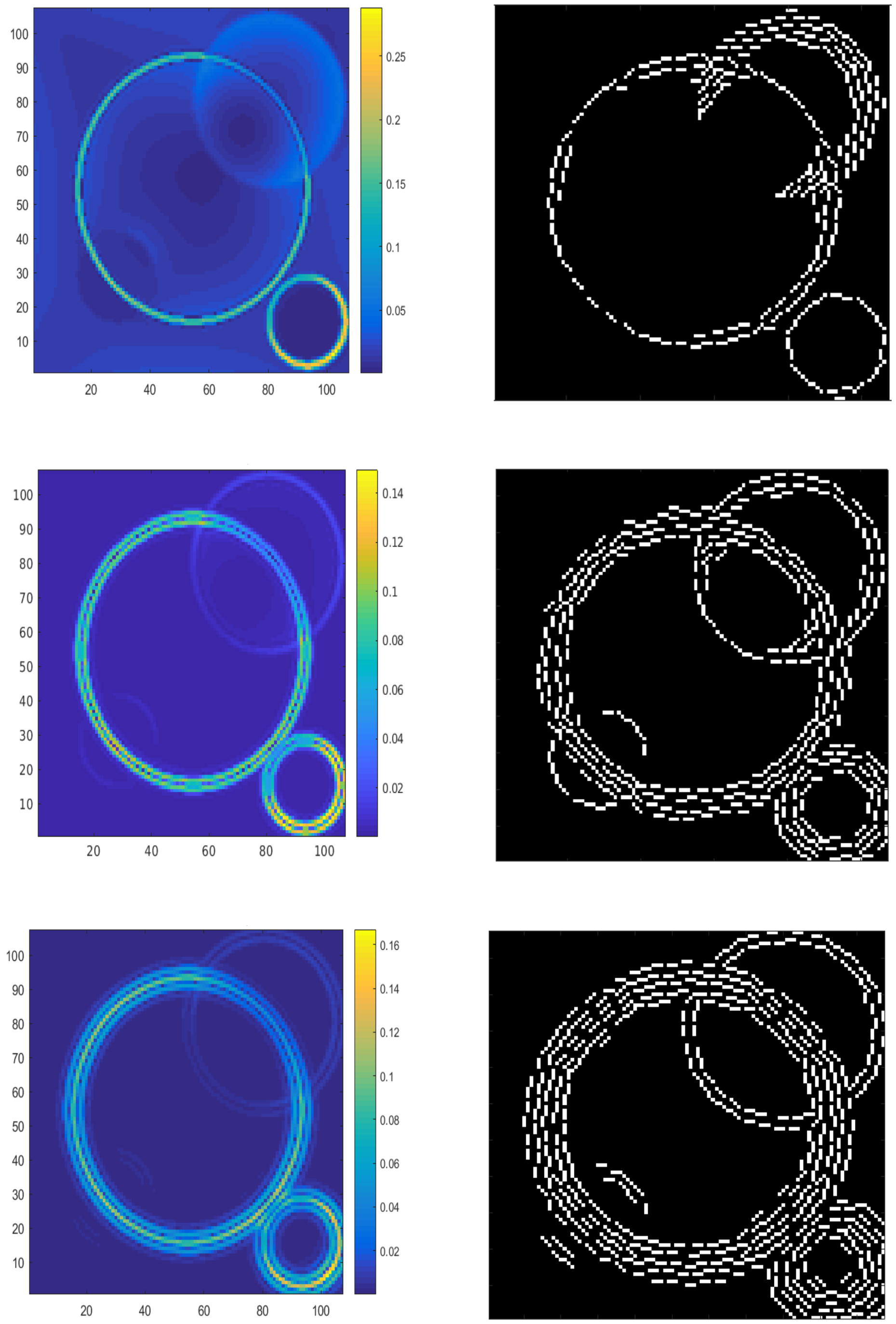

Figure 5. *Test* 1. *Implementation of Algorithm* 6.1. *In the first column we represent* $|\nabla^m u|$, *where* $m = 1, 2, 3$. *The second column gives the numerical outcomes of constructed line segments. The results are related to* $m = 1$ *in the first row,* $m = 2$ *in the second row, and* $m = 3$ *in the third row. We emphasize that with* $m = 1$ *the small circle at the left bottom part of the image is not identified. Instead, it appears in the reconstruction with* $m = 2$ *and* $m = 3$.

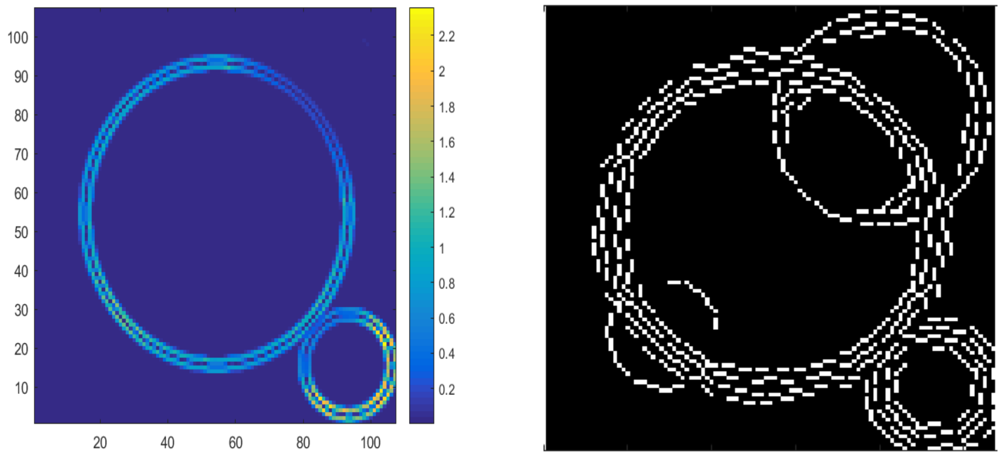

Figure 6. *Test* 2*. Implementation of Algorithm* 6.2*; i.e., we are using the criteria in Remark* 6.2 *to draw the line segments.*

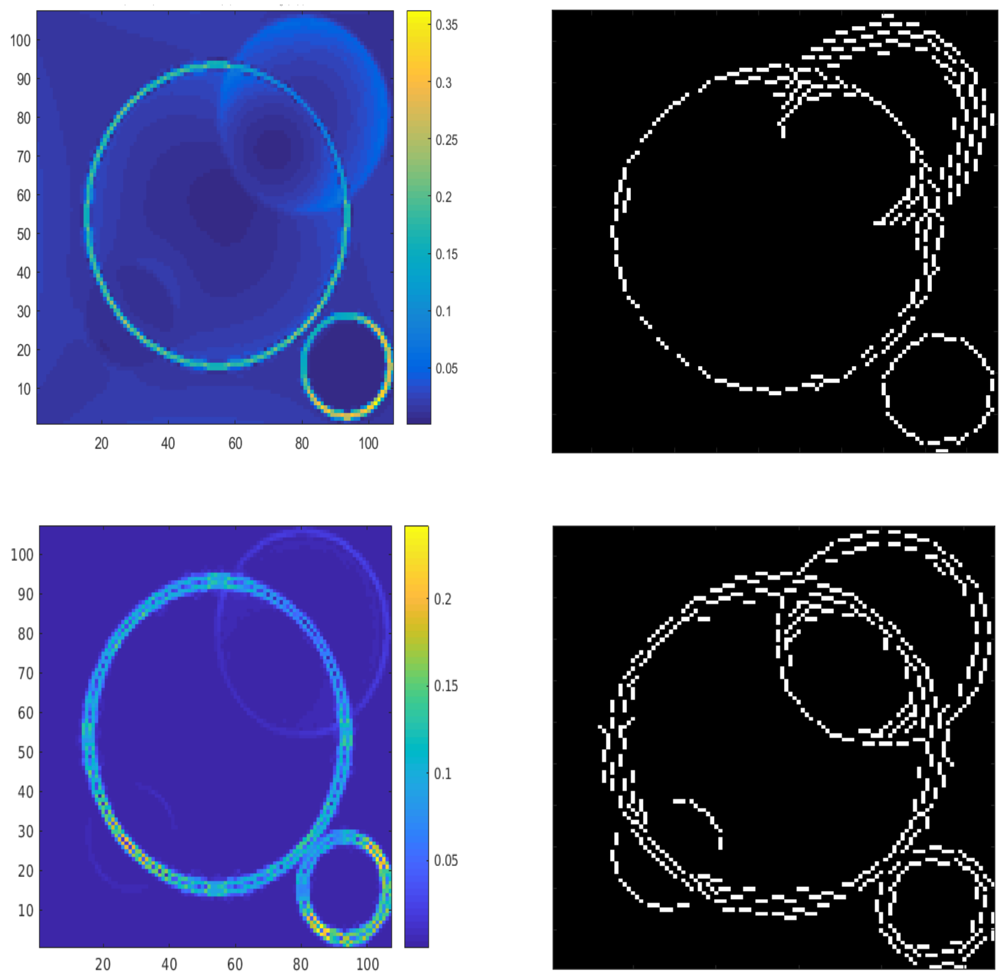

Figure 7. *Test* 3. *Implementation of Algorithm 6.3. In the first row, we have the results given by the second order elliptic equation; i.e., we choose* $m = 1$ *in the algorithm. In the second row, we provide the numerical outcomes of the fourth order elliptic equation; i.e., we choose* $m = 2$ *in the algorithm. For* $m = 1$*, we set* $s = 40$*. For* $m = 2$*, we use* $s = 50$*.*

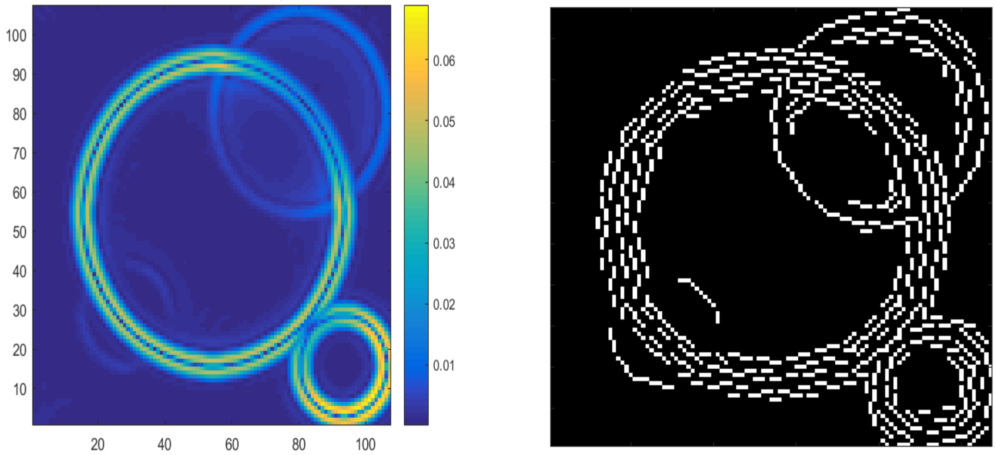

Figure 8. *Test* 4. *Implementation of Algorithm* 6.1 *in the case where the qPAT data are corrupted by a Gaussian noise of* 0.1%.

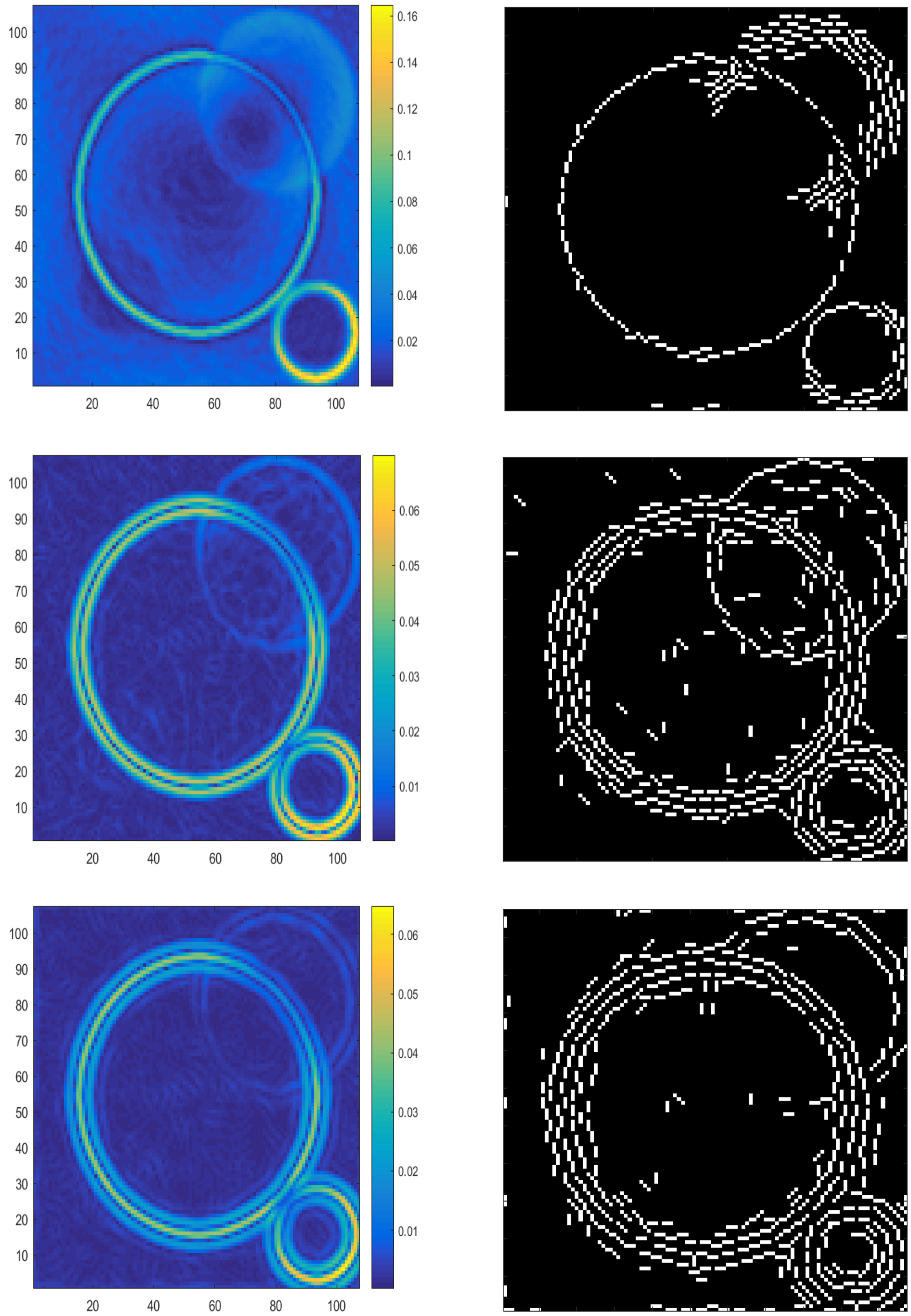

Figure 9. *Test* 5. *Implementation of Algorithm* 6.1 *in the case where the qPAT image is corrupted by a Gaussian noise of* 2%. *We provide the results for* $m = 1$ *in the first row, for* $m = 2$ *in the second row, and, finally, for* $m = 3$ *in the third row.*

Conclusion. In this paper we applied the method of asymptotic expansion of line segments for detection of *discontinuities in derivatives* of some data f. Previously, asymptotic expansions have been considered for detecting discontinuities in data (and not the derivatives). For numerical tests, we considered quantitative photoacoustic tomography with piecewise constant material parameters, which has also been considered before in [14]. In [14], the algorithm for the detection on discontinuities is based on a differential Canny edge detector. This method requires a presmoothing step, which is dependent on the order of discontinuity to be recovered and is very sensitive to the presence of noise. Conceptually our new approach includes filtering in the detection algorithms, delivers a tangential direction of the edge, and is more stable with respect to the presence of noise.

Appendix A. Proof of Lemma 3.1.

To prove the well-posedness of (3.2) and (3.3), we need the following generalized version of the Poincaré-type inequality.

Lemma A.1 (m-order Poincaré inequality [1]). *The two norms $\|\cdot\|_{H^m(\Omega)}$ and $\|\nabla^m\cdot\|_{L^2(\Omega)}$ are equivalent in $H_0^m(\Omega)$; i.e., there exists a positive constant C such that*

$$\|\nabla^m u\|_{L^2(\Omega)} \leq \|u\|_{H^m(\Omega)} \leq C\,\|\nabla^m u\|_{L^2(\Omega)} \quad \textit{for all } u \in H_0^m(\Omega). \tag{A.1}$$

Proof. We only concentrate on (3.2) because the argument of the proof is identical for (3.3).

First, we find the weak formulation of (3.2), and we study its well-posedness by applying the Lax–Milgram theorem. Second, we show that the weak formulation of (3.2) is in fact the optimality condition satisfied by the minimum of the functional (3.1) (where $\zeta = v$), and we prove the equivalence between the minimum problem for the functional (3.1) and the weak solution of (3.2).

Well-posedness of (3.2). Multiplying (3.2) by a test function $\varphi \in H_0^m(\Omega)$, then integrating by parts m-times and using the boundary conditions (see (3.2)), we find the weak formulation of the problem: Find $u \in H_0^m(\Omega)$ such that

$$\alpha \int_\Omega v\nabla^m u \cdot \nabla^m \varphi\, dx + \int_\Omega u\varphi\, dx = \int_\Omega f\varphi\, dx \qquad \text{for all } \varphi \in H_0^m(\Omega), \tag{A.2}$$

which can be equivalently written in the form

$$a(u,\varphi) = F(\varphi) \quad \text{ for all } \varphi \in H_0^m(\Omega),$$

where a and F denote the bilinear form and the linear functional, respectively,

$$a(u,\varphi) := \alpha \int_\Omega v\nabla^m u \cdot \nabla^m \varphi\, dx + \int_\Omega u\varphi\, dx \quad \text{ and } \quad F(\varphi) := \int_\Omega f\varphi\, dx.$$

In order to apply the Lax–Milgram theorem, we need prove continuity and coercivity of a and continuity of F. Continuity follows by the application of the Cauchy–Schwarz inequality; in fact, for all $\psi, \varphi \in H_0^m(\Omega)$, we have

$$|a(u,\varphi)| \leq C\,\|u\|_{H^m(\Omega)}\,\|\varphi\|_{H^m(\Omega)}, \qquad |F(\varphi)| = \left|\int_\Omega f\varphi\, dx\right| \leq C\,\|f\|_{L^\infty(\Omega)}\,\|\varphi\|_{H^m(\Omega)}.$$

Coercivity of a on $H_0^m(\Omega)$ follows from the Poincaré inequality (A.1) and the definition of v given in (2.5):

$$a(u,u) \geq C(\|\nabla^m u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}) \geq C\,\|u\|^2_{H^m(\Omega)}\,.$$

Hence, by the Lax–Milgram lemma (see [7]) there exists a unique weak solution $u \in H_0^m(\Omega)$ to

$$a(u,\varphi) = F(\varphi) \ \text{ for all } \varphi \in H_0^m(\Omega)$$

and hence of (3.3). The energy estimate follows by means of the coercivity of a and the continuity of F, i.e.,

$$c\,\|u\|^2_{H^m(\Omega)} \leq |a(u,u)| = |F(u)| \leq C\,\|f\|_{L^\infty(\Omega)}\,\|u\|_{H^m(\Omega)}\,.$$

Equivalence of the problems. Let us assume that u is the minimum of the functional $\mathcal{J}(u;v)$. Then we choose $\xi \in \mathbb{R}$, and for all $h \in H_0^m(\Omega)$ we define $w := u + \xi h$. Trivially it holds that $\mathcal{J}(u;v) \leq \mathcal{J}(w;v)$. Then, by simple calculations, we have that

$$\frac{\mathcal{J}(u+\xi h;v) - \mathcal{J}(u;v)}{\xi} = \int_\Omega (u-f)h\,dx + \alpha \int_\Omega v\nabla^m u \cdot \nabla^m h\,dx + \mathcal{O}(\xi),$$

which gives, as $\xi \to 0$, the weak formulation (A.2). On the contrary, assuming that u is the solution to (A.2), then, taking $\varphi = \xi h$ for all $h \in H_0^m(\Omega)$ and $\xi \in \mathbb{R}$, we find

$$\mathcal{J}(u+\xi h;v) = \mathcal{J}(u;v) + \frac{1}{2}\left(\int_\Omega \xi^2 h^2\,dx + \alpha\int_\Omega \xi^2 v|\nabla^m h|^2\,dx\right).$$

Now, noticing that $\int_\Omega \xi^2 h^2\,dx + \alpha\int_\Omega \xi^2 v|\nabla^m h|^2\,dx \geq 0$ for all $h \in H_0^m(\Omega)$, $\alpha > 0$, and $v \in L^\infty_+(\Omega)$, we get

$$\mathcal{J}(u+\xi h;v) - \mathcal{J}(u;v) \geq 0.$$

■

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press, New York, 1975.

[2] G. S. Alberti and H. Ammari, *Disjoint sparsity for signal separation and applications to hybrid inverse problems in medical imaging*, Appl. Comput. Harmon. Anal., 42 (2017), pp. 319–349.

[3] L. Ambrosio and V. M. Tortorelli, *Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence*, Comm. Pure Appl. Math., 43 (1990), pp. 999–1036.

[4] H. Ammari, E. Bretin, J. Garnier, H. Kang, H. Lee, and A. Wahab, *Mathematical Methods in Elasticity Imaging*, Princeton Ser. Appl. Math., Princeton University Press, Princeton, NJ, 2015.

[5] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math. 1846, Springer-Verlag, Berlin, 2004.
[6] S. Amstutz, A. A. Novotny, and N. Van Goethem, *Topological sensitivity analysis for elliptic differential operators of order* $2m$, J. Differential Equations 256 (2014), pp. 1735–1770.
[7] I. Babuska, *Error-bounds for finite element method*, Numer. Math., 16 (1971), pp. 322–333.
[8] G. Bal and G. Uhlmann, *Inverse diffusion theory of photoacoustics*, Inverse Problems, 26 (2010), 085010.
[9] G. Bal and G. Uhlmann, *Reconstructions for some coupled-physics inverse problems*, Appl. Math. Lett., 25 (2012), pp. 1030–1033.
[10] A. Barton, *Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients*, Manuscripta Math., 151 (2016), pp. 375–418.
[11] E. Beretta and E. Francini, *An asymptotic formula for the displacement field in the presence of thin elastic inhomogeneities*, SIAM J. Math. Anal., 38 (2006), pp. 1249–1261, https://doi.org/10.1137/050648596.
[12] E. Beretta, E. Francini, and M. S. Vogelius, *Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities, a rigorous error analysis*, J. Math. Pures Appl., 82 (2003), pp. 1277–1301.
[13] E. Beretta, M. Grasmair, M. Muszkieta, and O. Scherzer, *A variational algorithm for the detection of line segments*, Inverse Probl. Imaging, 8 (2014), pp. 389–408.
[14] E. Beretta, M. Muszkieta, W. Naetar, and O. Scherzer, *A variational method for quantitative photoacoustic tomography with piecewise constant coefficients*, in Variational Methods in Imaging and Geometric Control, M. Bergounioux, G. Peyre, C. Schnörr, J. Caillau, and T. Haberkorn, eds., Radon Ser. Comput. Appl. Math. 18, De Gruyter, Berlin, 2017, pp. 202–224.
[15] E. Beretta, E. Bonnetier, E. Francini, and A. L. Mazzucato, *Small volume asymptotics for anisotropic elastic inclusions*, Inverse Probl. Imaging, 6 (2012), p. 1–23.
[16] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Ser. Comput. Math. 44, Springer, Heidelberg, 2013.
[17] Y. Capdeboscq and M. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, Math. Model. Numer. Anal., 37 (2003), pp. 159–173.
[18] Y. Capdeboscq and M. Vogelius, *Pointwise polarization tensor bounds, and applications to voltage perturbations caused by thin inhomogeneities*, Asymptot. Anal., 50 (2006), pp. 175–204.
[19] P. Comon, G. Golub, L. Lim, and B. Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 1254–1279, https://doi.org/10.1137/060661569.
[20] B. T. Cox, J. G. Laufer, and P. C. Beard, *The challenges for quantitative photoacoustic imaging*, Proc. SPIE, 7177 (2009), 717713.
[21] B. T. Cox, J. G. Laufer, S. R. Arridge, and P. C. Beard, *Quantitative spectroscopic photoacoustic imaging: A review*, J. Biomed. Opt., 17 (2012), 061202.
[22] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic Boundary Value Problems*, Springer, Berlin, Heidelberg, 2010.
[23] M. Grasmair, M. Muszkieta, and O. Scherzer, *An approach to the minimization of the Mumford-Shah functional using* Γ*-convergence and topological asymptotic expansion*, Interfaces Free Bound., 15 (2013), pp. 141–166.
[24] L. Huang, L. Yao, L. Liu, J. Rong, and H. Jiang, *Quantitative thermoacoustic tomography: Recovery of conductivity maps of heterogeneous media*, Appl. Phys. Lett., 101 (2012), 244106.
[25] L. Huang, J. Rong, L. Yao, W. Qi, D. Wu, X. Jin-Yu, and H.-B. Jiang, *Quantitative thermoacoustic tomography for ex vivo imaging conductivity of breast tissue*, Chinese Phys. Lett., 30 (2013), 124301.
[26] T. G. Kolda and B. W. Bader, *Tensor Decompositions and Applications*, SIAM Rev., 51 (2009), pp. 455–500, https://doi.org/10.1137/07070111X.
[27] P. Kuchment, *The Radon Transform and Medical Imaging*, CBMS-NSF Reg. Conf. Ser. Appl. Math. 85, SIAM, Philadelphia, 2014, https://doi.org/10.1137/1.9781611973297.
[28] P. Kuchment and L. Kunyansky, *Mathematics of thermoacoustic tomography*, European J. Appl. Math., 19 (2008), pp. 191–224.

[29] S. Larnier, J. Fehrenbach, and M. Masmoudi, *The topological gradient method: From optimal design to image processing*, Milan J. Math., 80 (2012), pp. 411–441.

[30] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications* I, Grundlehren Math. Wiss. 181, Springer-Verlag, New York, 1972.

[31] A. Mamonov and K. Ren, *Quantitative photoacoustic imaging in radiative transport regime*, Commun. Math. Sci., 12 (2014), pp. 201–234.

[32] M. Muszkieta, *Optimal edge detection by topological asymptotic analysis*, Math. Models Methods Appl. Sci., 19 (2009), pp. 2127–2143.

[33] W. Naetar and O. Scherzer, *Quantitative photoacoustic tomography with piecewise constant material parameters*, SIAM J. Imaging Sci., 7 (2014), pp. 1755–1774, https://doi.org/10.1137/140959705.

[34] J. Nestruev, *Smooth Manifolds and Observables*, Grad. Texts in Math. 220, Springer-Verlag, New York, 2003.

[35] C. Pozrikidis, *Introduction to Finite and Spectral Elements Methods using MATLAB*, CRC Press, Boca Raton, FL, 2014.

[36] A. Pulkkinen, B. Cox, S. Arridge, J. Kaipio, and T. Tarvainen, *Quantitative photoacoustic tomography using illuminations from a single direction*, J. Biomed. Opt., 20 (2015), 036015.

[37] L. Qi, H. Chen, and Y. Chen, *Tensor Eigenvalues and Their Applications*, Springer, Singapore, 2018.

[38] K. Ren, H. Gao, and H. Zhao, *A hybrid reconstruction method for quantitative PAT*, SIAM J. Imaging Sci., 6 (2013), pp. 32–55, https://doi.org/10.1137/120866130.

[39] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[40] P. Soille, *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.

[41] J. Sokołowski and A. Żochowski, *On the topological derivative in shape optimization*, SIAM J. Control Optim., 37 (1999), pp. 1251–1272, https://doi.org/10.1137/S0363012997323230.

[42] J. Sokołowski and J.-P. Zolesio, *Introduction to Shape Optimization*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[43] L. V. Wang, ed., *Photoacoustic Imaging and Spectroscopy*, Optical Science and Engineering. CRC Press, Boca Raton, FL, 2009.

[44] L. V. Wang and H. Wu, eds., *Biomedical Optics: Principles and Imaging*, Wiley-Interscience, New York, 2007.

[45] L. Yao, Y. Sun, and H. Jiang, *Quantitative photoacoustic tomography based on the radiative transfer equation*, Optics Lett., 34 (2009), pp. 1765–1767.

[46] Z. Yuan and H. Jiang, *Quantitative photoacoustic tomography: Recovery of optical absorption coefficient maps of heterogeneous media* Appl. Phys. Lett., 88 (2006), 231101.