VU Kontinuierliche Optimierung

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Exercise Sheet 1

1. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = (1-x)^2 + 100(y-x^2)^2 \ .$$

   (a) Compute the Gradient ∇f and the Hessian H_f of f.

   (b) Show that f is not convex but it has a unique global minimum.

   (c) Compute the condition number of H_f (the ratio of the biggest to the smallest eigenvalue of H_f) at the minimum of f.

2. Consider the non-linear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x,y) = \begin{pmatrix} x^3 - 3xy + 1 \\ 3x^2y - y^3 \end{pmatrix}.$$

   To solve the equation

$$F(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   (a) construct an iterative scheme, where the next step $(x^{(k+1)},\, y^{(k+1)})^T$ is computed by approximating the function F linearly around $(x^{(k)},\, y^{(k)})^T$.

   (b) Apply two iterations of the above method with initial vector $(x^{(0)},\, y^{(0)})^T = (1,\, 1)^T$.

3. Consider the quadratic function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1,x_2) = \frac{1}{2}(x_1^2 + c\,x_2^2), \quad c > 0.$$

   (a) Compute the optimal point x^* such that $\nabla f(x^*) = 0$ and the optimal value of f.

(b) Show that the condition number of the Hessian is given by

$$\kappa(H_f) = \max\{c, 1/c\}$$

(c) Apply the steepest descent method with exact line search, starting with initial vector $(x_1^{(0)}, x_2^{(0)}) = (c,\, 1)^T$. Show that the iterates $x^{(k)}$ obtain the following expressions

$$x_1^{(k)} = c\left(\frac{c-1}{c+1}\right)^k, \qquad x_2^{(k)} = (-1)^k\left(\frac{c-1}{c+1}\right)^k$$

and

$$f(x_1^{(k)}, x_2^{(k)}) = \left(\frac{c-1}{c+1}\right)^{2k} f(x_1^{(0)}, x_2^{(0)}).$$

4. Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \sqrt{1+x^2}$$

(a) Show that the function f is strictly convex.

(b) Find the minimum x^* of f and show that Newton's method (without line search) for starting value $x^{(0)} \in \mathbb{R}$ with $|x^{(0)}| \geq 1$ does not converge to x^*.

5. Implement in Matlab or Octave the steepest descent method with exact line search for the specific function f of the second exercise using the expressions from 2(c) and values $c = 2,\, 10$. The output should be a contour plot of the function f and the iterates $x^{(k)}$ of the steepest descent method.

6. Implement in Matlab or Octave the line search method with Armijo's rule and Goldstein and Price rule. For computing the step size use the simplest methods as discussed in the course.

7. Implement in Matlab or Octave the Newton's method with line search. Terminate the iteration of the method when the norm of the gradient $\|\nabla f(x^{(k)})\|$ is smaller than a given values $\epsilon > 0$. Prescribe also a maximum number of iterations.