Exercise Sheet 4

1. Consider the problem

$$\text{minimise} \quad f(x) = (x_1 - 2x_2)^2 + (x_1 - 2)^2$$

   Solve the above problem using Algorithm 1 (page 4). Perform two steps (by hand) with $x^{(0)} = (0,\, 0)^T$.

   Hint: In Algorithm 1, in order to compute the step size, consider f as a function of t and search for the minimum.

2. Consider the problem

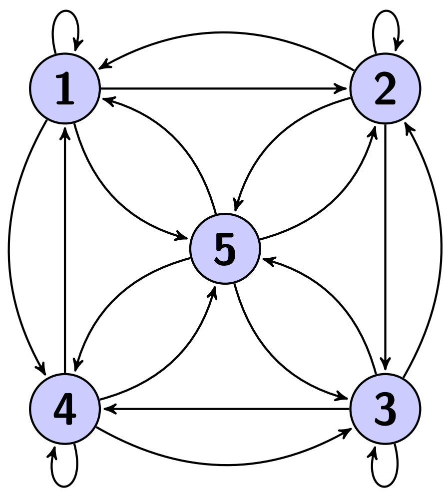
$$\text{minimise} \quad f(x) = \frac{1}{2}(x_1 - 2x_2)^2 + x_1^4$$

   Solve the above problem using Algorithm 2 (page 4). Perform two steps (by hand) with $x^{(0)} = (2,\, 1)^T$ and $m = 1/2$.

3. Determine the classes and specify if they are recurrent or transient for the following transition matrices:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

4. Consider the following random walk with states $S = \{1, 2, 3, 4, 5\}$. Assume that at each state the transition probabilities to other adjacent states are all equal and the probability to stay at the same state in the next transition is zero.



(a) Construct the transition probability matrix P.

(b) Show that the Markov chain of the random walk is irreducible and all the states are recurrent.

(c) Find the steady-state probability distribution.

5. (a) Consider the transition probability matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

i. Show that the Markov chain defined by P is irreducible but not regular.

ii. Find the unique stationary distribution. Does this Markov chain converge to the stationary distribution?

(b) Consider the transition probability matrix

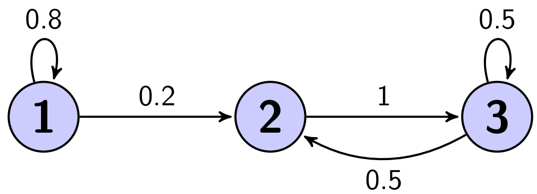
$$P = \begin{pmatrix} 0 & 1 \\ 2/3 & 1/3 \end{pmatrix}.$$

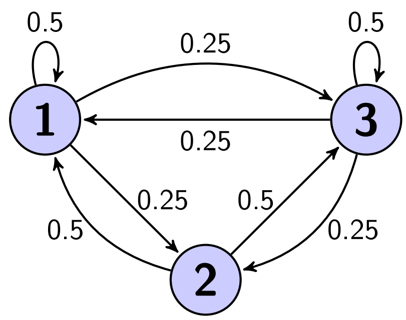
The stationary distribution π is given by

$$\lim_{n\to\infty} P^n X^{(0)} = \pi,$$

for any initial distribution $X^{(0)}$. Compute π using the eigen-decomposition of P.

6. Consider the following two random walks. Construct the transition probability matrices and find the steady-state probability distributions.





7. Consider a game with five levels, where the 5^{th} level is the highest. A player starts at the lowest (1^{st} level) and every time he flips a coin. If it turns up head, the player moves up one level. If tails, he moves down to the 1^{st} level. When the player reaches the highest level, if it turns up heads he stays there and if tails he moves to the lowest level.

   (a) Find the transition probability matrix.

   (b) What is the probability that the player will be in the 3^{rd} level after his second flipping if he started at the 2^{nd} level?

   (c) What is the probability that the player will be in the 2^{nd} level after his third flipping for any starting level?

   (d) Find the steady-state distribution of the Markov chain (by hand).

Algorithm 1: Steepest descent with exact line search

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Input:  cost function f: ℝ^n → ℝ, initial guess x^(0), k = 0
While convergence criterion not yet satisfied do
  compute d^(k) = −∇f(x^(k))
  compute t^(k) := arg min_{t≥0} {f(x^(k) + t d^(k))}
  define x^(k+1) := x^(k) + t^(k) d^(k)
  k ← k + 1
end
define x* := x^(k)
```

Algorithm 2: Newton's method with line search (Armijo's rule)

Input: cost function $f\colon \mathbb{R}^n \to \mathbb{R}$, initial guess $x^{(0)}$,
initial step $t^{(0)} = 1$, parameter $0 < m < 1,\ k = 0$
While *convergence criterion not yet satisfied* do
  define $d^{(k)}$ by solving the equation

$$H_f(x^{(k)})\, d^{(k)} = -\nabla f(x^{(k)})$$

  define $x^{(k+1)} := x^{(k)} + t^{(k)} d^{(k)}$
  if $f(x^{(k+1)}) - f(x^{(k)}) > m t^{(k)} \nabla f(x^{(k)})^T d^{(k)}$ then
    define $t^{(k+1)} = t^{(k)}/2$
  else $t^{(k+1)} = t^{(k)}$
  $k \leftarrow k + 1$
end
define $x^* := x^{(k)}$