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Exercise Sheet 2

1. Consider the symmetric matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

   (a) Show that the matrix A is positive definite.

   (b) Show that the Jacobi method for solving the linear system $Ax = b$ for any choice of b and staring vector $x^{(0)}$ diverges.

2. Consider the Jacobi method to solve the linear equation $Ax = b$, for a given diagonal dominant matrix $A = (a_{ij})_{i,j=1}^n$ of the form

$$A = \begin{pmatrix} 1 & 0 & \alpha \\ \beta & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix}, \quad \alpha, \beta, \gamma \in (0,1)$$

and a vector $b \in \mathbb{R}^n$. Instead, we can equivalently consider the equation

$$PAx = P\,b,$$

where P is an invertible matrix of the form

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\ell_2 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ -\ell_{n-1} & \vdots & \ddots & 1 & 0 \\ -\ell_n & 0 & \dots & 0 & 1 \end{pmatrix}, \quad \ell_i = \frac{a_{i1}}{a_{11}}, \quad i = 2, \dots, n.$$

Show that the convergence of the Jacobi method is improved using this so-called preconditioning for the given matrix P.

Hint: The smaller the spectral radius of the iteration matrix is, the faster the iteration method converges.

3. Let
$$A = \begin{pmatrix} 4 & 1/2 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$
Examine if the Gauss-Seidel method converges to the solution of $Ax = b$, $b \in \mathbb{R}^3$ for every starting vector $x^{(0)} \in \mathbb{R}^3$.

4. Consider the matrix
$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$
Show that the iterative methods, Jacobi, Gauss-Seidel and SOR converge to the solution of the linear equation $Ax = b$, $b \in \mathbb{R}^2$. Compare the speed of convergence of the three methods.

   Hint: For the SOR method, consider the optimal value of the over-relaxation parameter ω which occurs when the two eigenvalues of the iteration matrix are equal.

5. Consider a matrix $A \in \mathbb{R}^{n\times n}$ with positive diagonal elements and the matrix
$$\tilde{A} = D^{-1/2} A D^{-1/2}, \quad \text{where} \quad D = \operatorname{diag}(a_{11}, ..., a_{nn}).$$
Then, show that the above scaling does not affect the spectral radii of the Jacobi and the SOR methods, this means,
$$\rho(M_J) = \rho(\tilde{M}_J) \quad \text{and} \quad \rho(M_{SOR}) = \rho(\tilde{M}_{SOR}),$$
where $M_J, \tilde{M}_J$ are the iteration matrices of the Jacobi method for A and $\tilde{A}$, respectively. Equivalently, $M_{SOR}, \tilde{M}_{SOR}$ are the iteration matrices of the SOR method for A and $\tilde{A}$, respectively.

6. Implement in MATLAB the Arnoldi iterative method (Lecture notes: Algorithm 6).

7. Write a MATLAB-PROGRAM, that if the input matrix is symmetric and positive definite approximates the solution $x \in \mathbb{R}^n$ of the linear system $Ax = b$, for a given vector $b \in \mathbb{R}^n$, using the conjugate gradient method, otherwise it returns an error message.