Exercise Sheet 1

1. Given the matrices,
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix}$$
compute the matrix norms $\|A\|_{\infty,1}, \|A\|_2, \|A\|_F$ and $\|B\|_{1,\infty}, \|B\|_F$.

2. Let $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. Show that $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$, where $\|A\|_2$ is the spectral norm.

3. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that,
$$\|A\|_2 = \max_k |\lambda_k(A)|,$$
where $\lambda_k(A)$ are the eigenvalues of A.

4. Compute the condition number of the matrix (with respect to the spectral norm)
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

5. Consider the linear system $Ax = b$ and its perturbation $A(x + \Delta x) = b + \Delta b$ with
$$A = \begin{bmatrix} 1 & 4 \\ 1 & 4.001 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \Delta b = \begin{bmatrix} 0,01 \\ 0 \end{bmatrix}.$$
Check if the system is well- or ill-posed and compute the relative error.

6. (a) Let $\{a_\varepsilon : \varepsilon \neq 0\}, \{b_\varepsilon : \varepsilon \neq 0\}$ and $\{c_\varepsilon : \varepsilon \neq 0\}$ be parametrized families of numbers satisfying $a_\varepsilon, b_\varepsilon, c_\varepsilon \neq 0$. Show that if $a_\varepsilon = \mathcal{O}(b_\varepsilon)$ and $c_\varepsilon = \mathcal{O}(a_\varepsilon)$ then $c_\varepsilon = \mathcal{O}(b_\varepsilon)$.

(b) Show that,

   i. $\log(1+x) = x + \mathcal{O}(x^2)$.

   ii. $\arctan(x) - x = o(x)$.

7. Consider the linear systems,

$$\text{(a)} \quad \begin{aligned} 2x_1 - 2x_2 + \ \ x_3 &= 6 \\ x_2 + 2x_3 &= 3, \\ 5x_1 + 3x_2 + \ \ x_3 &= 4 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 - 3x_3 &= 10 \\ 5x_1 + 6x_2 - \ \ x_3 &= 2 \\ x_1 - x_2 - \ \ x_3 &= 6 \end{aligned}$$

Solve the above systems using the Gauss–Elimination algorithm.