Exercise Sheet 4

1. Show that the stability function R of an s-stage Runge-Kutta method (A, b, c) can be written in the form
$$R(\zeta) = \frac{\det(\mathbb{1}_{s\times s} - \zeta A + \zeta 1_{s\times 1} b^{\mathrm{T}})}{\det(\mathbb{1}_{s\times s} - \zeta A)},$$
for all $\zeta \in \mathbb{C}$ such that $\frac{1}{\zeta} \notin \sigma(A)$, where $\mathbb{1}_{s\times s}$ denotes the $s \times s$ identity matrix, $1_{s\times 1}$ is the vector whose entries are all one: $1_{s\times 1} = (1)_{k=1}^{s}$, and $\sigma(A)$ is the spectrum of the matrix A.

2. Consider an arbitrary explicit 2-stage Runge-Kutta method of order 2.
   (a) Calculate the stability function of such a method.
   (b) Sketch the region of stability.
   (c) How small do we have to pick the step size to ensure stability if we apply the method to the differential equation
$$y'(t) = \begin{pmatrix} -5 & -2 \\ 2 & -5 \end{pmatrix} y(t), \quad t > 0?$$

3. Calculate the stability function and the region of stability for the implicit Runge-Kutta method (A, b, c) given by
$$\begin{array}{c|c} c & A \\ \hline & b^{\mathrm{T}} \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

4. Let an A-stable implicit s-stage Runge-Kutta method (A, b, c) be given. We now solve the implicit equations
$$\eta_j = F_j(\eta_1, \ldots, \eta_s) := y_i + h \sum_{k=1}^{s} a_{jk} f(t_i + c_k h, \eta_k), \quad j = 1, \ldots, s,$$
appearing in this method for every step size h at every point (t_i, y_i) not exactly, but with m steps of a fixed point iteration starting with $\eta_j^{(0)} = y_i$ for all j, i.e.
$$\eta_j^{(\ell)} = F_j(\eta_1^{(\ell-1)}, \ldots, \eta_s^{(\ell-1)}), \quad j = 1, \ldots, s, \quad \ell = 1, \ldots, m.$$

Is the method defined by

$$y_{i+1} = y_i + h \sum_{k=1}^{s} b_k f(t_i + c_k h, \eta_k^{(m)})$$

A-stable?

5. Write a program that draws the region of stability (inside a given square $Q \subset \mathbb{C}$) for a given s-stage Runge-Kutta method (A, b, c).

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