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Inverse Problems

Lecture Notes

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Contents

Part I

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Chapter 1

Transforms

1.1 Distributions

Definition 1.1. $\mathcal{D} := C_0^\infty(\mathbb{R}^n;\mathbb{C})$ *denotes the space of test functions.*

The linear functional

$$T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$$

is called a distribution if

1. $T(\phi_1 + \phi_2) = T\phi_1 + T\phi_2\,, \quad \forall \phi_1, \phi_2 \in \mathcal{D},$
2. $T(\lambda\phi) = \lambda T\phi\,, \quad \forall \phi \in \mathcal{D}$ *and* $\lambda \in \mathbb{C},$
3. *If* $\phi_j \to \phi$ *in* $\mathcal{D}$*, then* $T\phi_j \to T\phi$*. Convergence means that the supports of* (ϕ_j) *are contained in a compact set, and* $\nabla^k\phi_j \to \nabla^k\phi$ *for every* n*-tupel* k*.*

The space of distributions is denoted by $\mathcal{D}'$*.*

Example 1.2. *1. Regular distribution: Let* f *be locally integrable on* $\mathbb{R}^n$ *- that is on every compact set* $K \subseteq \mathbb{R}^n$ *we have* $f \in L^1(K;\mathbb{C})$*. Then* T_f *defined by*

$$T_f\phi = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx$$

is a distribution on $\mathbb{R}^n$*.*

Sometimes it is also convenient to write this in a complex L^2*-inner product form:*

$$T_f\phi = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx = \left\langle f, \overline{\phi} \right\rangle \ .$$

2. *The Dirac distribution T_δ is defined as follows:*

$$T_\delta \phi = \phi(0) \ .$$

Note that T_δ is not a regular distribution and δ is not a function.

Nevertheless it is common to notionally identify T_δ with a function δ, or in other words use the notional identification

$$T_\delta \phi = \int_{\mathbb{R}^n} \delta(x)\phi(x)\, dx = \left\langle \delta, \overline{\phi} \right\rangle \ .$$

The δ-distribution has a series of properties:

- *Let $x, \alpha \in \mathbb{R}$. Then*
$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x) \ . \tag{1.1}$$
- *Let $x \in \mathbb{R}$. Then*
$$\delta'(x) = -\frac{\delta(x)}{x} \ . \tag{1.2}$$

3. *Let $a \in \mathbb{R}^n$ and $k = (k_1, \ldots, k_n) \in \{0, 1, \ldots, |k|\}^n$ with $|k| = \sum_{i=1}^n k_i$. Then*

$$T_{\nabla^k \delta}\phi = \nabla^k \phi(a)$$

is a distribution. Here, for a $|k|$-times continuously differentiable function f we have

$$\nabla^k f(x) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} f(x) \ .$$

An equivalent definition of a distribution is that T is a linear functional on $\mathcal{D}$ such that for every compact set $K \subseteq \mathbb{R}^n$ there exists a constant C_K and an integer m satisfying

$$|\langle T, \phi\rangle| \leq C_K \sum_{|k|\leq m} \sup \left| D^k \phi \right| \ , \forall \phi \in \mathcal{D} \text{ with } \operatorname{supp}(\phi) \subseteq K \ .$$

Remark 1.3. *Distributions are often also called generalized functions. Note that by the Riesz's theorem every function in L^2 can be identified with a linear operator on L^2. In contrast to that the linear operators cannot be identified with functions anymore.*

1.2 Fourier transform

Definition 1.4. *The one-dimensional Fourier transform of a function* $f : \mathbb{R} \to \mathbb{C}$ *is defined as*

$$\boxed{\mathcal{F}[f](\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} f(t)\, dt\,, \quad \omega \in \mathbb{R}\,.} \tag{1.3}$$

The integral in the definition of the Fourier transform (1.3) *is considered an improper integral.*

The n*-dimensional Fourier transform of a function* $f : \mathbb{R}^n \to \mathbb{C}$ *is defined as*

$$\boxed{\mathcal{F}[f](\omega) := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{\mathrm{i}\omega\cdot x} f(x)\, dx\,, \quad x \in \mathbb{R}^n\,.} \tag{1.4}$$

There are many different notations for the Fourier-transform. For the one-dimensional Fourier transform we use also the abbreviation

$$\hat{f}(\omega) := \mathcal{F}[f](\omega)\,, \quad \omega \in \mathbb{R}\,. \tag{1.5}$$

Later on, it will be convenient to specify the coordinate transformation induced by the Fourier transform: Then $\mathcal{F}[f]$ is also written as

$$\boxed{\mathcal{F}[f] := \mathcal{F}_{(x_1,\dots,x_n)\to(\omega_1,\dots,\omega_n)}[f]\,.} \tag{1.6}$$

The definitions of the Fourier-transform are not unique: Sometimes instead of the factor $\frac{1}{\sqrt{2\pi}}$ the factor 1 or the factor $\frac{1}{2\pi}$ is used, respectively. Moreover, in the definition of the Fourier-transform instead of $e^{\mathrm{i}\omega t}$ it is also common to use the term $e^{-\mathrm{i}\omega t}$ instead. The Fourier-transform can also be written as an inner product:

$$\sqrt{2\pi}\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} f(t)\, dt = \left\langle f, e^{-\mathrm{i}\omega\cdot} \right\rangle\,. \tag{1.7}$$

The Fourier-transform has remarkable properties:

Lemma 1.5. *Linearity: Let* $f, g : \mathbb{R}^n \to \mathbb{C}$ *be functions and* $a, b \in \mathbb{C}$*. Then*

$$\mathcal{F}[af + bg](\omega) = a\mathcal{F}[f](\omega) + b\mathcal{F}[g](\omega)\,.$$

Differentiation of the n-dimensional Fourier-transform:

$$\boxed{\begin{aligned}\nabla^\alpha \mathcal{F}[f(x)](\omega) &= \mathrm{i}^{|\alpha|}\mathcal{F}[x^\alpha f(x)](\omega)\,,\\ \mathcal{F}[\nabla^\alpha f(x)](\omega) &= (-\mathrm{i})^{|\alpha|}\omega^\alpha \mathcal{F}[f](\omega)\,.\end{aligned}} \tag{1.8}$$

Proof. We do the proof in 1D and for $\alpha = 1$. For all other cases it follows by induction.

- From the definition of the Fourier transform it follows:

$$\frac{d}{d\omega}\mathcal{F}[f(x)](\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{i}x e^{\mathrm{i}\omega x} f(x)\,dx = \mathrm{i}\mathcal{F}[xf(x)](\omega)\,.$$

- With integration by parts we get

$$\begin{aligned}\mathcal{F}[f'(x)](\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega x} f'(x)\,dx\\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{i}\omega e^{\mathrm{i}\omega x} f(x)\,dx\\ &= -\mathrm{i}\omega\mathcal{F}[f](\omega)\,.\end{aligned}$$

□

Sign change:

$$\begin{aligned}\mathcal{F}[g(-\cdot)](\omega) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} g(-t)\,dt\\ \underbrace{=}_{\tau=-t,\ -d\tau=dt} & -\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\mathrm{i}\omega\tau} g(\tau)\,d\tau\\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}(-\omega)\tau} g(\tau)\,d\tau\\ =& \mathcal{F}[g](-\omega)\,.\end{aligned} \tag{1.9}$$

Fixed point: Let $\phi(x) = \frac{1}{\sqrt{2\pi}^n} e^{-\frac{\|x\|^2}{2}}$ (n-dimensional Gauß-distribution). Then

$$\mathcal{F}[\phi](x) = \phi(x)\,.$$

In particular ϕ is an eigenfunction of the Fourier-transform with eigenvalue 1.

We do the proof in one space dimension. In higher dimensions it is proven by integrating each dimension separately: We have

$$\frac{d}{dx}\phi(x) = -x\phi(x)\ . \tag{1.10}$$

Application of the Fourier transform on both sides gives:

$$\begin{aligned} -\mathrm{i}\omega\mathcal{F}[\phi](\omega) \underbrace{=}_{(1.8)} &\mathcal{F}\left[\frac{d}{dx}\phi\right](\omega) \\ = &-\mathcal{F}[x\phi(x)](\omega) \\ \underbrace{=}_{(1.8)} &-\frac{1}{\mathrm{i}}\frac{d}{d\omega}\mathcal{F}[\phi](\omega)\ . \end{aligned}$$

Thus we have

$$-\omega\mathcal{F}[\phi](\omega) = \frac{d}{d\omega}\mathcal{F}[\phi](\omega)\ .$$

Thus $\mathcal{F}[Phi]$ and ϕ satisfy the same differential equation (cf. (1.10)). The unique solution of this ordinary differential equation is $\omega \to \phi(\omega)$.

Symmetry:

$$\mathcal{F}[\mathcal{F}[f]](x) = f(-x)\,, \quad \forall x \in \mathbb{R}^n\ .$$

Inversion: The operator

$$\boxed{\mathcal{F}^{-1}[f](x) := \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} e^{-\mathrm{i}\omega\cdot x} g(\omega)\,d\omega\,, \quad x \in \mathbb{R}^n} \tag{1.11}$$

is the inverse of the Fourier-transform. That means

$$\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x) \text{ and } \mathcal{F}[\mathcal{F}^{-1}[g]](\omega) = g(\omega)\ . \tag{1.12}$$

Convolution:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,, \quad \forall x \in \mathbb{R}^n$$

denotes the convolution of f and g. Then

$$\mathcal{F}[f*g](x) = \sqrt{2\pi}^n\mathcal{F}[f](x)\mathcal{F}[g](x)\ . \tag{1.13}$$

Example 1.6. *We summarize a few examples of Fourier transforms: Let $a \in \mathbb{R}$ be a parameter, then the Fourier-transform of the following functions are again functions:*

$f(t)$	$\hat{f}(\omega)$	*Comment*
$e^{-at^2/2}$	$\frac{1}{\sqrt{a}} e^{-\omega^2/(2a)}$	
$\chi_{(-0.5,0.5)}(at)$	$\frac{1}{\sqrt{2\pi}\lvert a\rvert} sinc\left(\frac{\omega}{2\pi a}\right)$	$sinc(x) = \frac{\sin(x)}{x}$
$e^{-a\lvert t\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{\omega^2+a^2}$	
$\frac{1}{t^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{1}{a} e^{-a\lvert\omega\rvert}$	
t^{-n}	$\mathrm{i}^n \omega^{n-1} \sqrt{\frac{\pi}{2}} \frac{1}{(n-1)!} sgn(\omega)$	
$sgn(t)$	$\sqrt{\frac{2}{\pi}} \frac{\mathrm{i}}{\omega}$	

The Fourier transform of $t \to t^{-n}$ is calculated explicitly as follows:

- *First, we note that by* (1.8)

  $$\mathcal{F}[t^{-n}](\omega) = -\mathrm{i}\omega \mathcal{F}\left[-\frac{1}{n-1} t^{-(n-1)}\right](\omega)\,,$$

  and thus by induction

  $$\mathcal{F}[t^{-n}](\omega) = (\mathrm{i}\omega)^{n-1} \frac{1}{(n-1)!} \mathcal{F}\left[t^{-1}\right](\omega)\,,$$

- *We apply the residue theorem:*

  Theorem 1.7. *Let $\omega > 0$ and $z \to h(z) := e^{\mathrm{i}\omega z} g(z)$ be analytic in $\mathbb{C}$ outside of finitely many poles. Moreover, we assume that g does have only simple poles $\{x_1, \ldots, x_m\}$ on the real axis and it satisfies for some M and R*

  $$|g(z)| \leq \frac{M}{|z|}\,, \qquad \forall \Im z \geq 0 \text{ and } |z| \geq R\,.$$

  Denoting by $\{z_1, \ldots, z_n\}$ the poles in the upper half plane, we have

  $$\int_{-\infty}^{\infty} h(x)\,dx = 2\pi\mathrm{i} \sum_{i=1}^{n} Res(h, z_i) + \pi\mathrm{i} \sum_{i=1}^{m} Res(h, x_i)\,.$$

Now, we apply this theorem to calculate the Fourier-transform of $t \to t^{-1}$*: We have*

$$\sqrt{2\pi}\mathcal{F}\left[t^{-1}\right](\omega) = \int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} t^{-1}\, dt \,.$$

– *Let* $\omega > 0$*, then the residual theorem implies that*

$$\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} t^{-1}\, dt = \pi \mathrm{i} Res\left(e^{\mathrm{i}\omega t} t^{-1}, 0\right) = \pi \mathrm{i} \,.$$

Thus we have

$$\mathcal{F}[t^{-n}](\omega) = \mathrm{i}^n \omega^{n-1} \frac{1}{(n-1)!}\sqrt{\frac{\pi}{2}}\,, \quad \forall \omega > 0 \,.$$

– *Let* $\omega < 0$*, then we substitute* $\hat{t} = -t$*:*

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t} t^{-1}\, dt &= \int_{-\infty}^{\infty} e^{\mathrm{i}(-\omega)(-t)} t^{-1}\, dt \\
&= \int_{\infty}^{-\infty} e^{\mathrm{i}(-\omega)\hat{t}} \frac{1}{\hat{t}}\, d\hat{t} \\
&= -\int_{-\infty}^{\infty} e^{\mathrm{i}(-\omega)\hat{t}} \frac{1}{\hat{t}}\, d\hat{t} \,.
\end{aligned}$$

Now, we can apply again the residue theorem and get

$$\begin{aligned}
\sqrt{2\pi}\mathcal{F}\left[t^{-1}\right](\omega) &= -\int_{-\infty}^{\infty} e^{\mathrm{i}(-\omega)\hat{t}} \frac{1}{\hat{t}}\, d\hat{t} \\
&= -\pi \mathrm{i} Res\left(e^{\mathrm{i}(-\omega)\hat{t}} \frac{1}{\hat{t}}, 0\right) \\
&= -\pi \mathrm{i} \,.
\end{aligned}$$

Thus we have

$$\mathcal{F}[t^{-n}](\omega) = -\mathrm{i}^n \omega^{n-1} \frac{1}{(n-1)!}\sqrt{\frac{\pi}{2}}\,, \quad \forall \omega < 0 \,.$$

The one-dimensional Fourier-transform can, however, be also be defined for generalized functions:

Definition 1.8. *The Schwartz space $\mathcal{S}$ consists of $C^\infty(\mathbb{R}^n;\mathbb{C})$ functions such that*

$$x \to x^l \partial_x^k \phi(x) \in L^\infty(\mathbb{R};\mathbb{C}) , \qquad \forall k, l \geq 0 .$$

The space of tempered distributions *is the dual space of $\mathcal{S}$, $\mathcal{S}'$.*

Remark 1.9. *Because $\mathcal{D} \subseteq \mathcal{S}$ we have $\mathcal{S}' \subseteq \mathcal{D}'$.*

Theorem 1.10. *The Fourier transform is an isomorphism of $\mathcal{S}$.*

Example 1.11. *Let $a \in \mathbb{R}$ be a parameter, then the Fourier-transform of the following functions are distributions:*

$f(t)$	$\hat{f}(\omega)$
$e^{\mathrm{i}at}$	$\sqrt{2\pi}\delta(\omega + a)$
$\cos(at)$	$\sqrt{\frac{\pi}{2}}(\delta(\omega - a) + \delta(\omega + a))$
$\sin(at)$	$\mathrm{i}\sqrt{\frac{\pi}{2}}(\delta(\omega - a) - \delta(\omega + a))$
t^n	$\mathrm{i}^n\sqrt{2\pi}d^n\delta(\omega)$

1.3 Mellin transform

Let

$$f : [0, \infty) \to \mathbb{R}$$

Then, the Mellin transform is defined by

$$M[f](z) := \int_0^\infty f(t)t^{z-1}\, dt , \qquad \forall z \in \mathbb{C} .$$

There exists a backprojection formula:

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} M[f](z)x^{-z}\, dz .$$

When substituting $t = e^x$, then

$$\begin{aligned} M[f](z) &= \int_{-\infty}^{\infty} f(e^x)e^{x(z-1)}e^x\, dx \\ &= \int_{-\infty}^{\infty} f(e^x)e^{xz}\, dx \end{aligned}$$

Thus

$$M[f](\mathrm{i}s) = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(e^x)e^{\mathrm{i}sx}\,dx$$
$$= \sqrt{2\pi}\mathcal{F}[f(e^x)](s)\ .$$

This show the relation between Fourier and Mellin transform.

1.4 Fourier cosine transform

Definition 1.12. *Let* $f : [0,\infty] \to \mathbb{R}$. *Then*

$$\mathcal{C}[f](k_y) := \mathcal{C}_{y\to k_y}[f](k_y) := \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\cos(k_y y)f(y)dy\,,\quad \forall k_y \in \mathbb{R}\ .$$

We summarize a few properties of the Fourier cosine transform:

- The function $\mathcal{C}[f](k_y)$ is symmetric with respect to 0.
- Extending $f : [0,\infty) \to \mathbb{R}$ by zero to $(-\infty, 0)$, and denoting the function by f_e we have

$$\mathcal{C}[f](k_y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(k_y y)f_e(y)dy = \Re(\mathcal{F}[f_e])(k_y)\,,\quad \forall k_y \in \mathbb{R}\ .$$

- Let us denote by

$$f_s(y) = f_e(y) + f_e(-y)\,,\qquad \forall y \in \mathbb{R}\ .$$

Then

$$\mathcal{C}[f](k_y) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\cos(k_y y)f(y)dy$$
$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(k_y y)f_s(y)dy$$
$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\mathrm{i}k_y y}f_s(y)dy$$
$$= \frac{1}{2}\mathcal{F}[f_s](k_y)\,,\quad \forall k_y \in \mathbb{R}\ .$$

- For $g : \mathbb{R} \to \mathbb{R}$ symmetric:

$$\boxed{\begin{aligned} \mathcal{C}^{-1}[g](y) &:= \mathcal{C}^{-1}_{k_y \to y}[g](y) \\ &:= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \cos(k_y y) g(k_y) dk_y \,, \\ &\forall y \in (0, \infty) \,. \end{aligned}} \tag{1.14}$$

is in fact the inverse of $\mathcal{C}$.

To see this note that, because g is symmetric

$$\begin{aligned} \mathcal{C}^{-1}[g](y) &= \mathcal{C}^{-1}_{k_y \to y}[g](y) \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \cos(k_y y) g(k_y) dk_y \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} (\cos(-k_y y) + \mathrm{i} \sin(-k_y y)) g(k_y) dk_y \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i} k_y y} g(k_y) dk_y \\ &= 2\mathcal{F}^{-1}[g](y) \,, \quad \forall y \in (0, \infty) \,. \end{aligned}$$

Thus for $g = \mathcal{C}[f]$ for $f : (0, \infty) \to \mathbb{R}$ we have

$$\begin{aligned} \mathcal{C}^{-1}[\mathcal{C}[f]](y) &= 2\mathcal{F}^{-1}[\mathcal{C}[f]](y) \\ &= \mathcal{F}^{-1}[\mathcal{F}[f_s]](y) \\ &= f_s(y) \\ &= f(y) \,, \quad \forall y \in (0, \infty) \,. \end{aligned}$$

The proof that $\mathcal{C}[\mathcal{C}^{-1}[g]] = g$ for all $g : \mathbb{R} \to \mathbb{R}$ symmetric is analogous.

1.5 Laplace transform

Definition 1.13. *The Laplace transform is defined as*

$$\boxed{\mathcal{L}[f](p) := \int_0^{\infty} f(t) e^{-pt} \, dt = \langle f(\cdot), e^{p\cdot} \rangle \,, \quad \forall p \in \mathbb{C} \,.} \tag{1.15}$$

We identify $f : [0, \infty) \to \mathbb{R}$ with the function $f : \mathbb{R} \to \mathbb{R}$ with $f(t) = 0$ for $f < 0$. The latter class of functions is called causal.

Remark 1.14. *Let f be causal, then for $p = -\mathrm{i}\eta$ and $\eta \in \mathbb{R}$ we have*

$$\mathcal{L}[f](\mathrm{i}\eta) := \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}\eta t}\,dt = \sqrt{2\pi}\mathcal{F}[f](\eta)\,, \quad \forall \eta \in \mathbb{R}\,. \tag{1.16}$$

Example 1.15. *In the following we summarize the Laplace transform of some functions:*

Assumption	f	$\mathcal{L}[f]$
	1	$\frac{1}{p}$, $\Re p > 0$
	t^m	$\frac{m!}{p^{m+1}}$, $\Re p > 0$
$\Re\alpha > 0,\ \alpha \in \mathbb{C}$	$t^{\alpha-1}$	$\frac{\Gamma(\alpha)}{p^\alpha}$, $\Re p > 0$
Heaviside function $c \geq 0$	$\chi_{[c,\infty)}$	$\frac{1}{p}e^{-cp}$, $\Re p > 0$
$\alpha \in \mathbb{C}$	$e^{-\alpha t}$	$\frac{1}{p+\alpha}$ *for* $\Re p > \Re(-\alpha)$
$a \in \mathbb{R}$	$\sin(at)$	$\frac{a}{p^2+a^2}$, $\Re p > 0$
$a \in \mathbb{R}$	$\cos(at)$	$\frac{p}{p^2+a^2}$, $\Re p > 0$

The Laplace transform can also be defined for distributions. Thereby one makes the following trick: Let $T \in \mathcal{D}'_+$ (support in $[0,\infty)$). such that

$$e^{-\zeta_0 t}T \in \mathcal{S}'(\mathbb{R};\mathbb{C})\,.$$

Then also

$$e^{-pt}T \in \mathcal{S}'(\mathbb{R};\mathbb{C})\,, \quad \forall p \in \mathbb{C} \text{ with } \Re p \geq \zeta_0\,.$$

Moreover, let $\alpha \in C^\infty(\mathbb{R};\mathbb{C})$ with $\mathrm{supp}(\alpha) \subseteq [c,\infty)$ and $\alpha \equiv 1$ in $[0,\infty)$. Then $t \to \alpha(t)e^{pt} \in \mathcal{S}(\mathbb{R})$ for all $p \in \mathbb{C}$ with $\Re p < 0$. Therefore

$$\left\langle e^{-\zeta_0 t}T, \overline{\alpha(t)e^{(-p+\zeta_0)t}} \right\rangle$$

exist, commonly abbreviated as $\langle T, \overline{e^{-pt}} \rangle$ for $\Re p > \zeta_0$.

Example 1.16. *In the following we summarize the Laplace transform of some functions: Let $a > 0$*

f	$\mathcal{L}[f]$
δ	1
$d^m\delta$	p^m , $p \in \mathbb{C}$
$\delta(\cdot - a)$	e^{-ap} , $p \in \mathbb{C}$

The Laplace transform has also remarkable properties Let f and g have support in $[0,\infty)$, then

-
$$\mathcal{L}[f*g]=\mathcal{L}[f]\mathcal{L}[g]\,. \tag{1.17}$$

-
$$\mathcal{L}[f'](p)=p\mathcal{L}[f](p)-f(0)\,,\quad \forall\Re p<\zeta_0 \tag{1.18}$$
under the assumption that $e^{-\zeta t}|f(t)|$ is integrable for $\zeta<\zeta_0$ and $\mathrm{supp} f\subseteq[0,\infty)$.

In the following we derive the inversion formula for the Laplace transform: Let $f:\mathbb{R}\to\mathbb{C}$ with $f(t)=0$ for all $t<0$ and $e^{-\zeta t}f(t)\in L^1(\mathbb{R};\mathbb{C})$ for all $\zeta>\zeta_0$. Then

$$\mathcal{L}[f](\zeta+\mathrm{i}\eta)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\sqrt{2\pi}f(t)e^{-\zeta t}e^{-\mathrm{i}\eta t}\,dt\,,\qquad\forall\zeta>\zeta_0\,.$$

If, for some ζ, $\eta\to\mathcal{L}[f](\zeta+\mathrm{i}\eta)\in L^1(\mathbb{R};\mathbb{C})$, then by taking the Fourier transform, it follows

$$\sqrt{2\pi}f(t)e^{-\zeta t}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\mathcal{L}[f](\zeta+\mathrm{i}\eta)e^{\mathrm{i}\eta t}\,d\eta\,.$$

Thus

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty\mathcal{L}[f](\zeta+\mathrm{i}\eta)e^{(\zeta+\mathrm{i}\eta)t}\,d\eta=\frac{1}{2\pi\mathrm{i}}\int_{\zeta-\mathrm{i}\infty}^{\zeta+\mathrm{i}\infty}\mathcal{L}[f](p)e^{pt}\,dp\,.$$

Definition 1.17. *We call the operator $\mathcal{L}^{-1}$ by*

$$g\to\frac{1}{2\pi\mathrm{i}}\int_{\zeta-\mathrm{i}\infty}^{\zeta+\mathrm{i}\infty}g(p)e^{pt}\,dp\,.$$

Note, that it can be considered just a left inverse.

Example 1.18. *Given some $0<\alpha<1$, and f with support in $[0,\infty)$ we are solving the Abel integral equation*

$$\boxed{\int_0^t(t-\tau)^{\alpha-1}\phi(\tau)\,d\tau=f(t)\,,\qquad\forall t\geq 0\,,} \tag{1.19}$$

for some ϕ *with support in* $[0,\infty)$. *Then, by using the Heaviside-function* $\chi_{[0,\infty)}$ *we can rewrite this equation to*

$$\begin{aligned}
\int_0^t (t-\tau)^{\alpha-1}\phi(\tau)\,d\tau \underbrace{=}_{\phi(\tau)=0 \quad \forall \tau<0} & \int_{-\infty}^t (t-\tau)^{\alpha-1}\phi(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} \underbrace{\chi_{[0,\infty)}(t-\tau)(t-\tau)^{\alpha-1}}_{=:\zeta(t-\tau)}\phi(\tau)\,d\tau \\
&= (\zeta * \phi)(t)\,, \qquad \forall t \geq 0.
\end{aligned}$$

Taking the Laplace transforms then gives, taking into account Example 1.15

$$\begin{aligned}
\mathcal{L}[f](p) = \mathcal{L}\left[\zeta * \phi\right](p) & \underbrace{=}_{(1.17)} \mathcal{L}[\zeta](p)\mathcal{L}\left[\phi\right](p) \\
&= \mathcal{L}[t^{\alpha-1}](p)\mathcal{L}\left[\phi\right](p) \\
& \underbrace{=}_{\textit{Example } 1.15} \frac{\Gamma(\alpha)}{p^\alpha}\mathcal{L}\left[\phi\right](p)\,, \qquad \forall p \geq 0\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{L}\left[\phi\right](p) &= \frac{p^\alpha \mathcal{L}[f](p)}{\Gamma(\alpha)} = \frac{(p\mathcal{L}[f](p))p^{\alpha-1}}{\Gamma(\alpha)} \\
& \underbrace{=}_{(1.18)} \frac{(\mathcal{L}[f'](p)+f(0))p^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{1}{\Gamma(\alpha)}\left(\mathcal{L}[f'](p)p^{\alpha-1}\chi_{[0,\infty)}(p) + f(0)p^{\alpha-1}\chi_{[0,\infty)}(p)\right)\,, \qquad \forall p \geq 0\,.
\end{aligned}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
\phi(t) &= \frac{1}{\Gamma(\alpha)}\left(\mathcal{L}^{-1}\left[\mathcal{L}[f']\right] * \mathcal{L}^{-1}[p^{\alpha-1}\chi_{[0,\infty)}(p)]\right)(t) + \frac{f(0)}{\Gamma(\alpha)}\mathcal{L}^{-1}[p^{\alpha-1}\chi_{[0,\infty)}(p)] \\
& \underbrace{=}_{\textit{Example } 1.15} \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(f' * (\cdot)^{-\alpha}\right)(t) + \frac{f(0)}{\Gamma(1-\alpha)\Gamma(\alpha)}t^{-\alpha} \\
& \underbrace{=}_{\Gamma(1-\alpha)\Gamma(\alpha)=\frac{\sin(\pi\alpha)}{\alpha}} \frac{\sin(\pi\alpha)}{\alpha}\left(\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\,d\tau + f(0)t^{-\alpha}\right)\,.
\end{aligned}$$

1.6 Hilbert transform

The Hilbert-transform is defined as

$$\mathcal{H}[f](z) := \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - z}\, d\zeta\,, \quad \forall z \in \mathbb{R}\,, \tag{1.20}$$

where the integral exists as Cauchy principal value, meaning,

$$PV \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \lim_{\epsilon \to 0^+} \int_{|\zeta - z| > \epsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

In the following we will always omit the PV in front of the integral to simplify the notation.

The Hilbert transform can be rewritten as a convolution (cf. (1.5)):

$$\mathcal{H}[f](z) = \frac{1}{\pi} f(z) * \frac{1}{z}\,, \quad \forall z \in \mathbb{R}\,. \tag{1.21}$$

Then from (1.13) it follows that

$$\mathcal{F}[\mathcal{H}[f]](\omega) = \frac{\sqrt{2\pi}}{\pi} \mathcal{F}[f](\omega) \mathcal{F}\left[\frac{1}{z}\right](\omega) = \mathrm{iSign}(\omega) \mathcal{F}[f](\omega)\,, \quad \forall \omega \in \mathbb{R}\,.$$

In other word

$$\mathcal{H}[f](z) = \mathcal{F}^{-1}[\mathrm{iSign}(\omega) \mathcal{F}[f](\omega)](z)\,. \tag{1.22}$$

1.6.1 Kramers-Kronig relation

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a real and causal function. That is $0 = \Im_\chi(t) =: \chi_\Im(t)$ and $\chi(t) = 0$ for $t < 0$. Let, $\widehat{\chi} = \mathcal{F}[\chi]$, then, the first assumption provides that

$$\widehat{\chi_\Re}(-\omega) = \widehat{\chi_\Re}(\omega) \quad (even), \quad \widehat{\chi_\Im}(-\omega) = -\widehat{\chi_\Im}(\omega) \quad (odd),$$

and the second, tells that $\widehat{\chi}$ is an analytic function in the upper half plane. Then, from the Cauchy integral theorem, we obtain the Kramers-Kronig relation

$$\widehat{\chi}(\omega) = \frac{1}{\mathrm{i}\pi} \int_{\mathbb{R}} \frac{\widehat{\chi}(\omega')}{\omega' - \omega}\, d\omega'\,, \tag{1.23}$$

which holds if $\widehat{\chi}$ vanishes at least as fast as $1/|\omega|$. Writing the Kramers-Kronig relation in component form, we have

$$\begin{aligned} \widehat{\chi_{\Re}}(\omega) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\widehat{\chi_{\Im}}(\omega')}{\omega'-\omega}\,d\omega' = \mathcal{H}[\widehat{\chi_{\Im}}](\omega)\,, \\ \widehat{\chi_{\Im}}(\omega) &= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\widehat{\chi_{\Re}}(\omega')}{\omega'-\omega}\,d\omega' = \mathcal{H}^{-1}[\widehat{\chi_{\Re}}](\omega)\,, \end{aligned} \tag{1.24}$$

where $\mathcal{H}^{-1} = -\mathcal{H}$. The Kramers-Kronig relation is widely used in Physics to extend real or imaginary parts to the respective other part of a function. However, in the above form, the K-K relations are not useful, since we have to consider negative frequencies!! Thus, we rewrite them in the following form, using the properties of $\widehat{\chi}$,

$$\begin{aligned} \widehat{\chi_{\Re}}(\omega) &= \frac{2}{\pi}\int_0^\infty \frac{\omega'\widehat{\chi_{\Im}}(\omega')}{\omega'^2-\omega^2}\,d\omega'\,, \\ \widehat{\chi_{\Im}}(\omega) &= -\frac{2\omega}{\pi}\int_0^\infty \frac{\widehat{\chi_{\Re}}(\omega')}{\omega'^2-\omega^2}\,d\omega'. \end{aligned} \tag{1.25}$$

1.7 Fractional integrals and differentials

Theorem 1.19. *The n-th primitive f_n of a function $f : \mathbb{R} \to \mathbb{R}$, which satisfies $f(t) = 0$ for $t < 0$ is given by*

$$f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau\,, \qquad \forall t > 0\,, \quad n \in \mathbb{N}\,. \tag{1.26}$$

f_n is extended by 0 for $t < 0$.

Proof. The proof is done by induction:

- For $n = 1$, (1.26) means

$$f_1(t) = \int_0^t f(\tau)\,d\tau\,, \qquad \forall t \in \mathbb{R}\,,$$

which is actually the definition of the first primitive.

- For $n \to n+1$, we assume that (1.26) is true for n and we prove it for $n+1$: By integration by parts it follows that

$$\begin{aligned}\frac{1}{n!}\int_0^t (t-\tau)^n f(\tau)\,d\tau &= \frac{1}{n!}\int_0^t n(t-\tau)^{n-1} f_1(\tau)\,d\tau \\ &= \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f_1(\tau)\,d\tau \\ &= (f_1)_{n-1}(t) \\ &= f_n(t) \qquad \forall t>0;.\end{aligned}$$

□

Because $(n-1)! = \Gamma(n)$ an extension of the definition (1.26) to $\alpha \in \mathbb{R}^+$ is as follows:

Definition 1.20 (Fractional integral of order $\alpha > 0$:)**.** *For a causal function $f : \mathbb{R} \to \mathbb{R}$ we define*

$$J_\alpha[f](t) := f_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau\,, \qquad \forall t>0\,, \quad \alpha \in \mathbb{R}^+\,. \tag{1.27}$$

Again f_α is extended by 0 for $t < 0$.

We summarize a few basic properties of the fractional integral:

1. Note that $\Gamma(n) = (n-1)!$ such that $f_n = J_n[f]$.

2. $J_\alpha J_\beta = J_{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

3. Power functions:

$$J_\alpha[t^\gamma](t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}\,, \quad \alpha > 0\,, \quad \gamma > -1, \quad t > 0\,. \tag{1.28}$$

   [6].

4. Denote by

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}\,, \quad \alpha > 0\,. \tag{1.29}$$

   Then

$$J_\alpha[f](t) = (\Phi_\alpha * f)(t)\,, \quad \alpha > 0\,.$$

In the following we define fractional derivatives. First of all we note that

$$D^n J_n = I\,,$$

where D^n denotes differentiation. However, in general, $J_n D^n \neq I$. That means that D^n is a left inverse. In fact we have

$$J_n D^n[f](t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}\,, \qquad \forall t > 0\,.$$

We also desire that D^α is a left inverse of J_α.

Definition 1.21 (Fractional derivative of order $\alpha > 0$:)**.** *Let $m-1 < \alpha \leq m$, then*

$$D^\alpha[f](t) := D^m J_{m-\alpha}[f](t)\,, \qquad \forall t > 0\,.$$

Taking into account the definition of $J_{m-\alpha}$ it follows that

$$D^\alpha[f](t) = \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau\right]\,, & m-1 < \alpha < m\,, \\ \frac{d^m}{dt^m}[f](t)\,, & \alpha = m\,. \end{cases} \tag{1.30}$$

Moreover, we define $D^0 = I_0 = I$.

We summarize a few basic properties of the fractional integral:

1. $D^\alpha J_\alpha = I$.
2. $$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}\,, \qquad \alpha > 0\,, \gamma > -1, t > 0\,.$$
3. Note that for $\gamma = 0$ the above formula provides:
$$D^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}\,, \qquad \alpha \geq 0\,, \quad t > 0\,.$$
Note that according to (1.30) this applies only when $\alpha \neq m$.

Remark 1.22. *The Abel integral equation* (1.19) *can be written as*

$$\Gamma(\alpha) J_\alpha[\phi](t) = f(t)\,.$$

Thus the Abel integral equation determines the α-th derivative of a function f.

Evaluating $J_\alpha[\phi]$ (differentiating ϕ) we call the forward problem. Solving the Abel equation is the inverse problem.

We emphasize that integration is stable and differentiation is unstable.

1.7.1 The Caputo Derivative

The Caputo derivative is slightly different defined as the fractional derivative:

Definition 1.23 (Caputo derivative of order $\alpha > 0$:)**.**

$$D_*^\alpha[f](t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{d^m f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\,, & m-1 < \alpha < m\,, \\ \frac{d^m}{dt^m}[f](t)\,, & \alpha = m\,. \end{cases} \tag{1.31}$$

We just summarize a few properties which highlight the differences of the derivatives:

- In general

$$D^\alpha f(t) := D^m J_{m-\alpha} f(t) \neq J_{m-\alpha} D^m f(t) = D_*^\alpha f(t)\,.$$

- $D_*^\alpha 1 \equiv 0$ for all $\alpha > 0$.
- Even more, we have $D_*^\alpha t^{\alpha-1} \equiv 0$, for all $\alpha > 0$ and $t > 0$.

Remark 1.24. *The different transforms are very useful in many inverse problems, because of the giant tables of explicit solutions.*

1.8 Abel transform

Definition 1.25. *The* Abel transform $\mathcal{A}[\psi]$ *of a smooth function* $\psi : \mathbb{R}_+ \to \mathbb{R}$*, which decays sufficiently fast to zero at* ∞*, is defined by*

$$\mathcal{A}[\psi](y) = \int_{-\infty}^{\infty} \psi(\sqrt{x^2+y^2})\, dx \quad y \geq 0. \tag{1.32}$$

From the definition of the Abel transform it follow that for given $y \geq 0$

$$\mathcal{A}[\psi](y) = \int_{-\infty}^{\infty} \psi(\sqrt{x^2+y^2})\, dx = 2\int_0^{\infty} \psi(\sqrt{x^2+y^2})\, dx\,.$$

Then, for $x, y > 0$ we substitute $r = \sqrt{x^2+y^2}$. This implies that

$$x = \sqrt{r^2-y^2} \text{ and } dr = \frac{x}{\sqrt{x^2+y^2}} dx = \frac{\sqrt{r^2-y^2}}{r} dx\,.$$

Thus
$$\mathcal{A}[\psi](y) = 2\int_0^\infty \psi(\sqrt{x^2+y^2})\,dx = 2\int_y^\infty \frac{r\psi(r)}{\sqrt{r^2-y^2}}\,dr\ .$$
Next, we express the Abel transform in reciprocal coordinates $y=\frac{1}{t}$ and find that
$$\mathcal{A}[\psi]\left(\frac{1}{t}\right) = 2\int_{\frac{1}{t}}^\infty \frac{r\psi(r)}{\sqrt{r^2-\frac{1}{t^2}}}\,dr = 2\int_{\frac{1}{t}}^\infty \frac{tr\psi(r)}{\sqrt{t^2r^2-1}}\,dr\ .$$
Substitution $r=\frac{1}{s}$, and thus $dr=-\frac{1}{s^2}ds$, it follows that

$$\mathcal{A}[\psi]\left(\frac{1}{t}\right) = 2\int_0^t \frac{t\frac{1}{s}\psi\left(\frac{1}{s}\right)}{\sqrt{\frac{t^2}{s^2}-1}}\frac{1}{s^2}\,ds = 2\int_0^t \frac{t\psi\left(\frac{1}{s}\right)}{s^2\sqrt{t^2-s^2}}ds\ . \tag{1.33}$$

Now, we use, what follows from (1.32) that for all $y \geq 0$

$$(\mathcal{A}[\psi])'(y) = \int_{-\infty}^\infty \psi(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}}\,dy\,.$$

Therefore it holds for all $v \geq 0$

$$\begin{aligned} &\int_{-\infty}^\infty \frac{(\mathcal{A}[\psi])'(\sqrt{u^2+v^2})}{\sqrt{u^2+v^2}}\,du \\ =&\int_{-\infty}^\infty \frac{1}{\sqrt{u^2+v^2}}\int_{-\infty}^\infty \sqrt{u^2+v^2}\frac{\psi'(\sqrt{x^2+u^2+v^2})}{\sqrt{x^2+u^2+v^2}}\,dx\,du \\ =&\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\psi'(\sqrt{x^2+u^2+v^2})}{\sqrt{x^2+u^2+v^2}}\,du\,dx\ . \end{aligned} \tag{1.34}$$

Substituting $x=\sqrt{\rho^2-v^2}\cos(\varphi)$ and $u=\sqrt{\rho^2-v^2}\sin(\varphi)$. we get

$$\frac{dx}{d\rho} = \frac{\rho}{\sqrt{\rho^2-v^2}}\cos(\varphi)\,,\quad \frac{dx}{d\varphi} = -\sqrt{\rho^2-v^2}\sin(\varphi)$$

and
$$\frac{du}{d\rho} = \frac{\rho}{\sqrt{\rho^2-v^2}}\sin(\varphi)\,,\quad \frac{du}{d\varphi} = \sqrt{\rho^2-v^2}\cos(\varphi)\ .$$

Thus
$$dudx = \rho d\varphi d\rho\ .$$

Using that $x^2+u^2+v^2=\rho^2$ it follows from (1.34) that

$$\int_{-\infty}^{\infty} \frac{(\mathcal{A}[\psi])'(\sqrt{u^2+v^2})}{\sqrt{u^2+v^2}}\,du = 2\pi \int_v^{\infty} \psi'(\rho)\,d\rho = -2\pi\psi(v).$$

Then, by using the definition of the Abel transform, it follows that:

$$\mathcal{A}\left[\frac{(\mathcal{A}[\psi])'(\cdot)}{\cdot}\right](v) = -2\pi\psi(v).$$

In other word this means that if we denote by $\mathcal{A}^{-1}[\psi]$ the *inverse Abel transform* of a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ that

$$\mathcal{A}^{-1}[\psi](y) = -\frac{1}{2\pi y}(\mathcal{A}[\psi])'(y). \tag{1.35}$$

For further expressing the inverse Abel transform we use that

$$\begin{aligned}
(\mathcal{A}[r^2\psi(r)])'(y) &= \partial_y\left(\int_{-\infty}^{\infty}(x^2+y^2)\psi(\sqrt{x^2+y^2})dx\right)\\
&= \partial_y\big(y^2\mathcal{A}[\psi](y)\big) + \partial_y\left(\int_{-\infty}^{\infty} x^2\psi(\sqrt{x^2+y^2})dx\right)\\
&= \partial_y\big(y^2\mathcal{A}[\psi](y)\big) + \int_{-\infty}^{\infty}\frac{x^2 y}{\sqrt{x^2+y^2}}\psi'(\sqrt{x^2+y^2})dx\\
&= \partial_y\big(y^2\mathcal{A}[\psi](y)\big) + y\int_{-\infty}^{\infty} x\left(\frac{x}{\sqrt{x^2+y^2}}\psi'(\sqrt{x^2+y^2})\right)dx
\end{aligned}$$

With integration by parts it then follows that

$$\begin{aligned}
(\mathcal{A}[r^2\psi(r)])'(y) &= \partial_y\big(y^2\mathcal{A}[\psi](y)\big) - y\int_{-\infty}^{\infty}\psi(\sqrt{x^2+y^2})dx\\
&= \partial_y\big(y^2\mathcal{A}[\psi](y)\big) - y\mathcal{A}[\psi](y)\\
&= 2y\mathcal{A}[\psi](y) + y^2\partial_y\big(\mathcal{A}[\psi](y)\big) - y\mathcal{A}[\psi](y)\\
&= y\left(\mathcal{A}[\psi](y) + y\partial_y\big(\mathcal{A}[\psi](y)\big)\right)\\
&= y\partial_y\big(y\mathcal{A}[\psi](y)\big)\ .
\end{aligned}$$

Using (1.35) we then have

$$-2\pi y\mathcal{A}^{-1}[r^2\psi(r)](y) = (\mathcal{A}[r^2\psi(r)])'(y) = y\partial_y\big(y\mathcal{A}[\psi](y)\big)$$

or in other words

$$\mathcal{A}^{-1}[r^2\psi](y) = -\frac{1}{2\pi}\partial_y\big(y\mathcal{A}[\psi](y)\big). \qquad (1.36)$$

The Abel-transform can be rewritten as the fractional integral $J_{1/2}$.

Lemma 1.26.

$$\boxed{\mathcal{A}[\psi](t^{-1/2}) = t^{1/2}J_{1/2}[\tau^{-3/2}\psi(\tau^{-1/2})](t)\ .} \qquad (1.37)$$

Proof. Using (1.33) it follows by the substitution $s^2 = \tau, 2s\,ds = d\tau$ that

$$\mathcal{A}[\psi]\left(\frac{1}{t}\right) = 2\int_0^t \frac{t\psi\left(\frac{1}{s}\right)}{s^2\sqrt{t^2-s^2}}\,ds = t\int_0^{t^2}\frac{\tau^{-3/2}\psi(\tau^{-1/2})}{\sqrt{t^2-\tau}}\,d\tau\ .$$

Thus, by putting $\hat{t} = t^2$ it follows that

$$\mathcal{A}[\psi]\left(\frac{1}{\sqrt{\hat{t}}}\right) = \hat{t}^{1/2}\int_0^{\hat{t}}\frac{\tau^{-3/2}\psi(\tau^{-1/2})}{\sqrt{\hat{t}-\tau}}\,d\tau = \hat{t}^{1/2}J_{1/2}[\tau^{-3/2}\psi(\tau^{-1/2})](\hat{t})\ .$$

□

Remark 1.27. *The identity* (1.37) *states that evaluation* $\mathcal{A}[\psi]$ *is equivalent to* $1/2$ *integration and solution of the Abel integral equation is* $1/2$*- times differentiation.*

Chapter 2

The Radon transform

We consider reconstructing $f : \mathbb{R}^2 \to \mathbb{R}$ given the values of integrals of f along lines. Thus the problem is to determine f from the *Radon Transform*

$$\boxed{\mathcal{R}[f](t,\theta) := \int_{L_{t,\theta}} f\,ds = \int_{s=-\infty}^{\infty} f\left(t\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + s\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right)\,ds\,,} \tag{2.1}$$

where $t \geq 0$ and $\theta \in [0, 2\pi)$ and

$$L_{t,\theta} = \left\{\left(t\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + s\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right) : s \in \mathbb{R}\right\}\,.$$

Because

$$\mathcal{R}[f](-t,\theta) = \mathcal{R}[f](t,\theta+\pi)\,,$$

We can extend the Radon transform for $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

2.1 Back-projection

Fundamental formulas for inverting the Radon transform are based on *back-projection*. Suppose we select a point $\binom{x_1}{x_2} = t\binom{\cos\theta}{\sin\theta}$, $t > 0$, then the vector $\vec{n} = \binom{\cos\theta}{\sin\theta}$ is orthogonal to the line $L_{t,\theta}$.

The first step in recovering f from the values of the Radon transform is the *back-projection*, which consists in calculating the average of all values of line integrals where the lines pass through $\binom{x_1}{x_2}$.

Definition 2.1. (back-projection) *Let $h := h(t, \theta)$ be a function in polar coordinates. The* back-projection *of h at a point $\vec{x} = \binom{x_1}{x_2}$ is defined by*

$$\mathcal{B}[h]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \frac{1}{\pi} \int_{\theta=0}^{\pi} h(x_1 \cos\theta + x_2 \sin\theta, \theta)\, d\theta \; .$$

Note that $t := x_1 \cos\theta + x_2 \sin\theta \binom{\cos\theta}{\sin\theta}$ is the projection of x on the normal vector $\binom{\cos\theta}{\sin\theta}$ of the Line $L_{t,\theta}$. See Figure 2.1. In particular the back-projection

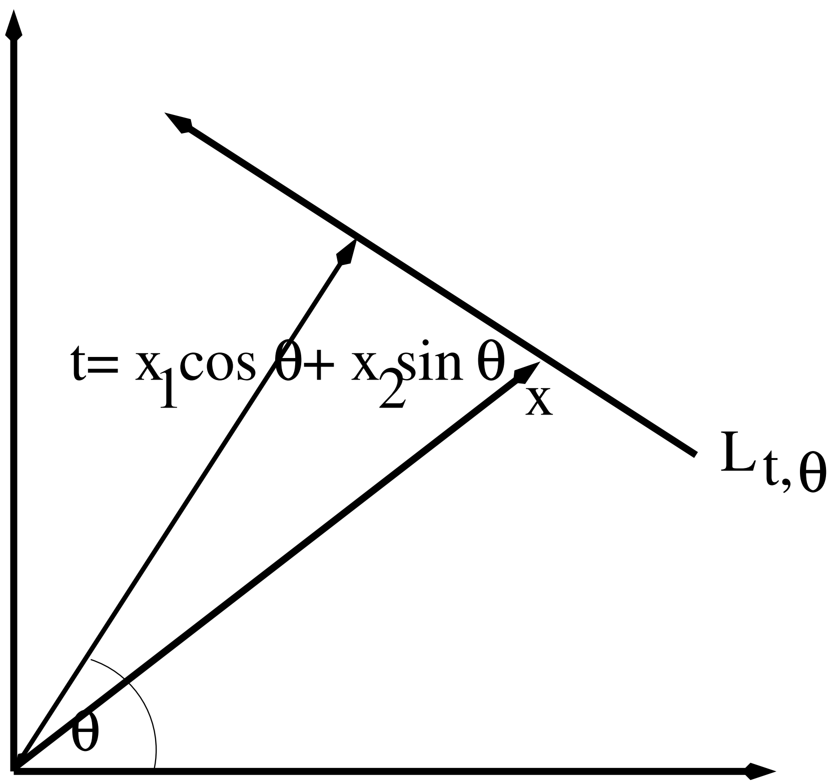

Figure 2.1: Visualization of back-projection

of the Radon transform is given by:

$$\mathcal{B}[\mathcal{R}[f]]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\pi} \int_{\theta=0}^{\pi} \mathcal{R}[f](x_1 \cos\theta + x_2 \sin\theta, \theta)\, d\theta \; .$$

In general just application of $\mathcal{B}$ to Radon data $\mathcal{R}[f]$ provides already an approximation of $2f$, however, it is not identical:

Example 2.2. • *Let*

$$f = \chi_{\{\vec{x}:1/4 \le |\vec{x}| \le 3/4\}} \ .$$

Then, again, for every line $L_{0,\theta}$ *through the origin, we have* $\mathcal{R}[f](0,\theta) = 1$. *Consequently* $1 = \mathcal{B}[\mathcal{R}[f]](0,0) \neq 2f(0,0) = 0$.

The Radon transform of the function

$$f = \chi_{\mathcal{B}_1(0)} \ .$$

is given by

$$\begin{aligned}\mathcal{R}[f](\tau,\theta) &= \int_{L_{\tau,\theta}} f(\vec{x})\, ds(\vec{x}) \\ &= \begin{cases} 2\sqrt{1-\tau^2} & \text{if } |\tau| \le 1\,, \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathcal{R}[f](x_1\cos\theta + x_2\sin\theta, \theta) \\ &= \int_{L_{x_1\cos\theta+x_2\sin\theta,\theta}} f(\vec{x})\, ds(\vec{x}) \\ &= \begin{cases} 2\sqrt{1-(x_1\cos\theta+x_2\sin\theta)^2} & \text{if } |x_1\cos\theta+x_2\sin\theta| \le 1\,, \\ 0 & \text{else.} \end{cases}\end{aligned}$$

For $\binom{x_1}{x_2}$ *in the support of* f *we have that* $|x_1\cos\theta + x_2\sin\theta| \le 1$.

Now, we apply back-projection and get

$$\begin{aligned}\mathcal{B}[\mathcal{R}[f]]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{1}{\pi}\int_{\theta=0}^{\pi} \mathcal{R}[f](x_1\cos\theta + x_2\sin\theta, \theta)\, d\theta \\ &= \frac{1}{\pi}\int_{\theta=0}^{\pi} 2\sqrt{1-(x_1\cos\theta+x_2\sin\theta)^2}\, d\theta \ .\end{aligned}$$

Plotting f *and* $\mathcal{B}[\mathcal{R}[f]]$ *reveals that the later is a smooth approximation.*

Note also, that for every line $L_{0,\theta}$ *through the origin, we have* $\mathcal{R}[f](0,\theta) = 2$. *Consequently* $\mathcal{B}[\mathcal{R}[f]](0,0) = 2 = 2f(0,0)$.

back-projection is a basic approximative algorithm for inverting the Radon transform.

2.2 Fourier slice theorem

Recall from (1.4) that the 2D Fourier transform of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}(\omega_1 x_1+\omega_2 x_2)} f(x_1,x_2)\, dx_1\, dx_2 \ . \tag{2.2}$$

Note that this is a composition of a Fourier transform from $t \to \omega_1$ and a Fourier transform from $s \to \omega_2$. All of the above integrals are, in general, understood as principal values.

The following fundamental relation between the Fourier transform and Radon transform is fundamental for exact inversion formulas:

Theorem 2.3. (Central slice theorem)

$$\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\begin{pmatrix}\omega\cos\theta\\ \omega\sin\theta\end{pmatrix} = \frac{1}{\sqrt{2\pi}}\mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) \ . \tag{2.3}$$

Proof. The 2D Fourier transform gives

$$\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\begin{pmatrix}\omega\cos\theta\\ \omega\sin\theta\end{pmatrix} = \frac{1}{2\pi}\int_{\mathbb{R}^2} f\begin{pmatrix}x_1\\ x_2\end{pmatrix} e^{\mathrm{i}\omega(x_1\cos\theta+x_2\sin\theta)}\, dx_1\, dx_2 \ . \tag{2.4}$$

We make the change of variables

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = t\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} + s\begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix} \ .$$

The determinant of the transformation matrix

$$\begin{bmatrix}\frac{\partial x_1}{\partial t} & \frac{\partial x_1}{\partial s}\\ \frac{\partial x_2}{\partial t} & \frac{\partial x_2}{\partial s}\end{bmatrix}.$$

is one, such that

$$dx_1\, dx_2 = ds\, dt \ .$$

Thus from (2.4) it follows that

$$\begin{aligned}
&\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\begin{pmatrix}\omega\cos\theta\\ \omega\sin\theta\end{pmatrix}\\
=&\frac{1}{2\pi}\int_{\mathbb{R}^2} f\begin{pmatrix}t\cos\theta - s\sin\theta\\ t\sin(\theta)+s\cos(\theta)\end{pmatrix} e^{\mathrm{i}\omega t}\,ds\,dt\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f\begin{pmatrix}t\cos\theta - s\sin\theta\\ t\sin(\theta)+s\cos(\theta)\end{pmatrix} ds\right) e^{\mathrm{i}\omega t}\,dt\\
=&\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{R}[f](t,\theta) e^{\mathrm{i}\omega t}\,dt\\
=&\frac{1}{\sqrt{2\pi}}\mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta)\;.
\end{aligned} \tag{2.5}$$

This shows the assertion. □

2.3 Filtered back-projection

The following theorem provides an exact formula for inverting of the Radon transform:

Theorem 2.4. (Filtered back-projection)

$$\boxed{f\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \frac{1}{2}\mathcal{B}\left(\mathcal{F}^{-1}_{\omega\to t}\left(|\omega|\,\mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta)\right)\right)\begin{pmatrix}x_1\\ x_2\end{pmatrix}\;.} \tag{2.6}$$

Proof. We use that $\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}$ and $\mathcal{F}^{-1}_{(\omega_1,\omega_2)\to(x_1,x_2)}$ are inverse to each other. Thus

$$\begin{aligned}
f\begin{pmatrix}x_1\\ x_2\end{pmatrix} =&\mathcal{F}^{-1}_{(\omega_1,\omega_2)\to(x_1,x_2)}\left[\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\right]\begin{pmatrix}x_1\\ x_2\end{pmatrix}\\
=&\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f]\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} e^{-\mathrm{i}(x_1\omega_1+x_2\omega_2)}\,d\omega_1\,d\omega_2
\end{aligned}$$

Using the change of variables

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \omega\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \text{ for } \omega\in\mathbb{R}, \theta\in[0,\pi)\,,$$

we get

$$d\omega_1 \, d\omega_2 = |\omega| \, d\omega \, d\theta \; .$$

Thus by the central slice theorem

$$\begin{aligned}
f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =& \frac{1}{2\pi} \int_0^\pi \int_{\mathbb{R}} \mathcal{F}_{(x_1,x_2)\to(\omega_1,\omega_2)}[f] \begin{pmatrix} \omega \cos\theta \\ \omega \sin\theta \end{pmatrix} e^{-\mathrm{i}\omega(x_1\cos\theta + x_2 \sin\theta)} \, |\omega| \; d\omega \, d\theta \\
=& \frac{1}{(2\pi)^{3/2}} \int_0^\pi \int_{\mathbb{R}} \mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) e^{-\mathrm{i}\omega(x_1\cos\theta + x_2 \sin\theta)} \, |\omega| \; d\omega \, d\theta \\
=& \frac{1}{(2\pi)} \int_0^\pi \mathcal{F}^{-1}_{\omega\to t} \left(|\omega| \, \mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) \right) (x_1\cos\theta + x_2\sin\theta, \theta) \, d\theta \\
=& \frac{1}{2} \mathcal{B} \left(\mathcal{F}^{-1}_{\omega\to t} \left(|\omega| \, \mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) \right) \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .
\end{aligned} \tag{2.7}$$

□

The name filtered back-projection is due to its close relation to back-projection, which misses out just the $|\omega|$ term. The multiplication by $|\omega|$ can be interpreted as a filtering of the data, which becomes more clear by the below calculations:

Let $f : \mathbb{R} \to \mathbb{R}$, then because of (1.8) we have

$$\mathcal{F}_{t\to\omega}[f'](\omega) = -\mathrm{i}\omega \mathcal{F}_{t\to\omega}[f](\omega) \, , \tag{2.8}$$

and therefore, in particular,

$$\mathcal{F}_{t\to\omega} \left[\frac{\partial}{\partial t} \mathcal{R}[f] \right] (\omega,\theta) = -\mathrm{i}\omega \mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) \; .$$

Thus

$$\mathrm{iSign}(\omega) \mathcal{F}_{t\to\omega} \left[\frac{\partial}{\partial t} \mathcal{R}[f] \right] (\omega,\theta) = |\omega| \, \mathcal{F}_{t\to\omega}[\mathcal{R}[f]](\omega,\theta) \; .$$

Consequently it follows from (2.7)

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \mathcal{B} \left[\mathcal{F}^{-1}_{\omega\to t} \left[\mathrm{iSign}(\omega) \mathcal{F}_{t\to\omega} \left[\frac{\partial}{\partial t} \mathcal{R}[f] \right] (\omega,\theta) \right] \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} . \tag{2.9}$$

Recall from (1.22) that the operator in the middle of the right hand side

$$\mathcal{H}[g](t) = \mathcal{F}^{-1}_{\omega\to t} \left(\mathrm{iSign}(\omega) \mathcal{F}_{t\to\omega}[g](\omega) \right) (t)$$

is the *Hilbert transform*. With this (2.9) rewrites to

$$\boxed{f\begin{pmatrix}x_1\\x_2\end{pmatrix} = \frac{1}{2}\mathcal{B}\left[\mathcal{H}\left[\frac{\partial}{\partial t}\mathcal{R}[f]\right]\right]\begin{pmatrix}x_1\\x_2\end{pmatrix}.} \tag{2.10}$$

This is the original Radon inversion formula [11].

Chapter 3

Wave equation and spherical means

We consider the initial value problem for the wave equation

$$\boxed{\begin{aligned} &\partial_{tt}u - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty)\,, \\ &u = g, \partial_t u = h \text{ in } \mathbb{R}^n \times \{t=0\}\ . \end{aligned}} \tag{3.1}$$

Definition 3.1. *Let $x \in \mathbb{R}^n$ and $r \geq 0$. The* spherical mean operator *in $\mathbb{R}^n$ of an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$\boxed{\mathcal{M}_n[f](x;r) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} f(x+r\theta)ds(\theta),} \tag{3.2}$$

where $|\mathbb{S}^{n-1}|$ denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and $ds(\theta)$ denotes the surface measure.

Lemma 3.2. *Let*

$$F(x;r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} f(y)\,ds(y) \text{ for } n = 2,3,\dots\ .$$

Then

$$F(x;r) = \mathcal{M}_n[f](x;r)\ .$$

Proof. Introducing the coordinate transformation

$$\begin{aligned} \Psi : \mathbb{S}^{n-1} &\to \partial B(x,r) \subseteq \mathbb{R}^n\,, \\ \theta &\to x + r\theta \end{aligned}$$

we then find that

$$\nabla_{\mathbb{S}^{n-1}} \Psi(\theta) = rI \ .$$

and consequently

$$|\det \nabla_{\mathbb{S}^{n-1}} \Psi(\theta)| = r^{n-1} \ .$$

Consequently it follows by change of variables:

$$\begin{aligned} F(x;r) &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} f(y)\, ds(y) \\ &= \frac{1}{r^{n-1}\, |\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(x + r\theta)\, |\det \nabla \Psi(\theta)| \; ds(\theta) \\ &= \mathcal{M}_n[f](x;r) \ . \end{aligned}$$

□

We have the following relation between the solution u of (3.1) and the spherical means of the initial data g and h (see [5]):

- For $n = 2$:

$$\begin{aligned} u(x;t) = \frac{1}{2\pi} \Bigg[\partial_t \left(\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}}\, dy \right) \\ + \int_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}}\, dy \Bigg] \ . \end{aligned} \tag{3.3}$$

In particular, if $h \equiv 0$ in (3.1) it follows that

$$\begin{aligned} u(x;t) &= \frac{1}{2\pi}\partial_t \left(\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\, dy \right) \\ &= \frac{1}{2\pi}\partial_t \left(\int_0^t \int_{\partial B(x,\tau)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\, ds(y)\, d\tau \right) \\ &= \frac{1}{2\pi}\partial_t \left(\int_0^t \frac{1}{\sqrt{t^2-\tau^2}} \int_{\partial B(x,\tau)} g(y)\, ds(y)\, d\tau \right) \\ &= \frac{1}{2\pi}\partial_t \left(\int_0^t \frac{2\pi\tau}{\sqrt{t^2-\tau^2}} \frac{1}{2\pi\tau} \int_{\partial B(x,\tau)} g(y)\, ds(y)\, d\tau \right) \\ &\underbrace{=}_{\text{Lemma3.2}} \partial_t \left(\int_0^t \frac{\tau}{\sqrt{t^2-\tau^2}} \mathcal{M}_2[g](x,\tau)\, d\tau \right). \end{aligned} \tag{3.4}$$

Defining

$$\tilde{g}_x(\tfrac{1}{r}) := r^3 \mathcal{M}_2[g](x,r)$$

we get an expression for the solution of the wave equation in terms of the Abel transform:

$$\begin{aligned} u(x;t) &= \partial_t \left(\int_0^t \frac{\tau \mathcal{M}_2[g](x;\tau)}{\sqrt{t^2-\tau^2}} d\tau \right) \\ &= \partial_t \left(\int_0^t \frac{\tau^3 \mathcal{M}_2[g](x;\tau)}{\tau^2\sqrt{t^2-\tau^2}} d\tau \right) \\ &= \partial_t \left(\int_0^t \frac{\tilde{g}_x\left(\frac{1}{\tau}\right)}{\tau^2\sqrt{t^2-\tau^2}} d\tau \right) \\ &= \partial_t \left(\frac{1}{2t} \left(2\int_0^t \frac{t\tilde{g}_x\left(\frac{1}{\tau}\right)}{\tau^2\sqrt{t^2-\tau^2}} d\tau \right) \right) \\ &\underbrace{=}_{(1.33)} \partial_t \left(\frac{1}{2t} \mathcal{A}[\tilde{g}_x](\tfrac{1}{t}) \right). \end{aligned} \tag{3.5}$$

- For $n = 3$:

$$\begin{aligned} u(x;t) &= \frac{1}{4\pi} \left[\partial_t \left(\frac{1}{t} \int_{\partial B(x,t)} g(y)\, ds(y) \right) + \frac{1}{t} \int_{\partial B(x,t)} h(y)\, ds(y) \right] \\ &= \partial_t \left(t\mathcal{M}_3[g](x;t) \right) + t\mathcal{M}_3[h](x;t)\,. \end{aligned} \tag{3.6}$$

In particular, if $h \equiv 0$ in (3.1) we have

$$\boxed{u(x;t) = \partial_t \left(t\mathcal{M}_3[g]\right)(x;t)\ .} \tag{3.7}$$

Remark 3.3. *The spherical mean operator can be expressed in terms of a* $1D - \delta$ *distribution:*

$$\boxed{\mathcal{M}_n[g](x,r) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{R}^n} g(z)\frac{\delta(r-|z-x|)}{|z-x|^{n-1}}dz\ .}$$

To see this, note that from Lemma 3.2 and the definition of the $1D - \delta$ *distribution it follows that*

$$\begin{aligned}\mathcal{M}_n[g](x,r) &= \int_0^\infty \frac{1}{\tilde{r}^{n-1}\,|\mathbb{S}^{n-1}|}\int_{\partial B(0,\tilde{r})} g(x+y)\,ds(y)\delta(\tilde{r}-r)\,d\tilde{r} \\ &= \frac{1}{|\mathbb{S}^{n-1}|}\int_0^\infty\int_{\partial B(0,\tilde{r})} \frac{1}{\tilde{r}^{n-1}} g(x+y)\delta(\tilde{r}-r)\,d\tilde{r}\,ds(y)\ .\end{aligned}$$

Now, setting $z = x + y$ *and noting that* $\tilde{r} = |y| = |z - x|$ *it follows that*

$$\mathcal{M}[g](x,r) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{R}^3}\frac{1}{|z-x|^{n-1}} g(z)\delta(|z-x|-r)\,dz\ .$$

The desired representation then follows from the property (1.1) *of the* δ*-distribution.*

Chapter 4

Photoacoustic imaging

Definition 4.1.

- *Let p satisfy the wave equation*

$$\begin{aligned} \partial_{tt}p - \Delta p = 0 \ \textit{in}\ \mathbb{R}^n \times (0,\infty)\,, \\ p = p_0, \quad \partial_t p = 0 \ \textit{in}\ \mathbb{R}^n \times \{t=0\}\ . \end{aligned} \tag{4.1}$$

Photoacoustic imaging consists in determining the initial datum p_0 in the wave equation from the following measurements of p:

$$m(x,t) \ \textit{for}\ x \in \Gamma \ \textit{and}\ t > 0\ .$$

- *The inverse problem of integral geometry consists in determining g from measurements of the spherical mean operator $\mathcal{M}_n[f](x,t)$ for $x \in \Gamma$ and $t > 0$.*

4.1 Point measurements along a line

Maybe the simplest reconstruction formulas can be derived for point detector measurement along the real line axis for photoacoustic imaging in $\mathbb{R}^2$:

$$m(x,t) = p(x,0,t) \text{ for } x \in \mathbb{R}\,, t > 0\ .$$

We assume that the support of p_0 is entirely above the x-axis, that is:

$$\operatorname{supp}(p_0) \subset \{(x,y) : y > 0\}\,. \tag{4.2}$$

Applying the Fourier transform to the Laplacian of p gives

$$\begin{aligned}
&\mathcal{F}_{(x,y)\to(k_x,k_y)}[\Delta p](k_x,k_y,t)\\
=&-(k_x^2+k_y^2)\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}k_x x+\mathrm{i}k_y y} p(x,y,t)\,dxdy\\
=&-\left|\vec{k}\right|^2 \mathcal{F}_{(x,y)\to(k_x,k_y)}[p](k_x,k_y,t) \qquad \forall \vec{k}\in\mathbb{R}^2, t>0.
\end{aligned}$$

Taking the Fourier-transform of the differential equation (4.1) with respect to (x,y) it follows that

$$\begin{aligned}
&\partial_{tt}\mathcal{F}_{(x,y)\to(k_x,k_y)}[p](k_x,k_y,t)+\left|\vec{k}\right|^2 \mathcal{F}_{(x,y)\to(k_x,k_y)}[p](k_x,k_y,t)=0\,,\\
&\qquad \forall \vec{k}\in\mathbb{R}^2, t>0.
\end{aligned} \tag{4.3}$$

This is an ordinary differential equation in t for $v := \mathcal{F}_{(x,y)\to(k_x,k_y)}[p]$, and has the following solution:

$$\begin{aligned}
\mathcal{F}_{(x,y)\to(k_x,k_y)}[p](k_x,k_y,t) = C_1(k_x,k_y)e^{\mathrm{i}\left|\vec{k}\right|t}+C_2(k_x,k_y)e^{-\mathrm{i}\left|\vec{k}\right|t}\,,\\
\forall \vec{k}\in\mathbb{R}^2, t>0.
\end{aligned} \tag{4.4}$$

At this point we incorporate the initial conditions. By taking the Fourier transform of the initial conditions in (4.1) it follows

$$\begin{aligned}
&C_1(k_x,k_y)+C_2(k_x,k_y)=\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y) \text{ and}\\
&C_1(k_x,k_y)-C_2(k_x,k_y)=0 \qquad \forall \vec{k}\in\mathbb{R}^2\,.
\end{aligned}$$

Therefore

$$C_1(k_x,k_y)=C_2(k_x,k_y)=\frac{1}{2}\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y)\,, \qquad \forall \vec{k}\in\mathbb{R}^2\,. \tag{4.5}$$

Because,

$$\frac{1}{2}\left(e^{\mathrm{i}\left|\vec{k}\right|t}+e^{-\mathrm{i}\left|\vec{k}\right|t}\right)=\cos\left(\left|\vec{k}\right|t\right)\,, \tag{4.6}$$

it follows from (4.5) in (4.4) that

$$\begin{aligned}
&\mathcal{F}_{(x,y)\to(k_x,k_y)}[p](k_x,k_y,t)\\
=&\frac{1}{2}\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y)\left(e^{\mathrm{i}\left|\vec{k}\right|t}+e^{-\mathrm{i}\left|\vec{k}\right|t}\right)\,,\\
=&\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y)\cos\left(\left|\vec{k}\right|t\right)\\
&\qquad \forall \vec{k}\in\mathbb{R}^2, t\geq 0.
\end{aligned} \tag{4.7}$$

Taking the inverse Fourier transform $\mathcal{F}^{-1}_{(x,y)\to(k_x,k_y)}$ in (4.7) and evaluating the result at $(x,0,t)$ for $x \in \mathbb{R}$ and $t > 0$ gives:

$$\begin{aligned} m(x,t) &:= p(x,0,t) \\ &= \mathcal{F}^{-1}_{(k_x,k_y)\to(x,y)}[\mathcal{F}_{(x,y)\to(k_x,k_y)}[p]](x,0,t) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathrm{i}k_x x - \mathrm{i}k_y y|_{y=0}} \cos\left(\left|\vec{k}\right| t\right) \mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y) dk_x dk_y \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathrm{i}k_x x} \cos\left(\left|\vec{k}\right| t\right) \mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y) dk_x dk_y \end{aligned} \tag{4.8}$$

We use the one-to-one transformation

$$\begin{aligned} S : \mathbb{R}^2 &\to A := \{(k_x,\omega) : |k_x| \le |\omega|, \omega \in \mathbb{R}\}\ , \\ (k_x,k_y) &\to \left(k_x, \omega := \mathrm{Sign}(k_y)\sqrt{k_x^2 + k_y^2}\right) \end{aligned}$$

which has the inverse

$$\begin{aligned} &S^{-1} : A \to \mathbb{R}^2\,, \\ &(k_x,\omega) \to \left(k_x, k_y = \mathrm{Sign}(\omega)\sqrt{\omega^2 - k_x^2}\right)\ , \forall (k_x,\omega) \in A\ . \end{aligned}$$

Note that we have for all $(k_x,\omega) \in A$:

$$\left|\vec{k}\right| = |\omega| \ \text{ and } dk_y = \frac{\mathrm{Sign}(\omega)\omega}{\sqrt{\omega^2 - k_x^2}} d\omega = \frac{|\omega|}{\sqrt{\omega^2 - k_x^2}} d\omega\ . \tag{4.9}$$

We define

$$\begin{aligned} &g : \mathbb{R}^2 \to \mathbb{R} \\ &(k_x,\omega) \to \begin{cases} \mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0]\left(k_x, \mathrm{Sign}(\omega)\sqrt{\omega^2 - k_x^2}\right) \frac{|\omega|}{\sqrt{\omega^2 - k_x^2}} & \text{for} \quad (k_x,\omega) \in A\,, \\ 0 & \text{else}\ . \end{cases} \end{aligned}$$

Therefore

$$
\begin{aligned}
&m(x,t) \\
\underbrace{=}_{(4.8)}&\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathrm{i}k_x x}\cos\left(\left|\vec{k}\right| t\right)\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y)dk_x dk_y \\
\underbrace{=}_{(4.9)}&\frac{1}{2\pi}\int_{A} e^{-\mathrm{i}k_x x}\cos(\omega t)\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,\mathrm{Sign}(\omega)\sqrt{\omega^2-k_x^2})\frac{|\omega|}{\sqrt{\omega^2-k_x^2}}d\omega dk_x \\
=&\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-\mathrm{i}k_x x}\cos(\omega t)g(k_x,\omega)d\omega dk_x \\
\underbrace{=}_{(1.9)}&\frac{1}{2}\mathcal{C}^{-1}_{\omega\to t}\left[\mathcal{F}^{-1}_{k_x\to x}[g]\right](x,t)\ .
\end{aligned}
$$

Thus we have shown that

$$
g(k_x,\omega) = 2\mathcal{C}_{t\to\omega}\left[\mathcal{F}_{x\to k_x}[m]\right](k_x,\omega)\,,\forall(k_x,\omega)\in\mathbb{R}^2\ . \tag{4.10}
$$

Now, we note that

$$
\frac{\omega}{\sqrt{\omega^2-k_x^2}} = \frac{\sqrt{k_x^2+k_y^2}}{k_y}
$$

and thus from (4.10) it follows that

$$
\begin{aligned}
&\mathcal{F}_{(x,y)\to(k_x,k_y)}[p_0](k_x,k_y) \\
=&2\frac{|k_y|}{\sqrt{k_x^2+k_y^2}}\mathcal{C}_{t\to\omega}\left[\mathcal{F}_{x\to k_x}[m]\right](k_x,\omega=\mathrm{Sign}(k_y)\sqrt{k_x^2+k_y^2}) \\
&\quad\forall\vec{k}\in\mathbb{R}^2\ .
\end{aligned}
$$

In other words:

$$
\boxed{
\begin{aligned}
&p_0(x,y) \\
=&2\mathcal{F}^{-1}_{(k_x,k_y)\to(x,y)}\left(\frac{|k_y|}{\sqrt{k_x^2+k_y^2}}\mathcal{C}_{t\to\omega}\left[\mathcal{F}_{x\to k_x}[m]\right](k_x,\omega=\mathrm{Sign}(k_y)\sqrt{k_x^2+k_y^2})\right)\ . \\
&\quad\forall(x,y)\in\mathbb{R}^2\ .
\end{aligned}
}
\tag{4.11}
$$

4.1.1 Stability estimates

In the following we cite some stability estimates for the spherical mean operator and the wave operator.

Definition 4.2. *The wave operator in* $\mathbb{R}^n$, $\mathcal{W}_n$ *maps the initial datum* p_0 *onto* $p_{\Gamma\times(0,\infty)}$. *That is*

$$\mathcal{W}_n[p_0] = p_{\Gamma\times(0,\infty)}\,, \tag{4.12}$$

where p *solves* (4.1).

Definition 4.3. *Let* C *be a bounded, open, and connected subset of* $\mathbb{R}^n$. *We denote by*

$$H_0^s(C) := \{f \in H^s(C) : supp(f) \subset C\}\ ,$$

the Sobolev space of s*-times differentiable functions which have support in* C.

Let $f : C \times (0,\infty) \to \mathbb{R}$, *then*

$$\|f\|^2_{H_0^{s,\rho}(C)} = \int_{C\times(0,\infty)} \|\nabla^s f\|^2\, dx\, dt + \int_{C\times(0,\infty)} \|\partial_t^\rho f\|^2\, dx\, dt\ .$$

Remark 4.4. *In particular let* Γ *be a closed simply connected curve in* $\mathbb{R}^n$, *and let* $\tilde{C}$ *be a bounded open set in* $(0,\infty)$. *Then for* $\hat{C} = \Gamma\times\tilde{C}$

$$H_0^s(\hat{C}) := \left\{f \in H^s(\hat{C}) : supp(f) \subset \hat{C}\right\}\ .$$

The following estimate from [2] concerns the spherical mean operator:

Theorem 4.5. *(Proposition 21 in [2]) Let*

$$C = \mathcal{B}(0,1-\varepsilon) \subseteq \mathbb{R}^n\,, \Gamma = \partial\mathcal{B}(0,1)\,, \tilde{C} = (\varepsilon, 2-\varepsilon)\,, \hat{C} = \Gamma\times\tilde{C}\,,$$

and $s \geq 0$ *and* $\varepsilon > 0$.

Then there exists a constant C_ε *such that for* $p_0 \in H_0^s(C)$

$$C_\varepsilon^{-1}\,\|\mathcal{M}_n[p_0]\|_{H_0^{s+(n-1)/2}(\hat{C})} \leq \|p_0\|_{H_0^s(C)} \leq C_\varepsilon\,\|\mathcal{M}_n[p_0]\|_{H_0^{s+(n-1)/2}(\hat{C})}\ . \tag{4.13}$$

4.2 Sectional Imaging

We assume that the laser pulse which illuminates the sample is perfectly focused onto the plane

$$\{x \in \mathbb{R}^3 : x_3 = 0\} .$$

In this case the initial pressure distribution $p_0 : \mathbb{R}^3 \to \mathbb{R}$ can be considered to be of the form

$$\boxed{p_0(\xi, z) = \hat{p}_0(\xi)\delta(z), \quad \xi \in \mathbb{R}^2,\ z \in \mathbb{R},} \tag{4.14}$$

for some smooth function $\hat{p}_0 : \mathbb{R}^2 \to \mathbb{R}$ and with δ denoting the 1D delta-function. Let us denote by

$$x = (\xi, z) \text{ with } \xi \in \mathbb{R}^2\,, z \in \mathbb{R}\,,$$

then (4.1) reads as follows

$$\begin{aligned} \partial_{tt} p(\xi, z; t) &= \Delta_{\xi,z} p(\xi, z; t), \\ \partial_t p(\xi, z; 0) &= 0, \\ p(\xi, z; 0) &= p_0(\xi, z) = \hat{p}_0(\xi)\delta(z)\ . \end{aligned} \tag{4.15}$$

Here $\Delta_{\xi,z} = \partial_{\xi_1\xi_1} + \partial_{\xi_2\xi_2} + \partial_{zz}$.

The goal of photoacoustic sectional imaging is to recover the function $\hat{p}_0$. We assume that the detectors collect measurements on the boundary $\partial\Omega$ of a convex domain $\Omega \subset \mathbb{R}^2$ in the illumination plane, where we additionally assume that $\hat{p}_0$ has compact support in Ω.

We will consider the following four different measurement setups and derive reconstruction formulas.

Vertical Line Detectors: The measurement data are

$$m_1(\xi; t) := \int_{-\infty}^{\infty} p(\xi, z; t) dz \text{ for all } \xi \in \partial\Omega,\ t > 0. \tag{4.16}$$

Practically this is realized with line detectors which measure the overall pressure along a line orthogonal to the illumination plane.

Point Detectors: The measurement data are

$$m_2(\xi; t) := p(\xi, 0; t) \text{ for all } \xi \in \partial\Omega,\ t > 0. \tag{4.17}$$

Practically this is realized with standard ultrasound detectors. This measurement geometry is used in [9].

For the other two measurement methods, we additionally impose that the domain $\Omega \subset \mathbb{R}^2$ is strictly convex and bounded.

Vertical Plane Detectors: The measurement data are

$$m_3(\theta;t) := \int_{P(\theta)} p(x;t)ds(x) \text{ for all } \theta \in \mathbb{S}^1,\ t>0, \tag{4.18}$$

where $P(\theta) \subset \mathbb{R}^3$ denotes the tangential plane of the cylinder $\partial\Omega \times \mathbb{R}$ orthogonal to the vector $(\theta, 0)$. Practically this is realized with planar detectors which are moved tangentially to $\partial\Omega$ around the object and measure the averaged pressure on the plane.

Horizontal Line Detectors: The measurement data are

$$m_4(\theta;t) := \int_{T(\theta)} p(\xi,0;t)ds(\xi) \text{ for all } \theta \in \mathbb{S}^1,\ t>0, \tag{4.19}$$

where $T(\theta) \subset \mathbb{R}^2$ denotes the tangential line of $\partial\Omega$ orthogonal to the vector θ, see (4.20). This is a realization using line detectors which measure the overall pressure on a line tangential to $\partial\Omega$ in the illumination plane, see [10]. (In these papers, they use for the reconstruction a phenomenologically motivated formula whose structure is very similar to the formula (4.28) which we derive for this sort of measurements.)

In those cases where the domain Ω is strictly convex and bounded, we parametrize the boundary $\partial\Omega$ with the map $\zeta : \mathbb{S}^1 \to \partial\Omega$ which associates to every unit vector $\theta \in \mathbb{S}^1$ the point $\zeta(\theta) \in \partial\Omega$ where the outward unit normal vector of $\partial\Omega$ coincides with θ, see Figure 4.1.

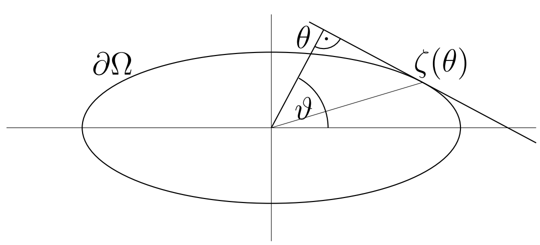

Figure 4.1: Definition of the point $\zeta(\theta)$, $\theta = (\cos\vartheta, \sin\vartheta)$.

Since the tangent line $T(\theta)$ of $\partial\Omega$ at $\zeta(\theta)$ is orthogonal to θ, we can define the family $T(r,\theta)$, $r \in \mathbb{R}$, of lines parallel to the tangent $T(\theta)$ by

$$T(r,\theta) = \zeta(\theta) + r\theta + \mathbb{R}\theta^{\perp} \subset \mathbb{R}^2, \quad T(\theta) = T(0,\theta), \quad \forall\theta \in \mathbb{S}^1\,, r \in \mathbb{R}\,. \tag{4.20}$$

Here, $\theta^{\perp} \in \mathbb{S}^1$ denotes a unit vector orthogonal to θ.

Moreover, we introduce the family $P(r,\theta)$, $r \in \mathbb{R}$, of planes parallel to the tangent plane $P(\theta)$ of the cylinder $\partial\Omega \times \mathbb{R}$ at $(\zeta(\theta), 0)$ by

$$P(r,\theta) = (T(\theta), 0) + (0, \mathbb{R}) \subset \mathbb{R}^3, \quad P(\theta) = P(0,\theta), \tag{4.21}$$

for every $\theta \in \mathbb{S}^1$ and $r \in \mathbb{R}$.

4.2.1 Reconstruction Methods

In the following, we derive reconstruction formulas for photoacoustic sectional imaging.

Measurements with Vertical Line Detectors

We introduce the function

$$\tilde{p}(\xi;t) = \int_{-\infty}^{\infty} p(\xi,z;t)dz, \quad \xi \in \mathbb{R}^2,\ t \geq 0. \tag{4.22}$$

Then the inital value problem (4.15) for the function p implies that the function $\tilde{p}$ satisfies the two-dimensional wave equation

$$\partial_{tt}\tilde{p}(\xi;t) = \Delta_{\xi}\tilde{p}(\xi;t) \quad \text{for all} \quad \xi \in \mathbb{R}^2,\ t > 0$$

with the initial conditions

$$\begin{aligned} \partial_t\tilde{p}(\xi;0) &= 0 && \text{for all} \quad \xi \in \mathbb{R}^2, \\ \tilde{p}(\xi;0) &= \hat{p}_0(\xi) && \text{for all} \quad \xi \in \mathbb{R}^2. \end{aligned}$$

The initially three-dimensional reconstruction problem therefore reduces to the two-dimensional problem of calculating $\hat{p}_0(\xi) = \tilde{p}(\xi;0)$, $\xi \in \mathbb{R}^2$, from the measurement data

$$m_1(\xi;t) = \tilde{p}(\xi;t), \quad \xi \in \partial\Omega,\ t > 0.$$

Reconstruction Formulas Based on Series Expansions

For special domains Ω, explicit reconstruction formulas are known: see the review [7] for Ω a circle and the half-space. The derivation for the ellipse is published in [4].

If Ω is the half-space $\{\xi \in \mathbb{R}^2 : \xi_2 > 0\}$, we get from (4.23) that

$$\begin{aligned}&\hat{p}_0(\xi_1,\xi_2)\\ =&2\mathcal{F}^{-1}_{(k_{\xi_1},k_{\xi_2})\to(\xi_1,\xi_2)}\left(\frac{|k_{\xi_2}|}{\sqrt{k_{\xi_1}^2+k_{\xi_2}^2}}\mathcal{C}_{t\to\omega}\left[\mathcal{F}_{\xi_1\to k_{\xi_1}}[m_1]\right](k_{\xi_1},\omega=\mathrm{Sign}(k_{\xi_2})\sqrt{k_{\xi_1}^2+k_{\xi_2}^2})\right).\\ &\quad\forall(\xi_1,\xi_2)\in\mathbb{R}^2\,.\end{aligned} \tag{4.23}$$

Measurements with Point Detectors

From equation (3.7), we know that the solution of the initial value problem (4.15) can be written as

$$p(x;t) = \partial_t\left(\frac{1}{4\pi t}\int_{\partial B(0,t)} f(x+y)ds(y)\right), \qquad \forall x \in \mathbb{R}^3 \text{ and } t > 0\,.$$

Parameterizing the sphere $\partial B(0,t)$ in cylindrical coordinates,

$$\partial B(0,t) = \left\{(\sqrt{t^2-h^2}\,(\cos(\theta),\sin(\theta)),h) \;:\; h\in[-t,t],\ \theta\in[0,2\pi)\right\},$$

we find for every $x=(\xi,z)$, $\xi\in\mathbb{R}^2$, $z\in\mathbb{R}$, and $t>0$ that

$$p(\xi,z;t) = \partial_t\left(\frac{1}{4\pi t}\int_{-t}^{t}\int_0^{2\pi}\hat{p}_0(\xi+\sqrt{t^2-h^2}(\cos(\theta),\sin(\theta)))\delta(z+h)td\theta dh\right).$$

Integrating out the δ-distribution, we get for $z\in[-t,t]$

$$p(\xi,z;t) = \partial_t\left(\frac{1}{4\pi}\int_0^{2\pi}\hat{p}_0(\xi+\sqrt{t^2-z^2}\,(\cos(\theta),\sin(\theta)))d\theta\right).$$

Using polar coordinate transformation, we find that

$$p(\xi,z;t) = \partial_t\left(\frac{1}{4\pi}\int_{\mathbb{S}^1}\hat{p}_0(\xi+\sqrt{t^2-z^2}\,\psi)ds(\psi)\right). \tag{4.24}$$

By the definition (3.2) of the spherical mean operator $\mathcal{M}_2$, this means

$$p(\xi, z; t) = \frac{1}{2}\partial_t\big(\mathcal{M}_2[\hat{p}_0](\xi; \sqrt{t^2 - z^2})\big) \quad \text{for} \quad z \in [-t, t]. \tag{4.25}$$

From the assumption that the support of $\hat{p}_0$ lies completely in Ω, we know that $\mathcal{M}_2[\hat{p}_0](\xi; 0) = \hat{p}_0(\xi) = 0$ for $\xi \notin \Omega$. Thus, we can integrate the relation (4.25) for $\xi \notin \Omega$ and find for every $z \in [-t, t]$ that

$$\mathcal{M}_2[\hat{p}_0](\xi; \sqrt{t^2 - z^2}) = 2\int_z^t p(\xi, z; \tilde{t})d\tilde{t}.$$

Setting $z = 0$, we get for every $\xi \in \partial\Omega$ and every $t > 0$ the relation

$$\mathcal{M}_2[\hat{p}_0](\xi; t) = 2\int_0^t m_2(\xi; \tilde{t})d\tilde{t}.$$

Having calculated the spherical mean of $\hat{p}_0$, we can now proceed as in Section 3.

Measurements with Vertical Plane Detectors

For every $\theta \in \mathbb{S}^1$, we define for $r \in \mathbb{R}$ and $t \geq 0$ the function

$$\tilde{p}_\theta(r; t) = \int_{P(r,\theta)} p(x; t)ds(x)\,,$$

where $P(r, \theta)$ denotes the plane as defined in (4.21).

Then, since the vectors $(\theta, 0)$, $(\theta^\perp, 0)$, and $(0, 0, 1)$ form an orthonormal basis of $\mathbb{R}^3$ and the Laplacian is rotationally invariant, we find from equation (4.15) that

$$\partial_{tt}\tilde{p}_\theta(r; t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta_x p(\zeta(\theta) + r\theta + u\theta^\perp, z; t)dudz = \partial_{rr}\tilde{p}_\theta(r; t)$$

for every $r \in \mathbb{R}$ and $t > 0$. Thus, $\tilde{p}_\theta$ solves the one-dimensional wave equation with the initial conditions

$$\begin{aligned} \partial_t\tilde{p}_\theta(r; 0) &= 0 && \text{for all } r \in \mathbb{R} \text{ and} \\ \tilde{p}_\theta(0; t) &= m_3(\theta; t) && \text{for all } t > 0 \end{aligned}$$

resulting from (4.15) and (4.18), respectively. Moreover, since $\hat{p}_0$ has its support inside Ω, we know that $\tilde{p}_\theta(r;0) = 0$ for $r \geq 0$.

With d'Alembert's formula for the solution of the one-dimensional wave equation, we find that the unique solution for this initial value problem is given by

$$\tilde{p}_\theta(r;t) = m_3(\theta; -t-r) + m_3(\theta; t-r), \quad r \in \mathbb{R},\ t > 0,$$

where we set $m_3(\theta;t) = 0$ for $t \leq 0$.

Finally, we have to recover from the values of $\tilde{p}_\theta$, $\theta \in \mathbb{S}^1$, the initial pressure distribution p_0 from equation (4.15). We have the relation

$$\tilde{p}_\theta(r;0) = \int_{-\infty}^{\infty} \hat{p}_0(\zeta(\theta) + r\theta + u\theta^\perp)du = \mathcal{R}[\hat{p}_0](r + \langle \zeta(\theta), \theta \rangle, \theta),$$

where $\mathcal{R}$ denotes the Radon transform as defined in (2.1). We can therefore recover $\hat{p}_0$ with an inverse Radon transform:

$$\hat{p}_0 = 2\mathcal{R}^{-1}[\tilde{m}_3], \quad \tilde{m}_3(r,\theta) = \begin{cases} m_3(\theta; \langle \zeta(\theta), \theta \rangle - r) & \text{if } r < \langle \zeta(\theta), \theta \rangle, \\ 0 & \text{if } r \geq \langle \zeta(\theta), \theta \rangle. \end{cases} \tag{4.26}$$

Equation (4.26) reveals an interesting property of integrating area detectors: For an arbitrary strictly convex measurement geometry Ω, exact reconstruction formulas exist. This is a property which is not known for conventional and other photoacoustic sectional imaging technologies.

Measurements with Horizontal Line Detectors

For every $\theta \in \mathbb{S}^1$, we define the function

$$\tilde{p}_\theta(r,z;t) = \int_{T(r,\theta)} p(\xi, z; t) ds(\xi),$$

where $T(r,\theta)$ is defined as in (4.20). Then, using that the vectors $(\theta, 0)$, $(\theta^\perp, 0)$, and $(0,0,1)$ are an orthonormal basis of $\mathbb{R}^3$ and that the Laplacian is rotationally invariant, the initial value problem (4.15) implies that $\tilde{p}_\theta$ solves for all $r, z \in \mathbb{R}$ and $t > 0$ the two-dimensional wave equation

$$\begin{aligned} \partial_{tt}\tilde{p}_\theta(r,z;t) &= \int_{-\infty}^{\infty} \Delta_x p(\zeta(\theta) + r\theta + u\theta^\perp, z; t)du \\ &= \partial_{rr}\tilde{p}_\theta(r,z;t) + \partial_{zz}\tilde{p}_\theta(r,z;t) \end{aligned}$$

with the initial conditions

$$\partial_t \tilde{p}_\theta(r,z;0) = 0,$$
$$\tilde{p}_\theta(r,z;0) = P_\theta(r)\delta(z), \quad P_\theta(r) = \int_{T(r,\theta)} \hat{p}_0(\xi)ds(\xi),$$

for every $r, z \in \mathbb{R}$.

From formula (3.7), we see that the solution of this initial value problem can be written as

$$\begin{aligned}\tilde{p}_\theta(r,z;t) &= \frac{1}{2\pi}\partial_t\left(\int_{B_t^2(0)} \frac{P_\theta(r+\rho)\delta(z+\zeta)}{\sqrt{t^2-\rho^2-\zeta^2}}ds(\rho,\zeta)\right)\\ &= \frac{1}{2\pi}\partial_t\left(\int_{-t}^{t}\delta(z+\zeta)\int_{-\sqrt{t^2-\zeta^2}}^{\sqrt{t^2-\zeta^2}} \frac{P_\theta(r+\rho)}{\sqrt{t^2-\rho^2-\zeta^2}}d\rho d\zeta\right)\end{aligned}$$

for all $r, z \in \mathbb{R}$ and $t > 0$. Integrating out the δ-function, we find for every $z \in [-t,t]$ that

$$\tilde{p}_\theta(r,z;t) = \frac{1}{2\pi}\partial_t\left(\int_{-\sqrt{t^2-z^2}}^{\sqrt{t^2-z^2}} \frac{P_\theta(r+\rho)}{\sqrt{t^2-z^2-\rho^2}}d\rho\right).$$

Since $\tilde{p}_\theta$ is related to the measurement m_4, given by (4.19), via $m_4(\theta;t) = \tilde{p}_\theta(0,0;t)$, and since $P_\theta(r) = 0$ for $r > 0$ by the assumption that $\hat{p}_0$ has support inside Ω, we find with the formula (1.19) for the Abel transform in reciprocal coordinates that

$$m_4(\theta;t) = \frac{1}{2\pi}\partial_t\left(\int_0^t \frac{P_\theta(-\rho)}{\sqrt{t^2-\rho^2}}d\rho\right) = \frac{1}{4\pi}\partial_t\left(\tfrac{1}{t}\mathcal{A}[\psi_\theta](\tfrac{1}{t})\right)$$

where $\psi_\theta(\frac{1}{\rho}) = \rho^2 P_\theta(-\rho)$. Switching to the reciprocal coordinate $s = \frac{1}{t}$ and using the identity (1.36), we see that this is of the form

$$\frac{2}{s^2}m(\theta;\tfrac{1}{s}) = -\frac{1}{2\pi}\partial_s\big(s\mathcal{A}[\psi_\theta](s)\big) = \mathcal{A}^{-1}[\tilde{\psi}_\theta](s)$$

with $\tilde{\psi}_\theta(\frac{1}{\rho}) = \frac{1}{\rho^2}\psi_\theta(\frac{1}{\rho}) = P_\theta(-\rho)$. Thus, we can directly solve the equation for P_θ and find

$$P_\theta(-\rho) = 2\mathcal{A}[\tilde{m}_\theta](\tfrac{1}{\rho}), \quad \tilde{m}_\theta(\tfrac{1}{t}) = t^2 m_4(\theta;t). \tag{4.27}$$

Since we have by definition

$$P_\theta(r) = \mathcal{R}[\hat{p}_0](r + \langle \zeta(\theta), \theta \rangle, \theta),$$

we finally get (remembering that $P_\theta(r) = 0$ for $r \geq 0$)

$$\hat{p}_0 = 2\mathcal{R}^{-1}[\tilde{P}], \quad \tilde{P}(r,\theta) = \begin{cases} \mathcal{A}[\tilde{m}_\theta]\left(\frac{1}{\langle \zeta(\theta),\theta \rangle - r}\right) & \text{if } r < \langle \zeta(\theta), \theta \rangle, \\ 0 & \text{if } r \geq \langle \zeta(\theta), \theta \rangle. \end{cases} \tag{4.28}$$

So, the reconstruction of $\hat{p}_0$ can be accomplished by an Abel transform of the rescaled measurements $\tilde{m}_\theta$, defined in (4.27), followed by an inverse Radon transform. Again, this reconstruction formula is valid for an arbitrary strictly convex measurement geometry Ω.

The attenuated Radon transform is defined by

$$\int_{\tau_-}^{\tau_+} e^{-Da(y,\theta)} f(y)\, ds(y)$$

It solves the PDE

$$\theta \cdot \nabla u(x,\theta) + a(x) u(x,\theta) = f(x).$$

Actually, we have

$$e^{-Da(y,\theta)} f(y) \Big|_{\tau_-(x,\theta)}^{\tau_-(x,\theta)} = \int_{\tau_-}^{\tau_+} e^{-Da(y,\theta)} f(y)\, ds(y)\,.$$

Part II

(L. Mindrinos)

Chapter 5

Inverse Acoustic Scattering Theory

This chapter is mainly based on [3] and partially on [8].

5.1 Introduction to Inverse Problems

5.1.1 Examples

1. Direct Problem: Find the zeros $x_1, ..., x_n$ of a given polynomial p of degree n. Then, the inverse problem reads: Find a polynomial p of degree n with given zeros $x_1, ..., x_n$. Here, the inverse problem is easier to solve. The solution is $p(x) = c(x - x_1) \cdots (x - x_n)$, $c \in \mathbb{R}$.

2. **(Scattering Problem)** Direct Problem: Calculate the scattered field for a given object and incident radiation. Given an incident wave $u^i(x)$, find the total field $u = u^i + u^s$. Then, the inverse problem is to find the shape or the properties of a scattering object given the intensity (and phase) of sound or the electromagnetic waves scattered by this object. More precise, let $D \subset \mathbb{R}^n$, $n = 2, 3$ be a bounded domain with smooth boundary ∂D describing the scattering object. Consider a plane incident wave,

$$u^i(x) = e^{ikd \cdot x},$$

where $k > 0$ is the wave number and d is a unit vector describing the incident direction. Then, the total fields solve the problem,

$$\Delta u + k^2 u = 0, \quad x \in \mathbb{R}^n \backslash \bar{D}$$
$$Bu = 0, \quad x \in \partial D$$
$$\frac{\partial u^s}{\partial r} - iku^s = \mathcal{O}(r^{-(n+1)/2}), \quad r = |x| \to \infty.$$

The last limit is considered uniformly in $x/|x|$. This system makes sense under the assumptions of time harmonic fields $u(x,t) = u(x)e^{-i\omega t}$, where ω is the frequency. For suitably polarized time harmonic electromagnetic scattering problems, Maxwell's equations reduce to the two-dimensional Helmholtz equation. The boundary conditions can be Dirichlet , Neumann or Robin depending on the nature of the medium. The radiation condition has to do with unique solvability of the direct problem and ensures the that the scattered field describes divergent wave with sources situated in a bounded domain. For a homogeneous medium, $k = \omega/c = \sqrt{\epsilon\mu}\,\omega$, where c is the speed of sound, ϵ the dielectric constant and μ the permeability. The radiation condition yields the asymptotic expansion,

$$u^s(x) = \frac{e^{ikr}}{r^{(n-1)/2}} u^\infty(\hat{x}) + \mathcal{O}(r^{-(n+1)/2}), \quad r \to \infty,$$

where $\hat{x} = x/|x|$. The inverse problem is to determine the shape of D when the far field pattern $u^\infty(\hat{x})$ is measured for all $\hat{x}$ on the unit sphere in $\mathbb{R}^n$.

For example, if $B = I$ is the identity, u is sufficient smooth and ψ is continuous density, let

$$u^s(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y), \quad x \in \mathbb{R}^n \setminus \partial D,$$

where,

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|), & n = 2 \\ \frac{e^{ik|x-y|}}{4\pi|x-y|}, & n = 3 \end{cases}$$

and $\frac{\partial}{\partial \nu}$ denotes then normal derivative where $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. Then, u^s solves the above problem provided ψ is a solution of the integral equation,

$$\psi(x) + 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y) ds(y) = -2u^i(x), \quad x \in \partial D.$$

In general, we can formulate the direct problem as the evaluation of an operator K acting on a known ϕ (model) in a space X and the inverse problem as the solution of the equation $K\phi = y$.

Direct problem: given ϕ (and K), evaluate $K\phi$.
Inverse problem: given y (and K), solve $K\phi = y$ for ϕ.

In order to formulate an inverse problem, the definition of the operator K, including its domain and range, has to be given. The formulation as an operator equation allows us to distinguish among finite, infinite-dimensional, linear and non-linear problems.

5.1.2 Preliminaries

Definition 5.1. *The operator $K : X \to Y$, mapping a vector space into a vector space is called linear if*

$$K(c_1\phi + c_2\psi) = c_1 K\phi + c_2 K\psi,$$

for all $\phi, \psi \in X$ and $c_1, c_2 \in \mathbb{C}$.

Theorem 5.2. *Let X and Y be normed spaces and $K : X \to Y$ a linear operator. Then K is continuous if it is continuous at one point.*

Proof. Suppose K is continuous at $\phi_0 \in X$. Then for every $\phi \in X$ and $\phi_n \to \phi$ we have that

$$K\phi_n = K(\phi_n - \phi + \phi_0) + K(\phi - \phi_0) \to K\phi_0 + K(\phi - \phi_0) = K\phi$$

since $\phi_n - \phi + \phi_0 \to \phi_0$. □

A linear operator $K : X \to Y$ from a normed space X into a normed space Y is called bounded if there exists a positive constant C such that

$$\|K\phi\| \leq C \|\phi\|$$

for every $\phi \in X$. The norm of K is the smallest such C, i.e.

$$\|K\| := \sup_{\|\phi\|=1} \|K\phi\|, \quad \phi \in X$$

If $Y = \mathbb{C}$, K is called a bounded linear functional. The space X^* of bounded linear functionals on a normed space X is called the dual space of X.

Theorem 5.3. *Let X and Y be normed spaces and $K : X \to Y$ a linear operator. Then K is continuous if and only if it is bounded.*

Proof. Let $K : X \to Y$ be bounded and let $\{\phi_n\}$ be a sequence in X such that $\phi_n \to 0$ as $n \to \infty$. Then $\|K\phi_n\| \leq C \|\phi_n\|$ implies that $K\phi_n \to 0$ as $n \to \infty$, i.e. K is continuous at $\phi = 0$. By Theorem 5.2 K is continuous for all $\phi \in X$. Conversely, let K be continuous and assume that there is no C such that $\|K\phi\| \leq C \|\phi\|$ for all $\phi \in X$. Then there exists a sequence $\{\phi_n\}$ with $\|\phi_n\| = 1$ such that $\|K\phi_n\| \geq n$. Let

$$\psi_n := \|K\phi_n\|^{-1} \phi_n.$$

Then $\psi_n \to 0$ as $n \to \infty$ and hence by the continuity of K we have that $K\psi_n \to K0 = 0$ which is a contradiction since $\|K\psi_n\| = 1$ for every integer n. Hence K must be bounded. □

Definition 5.4. *The operator $K : X \to Y$ is called compact if it maps every bounded set S into a relatively compact set $K(S)$.*

The set of all compact operators from X into Y is a closed subspace of $\mathcal{L}(X, Y)$ (the space of all linear bounded mappings from X to Y).

Theorem 5.5.

1. *If K_1 and K_2 are compact from X into Y, then so are $K_1 + K_2$ and λK_1 for every $\lambda \in \mathbb{C}$.*

2. *Let $K_n : X \to Y$ be a sequence of compact operators between Banach spaces X and Y. Let $K : X \to Y$ be bounded, and let K_n converge to K in the operator norm, i.e.,*

$$\|K_n - K\| := \sup_{x \neq 0} \frac{\|K_n x - Kx\|}{\|x\|} \to 0, \quad n \to \infty.$$

   Then K is also compact.

3. *If $L \in \mathcal{L}(X, Y)$ and $K \in \mathcal{L}(Y, Z)$, and L or K is compact, then KL is also compact.*

Theorem 5.6. *Let X be a normed space. Then the identity operator $I : X \to X$ is a compact operator if and only if X has finite dimension.*

Theorem 5.7 (Riesz Theorem). *Let $K : X \to X$ be a compact operator on a normed space X. Then either*

1. *the homogeneous equation*

$$\phi - K\phi = 0$$

   has a nontrivial solution $\phi \in X$ *or*

2. *for each* $f \in X$ *the equation*

$$\phi - K\phi = f$$

   has a unique solution $\phi \in X$*. If* $I - K$ *is injective (and hence bijective), then* $(I-K)^{-1} : X \to X$ *is bounded.*

Let $K : X \to X$ be a compact operator of a normed space into itself. A complex number λ is called an eigenvalue of K with eigenfunction $\phi \in X$ if there exists $\phi \in X, \phi \neq 0$, such that $K\phi = \lambda\phi$. It is easily seen that eigenfunctions corresponding to different eigenvalues must be linearly independent.

We call the dimension of the null space of $L_\lambda := \lambda I - K$ the multiplicity of λ. If $\lambda = 0$ is not an eigenvalue of K, it follows from the Riesz theorem that the resolvent operator $(\lambda I - K)^{-1}$ is a well defined bounded linear operator mapping X onto itself. On the other hand, if $\lambda = 0$ then K^{-1} cannot be bounded on $K(X)$ unless X is finite dimensional since if it were then $I = K^{-1}K$ would be compact.

5.1.3 Ill-posed problem

There is a fundamental difference between the direct and the inverse problems. In all cases, the inverse problem is ill-posed or improperly-posed in the sense of Hadamard, while the direct problem is well-posed.

We formulate the notion of well-posedness in the following way.

Definition 5.8 (well-posedness). *Let* X *and* Y *be normed spaces,* $K : X \to Y$ *a (linear or non-linear) mapping. The equation* $Kx = y$ *is called properly-posed or well-posed if the following holds:*

1. *Existence: For every* $y \in Y$ *there is (at least one)* $x \in X$ *such that* $Kx = y$*.*

2. *Uniqueness: For every* $y \in Y$ *there is at most one* $x \in X$ *with* $Kx = y$*.*

3. *Stability: The solution x depends continuously on y, that is, for every sequence $(x_n) \subset X$ with $Kx_n \to Kx$ as $n \to \infty$, it follows that $x_n \to x$ as $n \to \infty$.*

Equations for which (at least) one of these properties does not hold are called improperly-posed or ill-posed. Existence and uniqueness depend only on the algebraic nature of the spaces and the operator, that is, whether the operator is onto or one-to-one. Stability, however, depends also on the topologies of the spaces, i.e., whether the inverse operator $K^{-1} : Y \to X$ is continuous.

5.2 Scattering Theory

Studying an inverse problem requires a good knowledge of the theory for the corresponding direct problem. Therefore, we begin by presenting the foundations of obstacle scattering problems for time-harmonic acoustic waves, that is, to exterior boundary value problems for the scalar Helmholtz equation. Our aim is to develop the analysis for the direct problems.

Consider the propagation of sound waves of small amplitude in a homogeneous isotropic medium in $\mathbb{R}^3$ viewed as an inviscid fluid. Let $v = v(x,t)$ be the velocity field and let $p = p(x,t), \rho = \rho(x,t)$ and $s = s(x,t)$ denote the pressure, density and specific entropy, respectively, of the fluid. The motion is then governed by Euler's equation

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \gamma v + \frac{1}{\rho}\nabla p = 0,$$

the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

the state equation

$$p = f(\rho, s),$$

and the adiabatic hypothesis

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0$$

where γ is the absorption coefficient, f is a function depending on the nature of the fluid.

The above system for v, p, ρ and s is non-linear. To linearize the above system, we assume that v, p, ρ and s are small perturbations of the static state $v_0 = 0$, and constant values p_0, ρ_0, s_0, i.e.,

$$v = \sum_{k=0}^{\infty} \epsilon^k v_k, \qquad p = \sum_{k=0}^{\infty} \epsilon^k p_k,$$
$$\rho = \sum_{k=0}^{\infty} \epsilon^k \rho_k, \qquad s = \sum_{k=0}^{\infty} \epsilon^k s_k,$$

for $0 < \epsilon \ll 1$. Then, we obtain the linearized version keeping only terms of first order,

$$\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0} \nabla p_1 = 0,$$
$$\frac{\partial \rho_1}{\partial t} + \rho_0 \mathrm{div}(v_1) = 0,$$
$$\frac{\partial f}{\partial \rho}(\rho_0, s_0) \frac{\partial \rho_1}{\partial t} = \frac{\partial p_1}{\partial t},$$

We define the speed of sound by

$$c^2 = \frac{\partial f}{\partial \rho}(\rho_0, s_0)$$

and from the linearized Euler equation, we observe that there exists a velocity potential $u = u(x, t)$ such that

$$v = \frac{1}{\rho_0} \nabla u, \quad p = -\frac{\partial u}{\partial t}.$$

Clearly, the velocity potential also satisfies the wave equation

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \Delta u. \tag{5.1}$$

For time-harmonic acoustic waves of the form

$$u(x, t) = \Re \left\{ u(x) e^{-i\omega t} \right\}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent part u satisfies the reduced wave equation or Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{5.2}$$

where the wave number k is given by the positive constant $k = \omega/c$.

In general, considering the scattering of acoustic and electromagnetic waves by an **inhomogeneous medium** of compact support, we set

$$\rho(x,t) = \rho_0(x) + \epsilon\rho_1(x,t) + \ldots$$
$$s(x,t) = s_0(x) + \epsilon s_1(x,t) + \ldots$$

Then, keeping only the terms of order ϵ, we obtain

$$\frac{\partial^2 p_1}{\partial t^2} = c^2(x)\rho_0(x)\mathrm{div}\left(\frac{1}{\rho_0(x)}\nabla p_1(x)\right),$$

where now

$$c^2(x) = \frac{\partial f}{\partial \rho}(\rho_0(x), s_0(x)).$$

If we now assume that terms involving $\nabla\rho_0$ are negligible and that p_1 is time harmonic,

$$p_1(x,t) = \Re\{u(x)e^{-i\omega t}\}$$

we see that u satisfies

$$\Delta u + \frac{\omega^2}{c^2(x)}u = 0.$$

The above equation governs the propagation of time harmonic acoustic waves of small amplitude in a slowly varying inhomogeneous medium. We still must prescribe how the wave motion is initiated and what is the boundary of the region containing the fluid. We shall only consider the simplest case when the inhomogeneity is of compact support. Assuming the inhomogeneous region is contained inside a ball B, i.e., $c(x) = c_0 = $ constant for $x \in \mathbb{R}^3 \setminus B$, we see that the scattering problem under consideration is now modeled by

$$\Delta u + k^2 n(x)u = 0,$$

where $k = \omega/c_0 > 0$ is the wave number and

$$n(x) := \frac{c_0^2}{c^2(x)},$$

is the refractive index. In the following we assume $n(x) = 1$ and the inhomogeneous case will be considered later.

In obstacle scattering we must distinguish between the two cases of impenetrable and penetrable objects.

1. For a **sound-soft** obstacle the pressure of the total wave vanishes on the boundary. Consider the scattering of a given incoming wave u^i by a sound-soft obstacle D. Then the total wave $u = u^i + u^s$, where u^s denotes the scattered wave, must satisfy the wave equation in the exterior $\mathbb{R}^3 \setminus \bar{D}$ of D and a Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial D.$$

2. Similarly, the scattering from **sound-hard** obstacles leads to a Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D,$$

where ν is the unit outward normal to ∂D since here the normal velocity of the acoustic wave vanishes on the boundary.

3. More generally, allowing obstacles for which the normal velocity on the boundary is proportional to the excess pressure on the boundary leads to an impedance boundary condition of the form

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \quad \text{on} \quad \partial D,$$

with a positive constant λ.

4. The scattering by a penetrable obstacle D with constant density ρ_D and speed of sound c_D differing from the density ρ and speed of sound c in the surrounding medium $\mathbb{R}^3 \setminus \bar{D}$ leads to a **transmission problem**. Here, in addition to the superposition $u = u^i + u^s$ of the incoming wave and the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Helmholtz equation with wave number $k = \omega/c$, we also have a transmitted wave v in D satisfying the Helmholtz equation with wave number $k_D = \omega/c_D \neq k$. The continuity of the pressure and of the normal velocity across the interface leads to the transmission conditions

$$u = v, \qquad \frac{1}{\rho}\frac{\partial u}{\partial \nu} = \frac{1}{\rho_D}\frac{\partial v}{\partial \nu}, \quad \text{on} \quad \partial D.$$

For the scattered wave u^s, the radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|,$$

introduced by Sommerfeld will ensure uniqueness for the solutions to the scattering problems. From the two possible spherically symmetric solutions

$$\frac{e^{ikr}}{r}, \qquad \frac{e^{-ikr}}{r}$$

to the Helmholtz equation, only the first one satisfies the radiation condition. Since via

$$\Re\left\{\frac{e^{ikr-i\omega t}}{r}\right\} = \frac{\cos(kr-\omega t)}{r}$$

this corresponds to an outgoing spherical wave, we observe that physically speaking the Sommerfeld radiation condition characterizes outgoing waves. Throughout $|x|$ we denote the Euclidean norm of a point x in $\mathbb{R}^3$.

5.2.1 Green's theorem and formula

We begin by giving some basic properties of solutions to the Helmholtz equation $\Delta u + k^2 u = 0$ with positive wave number k. Most of these can be deduced from the fundamental solution

$$\Phi(x, y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y \tag{5.3}$$

Straightforward differentiation shows that for fixed $y \in \mathbb{R}^3$ the fundamental solution satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$.

Definition 5.9. *A domain $D \subset \mathbb{R}^3$, i.e., an open and connected set, is said to be of class C^k, $k \in \mathbb{N}$, if for each point z of the boundary ∂D there exists a neighborhood V_z of z with the following properties:*

1. *the intersection $V_z \cap \bar{D}$ can be mapped bijectively onto the half ball $\{x \in \mathbb{R}^3 : |x| < 1, x_3 \geq 0\}$, this mapping and its inverse are k-times continuously differentiable.*

2. *the intersection $V_z \cap \partial D$ is mapped onto the disk $\{x \in \mathbb{R}^3 : |x| < 1, x_3 = 0\}$.*

We will express the property of a domain D to be of class C^k also by saying that its boundary ∂D is of class C^k. By $C^k(D)$ we denote the linear space of real or complex valued functions defined on the domain D which are

$k-$times continuously differentiable. By $C^k(\bar{D})$ we denote the subspace of all functions in $C^k(D)$ which together with all their derivatives up to order k can be extended continuously from D into the closure $\bar{D}$.

One of the basic tools in studying the Helmholtz equation is provided by Green's integral theorems. Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Then, for $u \in C^1(D)$ and $v \in C^2(D)$ we have Green's first theorem

$$\int_D (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial D} u \frac{\partial v}{\partial \nu} ds \tag{5.4}$$

and for $u, v \in C^2(D)$ we have Green's second theorem

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds \tag{5.5}$$

Theorem 5.10. *Let D be a bounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $u \in C^2(D) \cap C(\bar{D})$ be a function which possesses a normal derivative on the boundary in the sense that the limit*

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to +0} \nu(x) \cdot \nabla u(x - h\nu(x)), \quad x \in \partial D$$

exists uniformly on ∂D. Then we have Green's formula

$$\begin{aligned} u(x) = &\int_{\partial D} \left(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y) \\ &- \int_D \left(\Delta u(y) + k^2 u(y) \right) \Phi(x,y) dy, \quad x \in D, \end{aligned} \tag{5.6}$$

where the volume integral exists as improper integral. In particular, if u is a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad in \quad D,$$

then,

$$u(x) = \int_{\partial D} \left(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y), \quad x \in D. \tag{5.7}$$

Proof. See theorem 2.1, p. 16 [3]. □

Obviously, Theorem 5.10 remains valid for complex values of k.

Theorem 5.11. *If u is a two times continuously differentiable solution to the Helmholtz equation in a domain D, then u is analytic.*

Proof. Let $x \in D$ and choose a closed ball contained in D with center x. Then Theorem 5.10 can be applied in this ball and the statement follows from the analyticity of the fundamental solution for $x \neq y$. □

As a consequence of Theorem 5.11, a solution to the Helmholtz equation that vanishes in an open subset of its domain of definition must vanish everywhere. In the sequel, by saying u is a solution to the Helmholtz equation we always tacitly imply that u is twice continuously differentiable, and hence analytic, in the interior of its domain of definition.

Definition 5.12. *A solution u to the Helmholtz equation whose domain of definition contains the exterior of some sphere is called radiating if it satisfies the Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|,$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Theorem 5.13. *Assume the bounded set D is the open complement of an unbounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation*

$$\Delta u + k^2 u = 0, \quad in \quad \mathbb{R}^3 \setminus \bar{D},$$

which possesses a normal derivative on the boundary in the sense that the limit

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\to+0} \nu(x) \cdot \nabla u(x + h\nu(x)), \quad x \in \partial D$$

exists uniformly on ∂D. Then we have Green's formula

$$u(x) = \int_{\partial D} \left(\frac{\partial \Phi(x,y)}{\partial \nu} u(y) - \Phi(x,y) \frac{\partial u(y)}{\partial \nu(y)} \right) ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{5.8}$$

Proof. See theorem 2.4, p. 18 [3]. □

From Theorem (5.13) we deduce that radiating solutions u to the Helmholtz equation automatically satisfy Sommerfeld's finiteness condition

$$u(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \quad |x| \to \infty$$

uniformly for all directions and that the validity of the Sommerfeld radiation condition is invariant under translations of the origin.

Solutions to the Helmholtz equation which are defined in all $\mathbb{R}^3$ are called entire solutions. An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically. This follows immediately from combining Green's formula (5.8) and Green's theorem (5.5).

We are now in a position to introduce the definition of the far field pattern or the scattering amplitude.

Theorem 5.14. *Every radiating solution u to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave*

$$u(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty \tag{5.9}$$

uniformly in all directions $\hat{x} = x/|x|$ where the function u, defined on the unit sphere Ω is known as the far field pattern of u. Under the assumptions of Theorem (5.13) *we have*

$$u_\infty(\hat{x}) = \int_{\partial D}\left(\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu}u(y) - e^{-ik\hat{x}\cdot y}\frac{\partial u(y)}{\partial \nu(y)}\right)ds(y), \quad \hat{x} \in \Omega. \tag{5.10}$$

Proof. From

$$|x-y| = \sqrt{|x|^2 - 2x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + \mathcal{O}\left(\frac{1}{|x|}\right)$$

we derive

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\hat{x}\cdot y} + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \tag{5.11}$$

and similarly

$$\frac{\partial}{\partial \nu(y)}\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}$$

uniformly for all $y \in \partial D$. Inserting this into (5.8), the theorem follows. □

The main problem will be to recover radiating solutions of the Helmholtz equation from a knowledge of their far field patterns. In terms of the mapping $F : u \mapsto u_\infty$, transferring the radiating solution u into its far field pattern u_∞, we want to solve the equation

$$Fu = u_\infty,$$

for a given u_∞.

5.2.2 The Far Field mapping

We establish the one-to-one correspondence between radiating solutions to the Helmholtz equation and their far field patterns.

Lemma 5.15 (**Rellich**). *Assume the bounded set D is the open complement of an unbounded domain and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 ds = 0.$$

Then, $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Rellich's lemma ensures uniqueness for solutions to exterior boundary value problems through the following theorem.

Theorem 5.16. *Let D be as in Lemma (5.15), let ∂D be of class C^2 with unit normal ν directed into the exterior of D and assume $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a radiating solution to the Helmholtz equation with wave number $k > 0$ which has a normal derivative in the sense of uniform convergence and for which*

$$\Im \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds \geq 0.$$

Then, $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof. From

$$\lim_{r\to\infty} \int_{\Omega_r} \left(|\frac{\partial u}{\partial \nu}|^2 + k^2 |u(x)|^2 \right) ds = -2k \Im \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds$$

we conclude that

$$\lim_{r\to\infty} \int_{\Omega_r} |u(x)|^2 ds = 0$$

and hence from theorem 5.15 follows the argument. □

Rellich's lemma also establishes the one-to-one correspondence between radiating waves and their far field patterns.

Theorem 5.17. *Let D be as in Lemma 5.15 and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically. Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

Proof. From (5.9) we deduce

$$\int_{|x|=r} |u(x)|^2 ds = \int_{\Omega} |u_\infty(\hat{x})|^2 ds + \mathcal{O}\left(\frac{1}{r}\right), \quad r \to \infty,$$

the assumption $u_\infty = 0$ on Ω implies that Rellich's Lemma can be applied. □

5.2.3 Single- and Double-Layer Potentials

We assume that D is the open complement of an unbounded domain of class C^2, that is, we include scattering from more than one obstacle in our analysis.

We first briefly review the basic jump relations and regularity properties of acoustic single- and double-layer potentials. Given an integrable function ϕ, the integrals

$$u(x) := \int_{\partial D} \Phi(x, y)\phi(y) ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D$$

and

$$v(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \phi(y) ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D$$

are called, respectively, acoustic single-layer and acoustic double-layer potentials with density ϕ. They are solutions to the Helmholtz equation in D and in $\mathbb{R}^3 \setminus \bar{D}$ and satisfy the Sommerfeld radiation condition. Green's formulas show that any solution to the Helmholtz equation can be represented as a combination of single- and double-layer potentials. For continuous densities, the behavior of the surface potentials at the boundary is described by the following jump relations. By $\|\cdot\|_\infty = \|\cdot\|_{\infty,G}$, we denote the usual supremum norm of real or complex valued functions defined on a set $G \subset \mathbb{R}^3$.

Theorem 5.18. *Let ∂D be of class C^2 and let ϕ be continuous. Then the single- layer potential u with density ϕ is continuous throughout $\mathbb{R}^3$ and*

$$\|u\|_{\infty,\mathbb{R}^3} \le C \|\phi\|_{\infty,\partial D}$$

for some constant C depending on ∂D. On the boundary we have

$$u(x) = \int_{\partial D} \Phi(x,y)\phi(y)ds(y), \quad x \in \partial D \tag{5.12}$$

and

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\phi(y)ds(y) \mp \frac{1}{2}\phi(x), \quad x \in \partial D \tag{5.13}$$

where

$$\frac{\partial u_\pm}{\partial \nu}(x) = \lim_{h\to+0} \nu(x)\cdot\nabla u(x \pm h\nu(x)),$$

is to be understood in the sense of uniform convergence on ∂D and where the integrals exist as improper integrals. The double-layer potential v with density ϕ can be continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with limiting values

$$v_\pm(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\phi(y)ds(y) \pm \frac{1}{2}\phi(x), \quad x \in \partial D \tag{5.14}$$

where

$$v_\pm(x) = \lim_{h\to+0} v(x \pm h\nu(x)),$$

and where the integrals exist as improper integrals. Furthermore,

$$\|v\|_{\infty,\mathbb{R}^3\setminus D} \le C\,\|\phi\|_{\infty,\partial D}\,, \quad \|v\|_{\infty,\bar{D}} \le C\,\|\phi\|_{\infty,\partial D}$$

for some constant C depending on ∂D.

For the direct values of the single- and double-layer potentials on the boundary ∂D, we have more regularity. This can be conveniently expressed in terms of the mapping properties of the single- and double-layer operators S and K, given by

$$(S\phi)(x) := 2\int_{\partial D} \Phi(x,y)\phi(y)ds(y), \quad x \in \partial D \tag{5.15}$$

and

$$(K\phi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\phi(y)ds(y), \quad x \in \partial D \tag{5.16}$$

and the normal derivative operators K' and T, given by

$$(K'\phi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\phi(y)ds(y), \quad x \in \partial D \tag{5.17}$$

and

$$(T\phi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\phi(y)ds(y), \quad x \in \partial D \tag{5.18}$$

Clearly, by interchanging the order of integration, we see that S is self adjoint and K and K' are adjoint with respect to the bilinear form

$$\langle\phi,\psi\rangle = \int_{\partial D}\phi\psi\,ds,$$

that is,

$$\langle S\phi,\psi\rangle = \langle\phi,S\psi\rangle$$

and

$$\langle K\phi,\psi\rangle = \langle\phi,K'\psi\rangle$$

for all $\phi,\psi \in C(\partial D)$. To derive further properties of the boundary integral operators, let u and v denote the double-layer potentials with densities ϕ and ψ in $C^{1,a}(\partial D)$, respectively. Then by the jump relations of Theorem 5.18, Green's theorem and the radiation condition we find that

$$\int_{\partial D} T\phi\,\psi\,ds = 2\int_{\partial D}\frac{\partial u}{\partial\nu}(v_+ - v_-)\,ds = 2\int_{\partial D}(u_+ - u_-)\frac{\partial v}{\partial\nu}\,ds = \int_{\partial D}\phi\,T\psi\,ds$$

that is, T also is self adjoint. Now, in addition, let w denote the single-layer potential with density $\phi \in C(\partial D)$. Then

$$\int_{\partial D} S\phi\,T\psi\,ds = 4\int_{\partial D} w\frac{\partial v}{\partial\nu}ds = 4\int_{\partial D} v_-\frac{\partial w_-}{\partial\nu}ds = \int_{\partial D}(K-I)\psi(K'+I)\phi\,ds$$

where I is the identity operator, hence

$$\int_{\partial D}\phi ST\psi\,ds = \int_{\partial D}\phi(K^2 - I)\psi\,ds$$

follows for all $\phi \in C(\partial D)$ and $\psi \in C^{1,a}(\partial D)$. Thus, we have proven the relation

$$ST = K^2 - I$$

and similarly, it can be shown the adjoint relation

$$TS = K'^2 - I$$

is also valid.

Theorem 5.19. *Let ∂D be of class C^2. Then the operator S, K, K' are bounded from $C(\partial D)$ into $C^{0,a}(\partial D)$. The operators S, K are also bounded from $C^{0,a}(\partial D)$ into $C^{1,a}(\partial D)$ and the operator T is bounded from $C^{1,a}(\partial D)$ into $C^{0,a}(\partial D)$.*

Theorem 5.20 (**Lax's theorem**). *Let X and Y be normed spaces both of which have a scalar product $(\cdot, \cdot)$ and assume that there exists a positive constant c such that*

$$|(\phi, \psi)| \leq \|\phi\| \, \|\psi\|$$

for all $\phi, \psi \in X$. Let $U \subset X$ be a subspace and let $A : U \to Y$ and $B : Y \to X$ be bounded linear operators satisfying

$$(A\phi, \psi) = (\phi, B\psi)$$

for all $\phi \in U$ and $\psi \in Y$. Then $A : U \to Y$ is bounded with respect to the norms induced by the scalar products.

Theorem 5.21. *Let ∂D be of class C^2 and let $H^1(\partial D)$ denote the usual Sobolev space. Then the operator S is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. Assume further that ∂D belongs to $C^{2,a}$. Then the operators K and K' are bounded from $L^2(\partial D)$ into $H^1(\partial D)$ and the operator T is bounded from $H^1(\partial D)$ into $L^2(\partial D)$.*

Proof. Theorem 3.6, p. 43 [3]. □

The jump relations of Theorem 5.18 can also be extended through the use of Lax's theorem from the case of continuous densities to L^2 densities.

5.2.4 Scattering from a Sound-Soft Obstacle

The scattering of time-harmonic acoustic waves by sound-soft obstacles leads to the following problem.

Direct Acoustic Obstacle Scattering Problem. Given an entire solution u^i to the Helmholtz equation representing an incident field, find a solution

$$u = u^i + u^s$$

to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ such that the scattered field u^s satisfies the Sommerfeld radiation condition and the total field u satisfies the boundary condition

$$u = 0, \quad \text{on} \quad \partial D$$

Clearly, after renaming the unknown functions, this direct scattering problem is a special case of the following Dirichlet problem.

Exterior Dirichlet Problem. Given a continuous function f on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$u = f, \quad \text{on} \quad \partial D$$

We briefly sketch uniqueness, existence and well-posedness for this boundary value problem.

Lemma 5.22. *Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ which satisfies the homogeneous Dirichlet boundary condition on ∂D. We define $D_R := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < R\}$ for sufficient large R. Then $\nabla u \in L^2(D_R)$ and*

$$\int_{|x|=R} u \frac{\partial \bar{u}}{\partial \nu} ds = \int_{D_R} \left(|\nabla u|^2 - k^2 |u|^2 \right) dx$$

Theorem 5.23. *The exterior Dirichlet problem has at most one solution.*

Proof. We have to show that solutions to the homogeneous boundary value problem $u = 0$ on ∂D vanish identically. The above lemma, justifies the application of Theorem 5.16 and hence $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. □

The existence of a solution to the exterior Dirichlet problem can be based on boundary integral equations. In the so-called layer approach, we seek the solution in the form of acoustic surface potentials. Here, we choose an approach in the form of a combined acoustic double- and single-layer potential

$$u(x) = \int_{\partial D} \left(\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right) \phi(y) ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D \tag{5.19}$$

with a density $\phi \in \partial D$ and a real coupling parameter $\eta \neq 0$. Then from the jump relations of Theorem 5.18 we see that the potential u solves the exterior Dirichlet problem provided the density is a solution of the integral equation

$$\phi + K\phi - i\eta S\phi = 2f. \tag{5.20}$$

Since $S, K : C(\partial D) \to C^{0,a}(\partial D)$ are bounded, from the compact imbedding $C^{0,a}(\partial D) \hookrightarrow C(\partial D)$, $0 < a \leq 1$ we see that $S, K : C(\partial D) \to C(\partial D)$ are compact. Therefore, the existence of a solution can be established by the Riesz-Fredholm theory for equations of the second kind with a compact operator.

Let ϕ be a continuous solution to the homogeneous form of (5.20). Then the potential u given by (5.19) satisfies the homogeneous boundary condition

$$u_+ = 0, \quad \text{on} \quad \partial D$$

hence by the uniqueness for the exterior Dirichiet problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations now yield

$$-u_- = \phi, \quad -\frac{\partial u_-}{\partial \nu} = i\eta\phi, \quad \text{on} \quad \partial D$$

Hence, using Green's theorem (5.4), we obtain

$$i\eta \int_{\partial D} |\phi|^2 ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu} ds = \int_D \left(|\nabla u|^2 - k^2|u|^2\right) dx$$

Taking the imaginary part of the last equation shows that $\phi = 0$. Thus, we have established uniqueness for the integral equation (5.20), that is, injectivity of the operator $I + K - i\eta S : C(\partial D) \to C(\partial D)$. Then, by the Riesz-Fkedholm theory, the inverse $(I + K - i\eta S)^{-1} : C(\partial D) \to C(\partial D)$ is bounded. Hence, the inhomogeneous equation (5.20) possesses a solution and this solution depends continuously on f in the maximum norm. From the representation (5.19) of the solution as a combined double- and single-layer potential, with the aid of the regularity estimates in Theorem 5.18, the continuous dependence of the density ϕ on the boundary data f shows that the exterior Dirichlet problem is well-posed, i.e., small deviations in f in the maximum norm ensure small deviations in u in the maximum norm on $\mathbb{R}^3 \setminus D$ and small deviations of all its derivatives in the maximum norm on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$.

We summarize these results in the following theorem.

Theorem 5.24. *The exterior Dirichlet problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^3 \setminus D$ *and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

Note that for $\eta = 0$ the integral equation (5.20) becomes non-unique if k is a so-called irregular wave number or internal resonance, i.e., if there exist nontrivial solutions u to the Helmholtz equation in the interior domain D satisfying homogeneous Neumann boundary conditions $\partial u/\partial\nu = 0$ on ∂D.

In order to be able to use Green's representation formula for the solution of the exterior Dirichlet problem, we need its normal derivative. However, assuming the given boundary values to be merely continuous means that in general the normal derivative will not exist. Hence, we need to impose some additional smoothness condition on the boundary data. This leads to an operator that transfers the boundary values, i.e., the Dirichlet data, into the normal derivative, i.e., the Neumann data, and therefore it is called the Dirichlet to Neumann map.

In general, for the scattering problem the boundary values are as smooth as the boundary since they are given by the restriction of the analytic function u^i to ∂D. In particular, for domains D of class C^2 our regularity analysis shows that the scattered field u^s is in $C^{1,a}(\mathbb{R}^3 \setminus D)$. Therefore, we may apply Green's formula 5.8 with the result

$$u^s(x) = \int_{\partial D} \left(\frac{\partial \Phi(x,y)}{\partial \nu} u^s(y) - \Phi(x,y) \frac{\partial u^s(y)}{\partial \nu(y)} \right) ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{5.21}$$

Green's theorem (5.5), applied to the entire solution u^i and $\Phi(x,.)$, gives

$$0 = \int_{\partial D} \left(\frac{\partial \Phi(x,y)}{\partial \nu} u^i(y) - \Phi(x,y) \frac{\partial u^i(y)}{\partial \nu(y)} \right) ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{5.22}$$

Adding these two equations and using the boundary condition $u^i + u^s = 0$ on ∂D gives the following theorem. The representation for the far field pattern is obtained with the aid of (5.11).

Theorem 5.25. *For the scattering of an entire field u^i from a sound-soft obstacle D we have*

$$u(x) = u^i(x) - \int_{\partial D} \Phi(x,y) \frac{\partial u(y)}{\partial \nu(y)} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

and the far field pattern of the scattered field u^s is given by

$$u_\infty(\hat{x}) = -\frac{1}{4\pi} \int_{\partial D} e^{-ik\hat{x}\cdot y} \frac{\partial u(y)}{\partial \nu(y)} ds(y), \quad \hat{x} \in \Omega,$$

In physics, the this representation for the scattered field is known as Huygen's principle.

5.2.5 Scattering from a Sound-hard Obstacle

Exterior Neumann Problem. Given a continuous function f on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = g, \quad \text{on} \quad \partial D$$

Uniqueness for the Neumann problem follows from Theorem (5.16). To prove existence we again use a combined single- and double-layer approach. We overcome the problem that the normal derivative of the double-layer potential in general does not exist if the density is merely continuous by incorporating a smoothing operator, that is, we seek the solution in the form

$$u(x) = \int_{\partial D} \left(\Phi(x,y)\phi(y) + i\eta \frac{\partial \Phi(x,y)}{\partial \nu(y)} (S_0^2 \phi)(y) \right) ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D \tag{5.23}$$

with a density $\phi \in \partial D$ and a real coupling parameter $\eta \neq 0$. By S_0 we denote the single-layer operator in the potential theoretic limit case $k = 0$. Note that by Theorem 5.19 the density $S_0^2\phi$ of the double-layer potential belongs to $C^{1,a}(\partial D)$. This operator is called a smoothing operator. From Theorem 5.18 we see that 5.23 solves the exterior Neumann problem provided the density is a solution of the integral equation

$$\phi - K'\phi - i\eta T S_0^2 \phi = -2g. \tag{5.24}$$

Since $K' + i\eta T S_0^2 : C^{0,a}(\partial D) \to C^{0,a}(\partial D)$ is compact the Riesz-Fredholm theory is available.

Let ϕ be a continuous solution to the homogeneous form of (5.24). Then the potential u given by (5.23) satisfies the homogeneous boundary condition

$$\frac{\partial u_+}{\partial \nu} = 0, \quad \text{on} \quad \partial D$$

hence by the uniqueness for the exterior Neumann problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations now yield

$$-u_- = i\eta S_0^2 \phi, \quad -\frac{\partial u_-}{\partial \nu} = -\phi, \quad \text{on} \quad \partial D$$

and, by interchanging the order of integration and using Green's integral theorem as above in the proof for the Dirichlet problem, we obtain

$$i\eta \int_{\partial D} |S_0\phi|^2 ds = i\eta \int_{\partial D} \phi S_0^2 \bar{\phi}\, ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu} ds = \int_D \left(|\nabla u|^2 - k^2 |u|^2 \right) dx$$

whence $S_0\phi = 0$ on ∂D follows. The single-layer potential S_0 with density ϕ and wave number $k = 0$ is continuous throughout $\mathbb{R}^3$, harmonic in D and in $\mathbb{R}^3 \setminus \bar{D}$ and vanishes on ∂D and at infinity. Therefore, by the maximum-minimum principle for harmonic functions, we have $S_0\phi = 0$ in $\mathbb{R}^3$ and the jump relation yields $\phi = 0$. Thus, we have established injectivity of the operator $I - K' - i\eta T S_0^2$ and, by the Riesz-Fredholm theory, $(I - K' - i\eta T S_0^2)^{-1}$ exists and is bounded in $C(\partial D)$. From this we conclude the existence of the solution to the Neumann problem for continuous boundary data g and the continuous dependence of the solution on the boundary data.

Theorem 5.26. *The exterior Neumann problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3 \setminus D$ and all its derivatives on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$.*

5.2.6 Scattering in inhomogeneous medium

Recall that if the inhomogeneous region is contained inside a ball B, i.e., $c(x) = c_0 =$ constant for $x \in \mathbb{R}^3 \setminus B$, we see that the scattering problem under consideration is now modeled by

$$\begin{aligned} \Delta u + k^2 n(x) u = 0, \quad \text{in} \quad \mathbb{R}^3 \\ u^i + u^s = u \\ \lim_{r\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \end{aligned} \tag{5.25}$$

where $k = \omega / c_0 > 0$ and

$$n(x) := \frac{c_0^2}{c^2(x)},$$

is the refractive index, positive, satisfying $n(x) = 1$ for $x \in \mathbb{R}^3 \setminus B$. we would also like to include the possibility that the medium is absorbing, i.e., the refractive index has an imaginary component.

The Lippmann-Schwinger Equation

The aim of this section is to derive an integral equation that is equivalent to the scattering problem (5.25) where $n \in C^1(\mathbb{R}^3)$ has the general form

$$n(x) = n_1(x) + i\frac{n_2(x)}{k}.$$

We set

$$m := 1 - n$$

with compact support and

$$n_1(x) > 0 \quad \text{and} \quad n_2(x) \geq 0$$

for all $x \in \mathbb{R}^3$. We set in addition $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$. To derive an integral equation equivalent to (5.25), we shall need to consider the volume potential

$$u(x) := \int_{\mathbb{R}^3} \Phi(x,y)\phi(y)dy, \quad x \in \mathbb{R}^3, \tag{5.26}$$

where Φ is the fundamental solution to the Helmholtz equation and ϕ is a continuous function in $\mathbb{R}^3$ with compact support, i.e., $\phi \in C_0(\mathbb{R}^3)$.

Theorem 5.27. *The volume potential u given by* (5.26) *exists as an improper integral for all $x \in \mathbb{R}^3$ and has the following properties. If $\phi \in C_0(\mathbb{R}^3)$ then $u \in C^{1,a}(\mathbb{R}^3)$ and the orders of differentiation and integration can be interchanged.*

If $\phi \in C_0(\mathbb{R}^3) \cap C^{1,a}(\mathbb{R}^3)$ then $u \in C^{2,a}(\mathbb{R}^3)$ and

$$\Delta u + k^2 u = -\phi, \quad in \quad \mathbb{R}^3.$$

In addition,

$$\|u\|_{2,a,\mathbb{R}^3} \leq C \|\phi\|_{a,\mathbb{R}^3},$$

where $C > 0$ depends only on the support of ϕ.

We now show that the scattering problem (5.25) is equivalent to the problem of solving an integral equation.

Theorem 5.28. *If $u \in C^2(\mathbb{R}^3)$ is a solution of* (5.25)*, then u is a solution of*

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)dy, \quad x \in \mathbb{R}^3. \tag{5.27}$$

Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (5.27) *then $u \in C^2(\mathbb{R}^3)$ and u is a solution of* (5.25).

Proof. Let $u \in C^2(\mathbb{R}^3)$ be a solution of (5.25). Let $x \in \mathbb{R}^3$ be an arbitrary point and choose an open ball B with exterior unit normal ν containing the support of m such that $x \in B$. From Green's formula (5.10) applied to u, we have

$$\begin{aligned} u(x) = &\int_{\partial B} \left(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y) \\ &- k^2 \int_B \Phi(x,y)m(y)u(y)dy, \end{aligned} \tag{5.28}$$

since $\Delta u + k^2 u = mk^2 u$. Note that in the volume integral over B we can integrate over all of $\mathbb{R}^3$ since m has support in B. Green's formula applied to u^i, gives

$$u^i(x) = \int_{\partial B} \left(\frac{\partial u^i}{\partial \nu}(y)\Phi(x,y) - u^i(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y). \tag{5.29}$$

Finally, from Green's theorem 5.5 and the radiation condition we see that

$$\int_{\partial B} \left(\frac{\partial u^s}{\partial \nu}(y)\Phi(x,y) - u^s(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right) ds(y) = 0$$

With the aid of $u = u^i + u^s$ we can now combine the above equations to conclude that (5.27) is satisfied.

Conversely, let $u \in C(\mathbb{R}^3)$ be a solution of (5.27) and define u^s by

$$u^s(x) := -k^2 \int_B \Phi(x,y)m(y)u(y)dy, \quad x \in \mathbb{R}^3$$

Since Φ satisfies the Sommerfeld radiation condition uniformly with respect to y on compact sets and m has compact support, it is easily verified that u^s satisfies the Sommerfeld radiation condition. Since $m \in C_0^1(\mathbb{R}^3)$, we can conclude from (5.27) and Theorem (5.27) that first $u \in C^1(\mathbb{R}^3)$ and then that $u^s \in C^2(\mathbb{R}^3)$ with $\Delta u^s + k^2 u^s = mk^2 u$. Finally, since $\Delta u^i + k^2 u^i = 0$, we can conclude that

$$\Delta u + k^2 u = mk^2 u$$

that is,

$$\Delta u + k^2 n u = 0.$$

in $\mathbb{R}^3$ and the proof is completed. □

We show that (5.27) is uniquely solvable for all values of $k > 0$. This result is nontrivial since it is based on the unique continuation principle.

Theorem 5.29. *Let G be a domain in $\mathbb{R}^3$ and suppose $u \in C^2(G)$ is a solution of*

$$\Delta u + k^2 n(x) u = 0$$

in G such that $n \in C(\bar{G})$ and u vanishes in a neighbourhood of some $x_0 \in G$. Then u is identically zero in G.

Theorem 5.30. *For each $k > 0$ there exists a unique solution to* (5.25) *and the solution u depends continuously with respect to the maximum norm on the incident field u^i.*

Proof. As previously discussed, to show existence and uniqueness it suffices to show that the only solution of

$$\Delta u + k^2 n(x) u = 0, \quad \text{in} \quad \mathbb{R}^3 \tag{5.30}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0$$

is u identically zero. If this is done, by the Riesz-Fredholm theory the integral equation (5.27) can be inverted in $C(\bar{B})$ and the inverse operator is bounded. From this, it follows that u depends continuously on the incident field u^i with respect to the maximum norm. Hence we only must show that the only solution of (5.30) is $u = 0$.

Recall that B is chosen to be a ball of radius a centered at the origin such that m vanishes outside of B. As usual ν denotes the exterior unit normal to ∂B. We begin by noting from Green's theorem (5.4) that

$$\int_{|x|=a} u\frac{\partial \bar{u}}{\partial \nu} ds = \int_{|x|\le a} \left(|\nabla u|^2 - k^2 \bar{n}|u|^2\right) dx$$

From this, since $\Im n > 0$, it follows that

$$\Im \int_{|x|=a} u\frac{\partial \bar{u}}{\partial \nu} ds = k^2 \int_{|x|\le a} \Im n\, |u|^2 dx \ge 0$$

Theorem (5.16) now shows that $u(x) = 0$ for $|x| \ge a$ and it follows by the previous Theorem that $u(x) = 0$ for all $x \in \mathbb{R}^3$. $\square$

Chapter 6

Regularization Theory

This chapter is mainly based on [8]. The last section follows [1].

6.1 General Regularization Scheme

Many inverse problems lead to integral equations of the first kind with continuous or weakly singular kernels. Such integral operators are compact with respect to any reasonable topology. These inverse problems can be formulated as operator equations of the form

$$Kx = y,$$

where, for now, K is a linear compact operator between Hilbert spaces X and Y over $\mathbb{R}$ or $\mathbb{C}$. Let K be also one-to-one and assume that that there exists a solution $x \in X$ of the unperturbed equation $Kx = y$. In other words, we assume that $y \in \mathcal{R}(K)$. The injectivity of K implies that this solution is unique.

In practice, the right-hand side $y \in Y$ is never known exactly but only up to an error of, say, $\delta > 0$. Therefore, we assume that we know $\delta > 0$ and $y^\delta \in Y$ with

$$\left\|y - y^\delta\right\| \leq \delta. \tag{6.1}$$

We want to "solve" the perturbed equation

$$Kx^\delta = y^\delta. \tag{6.2}$$

In general, Equation (6.2) is not solvable because we cannot assume that the measured data y^δ are in the range $\mathcal{R}(K)$ of K. Therefore, the best we can hope is to determine an approximation $x^\delta \in X$ to the exact solution x.

Definition 6.1. *Let $K : X \to Y$ be a linear bounded operator between Banach spaces, $X_1 \subset X$ a subspace, and $\|\cdot\|_1$ a "stronger" norm on X_1, that is, there exists $c > 0$ such that $\|x\| \leq c\,\|x\|_1$ for all $x \in X_1$. Then, we define*

$$\mathcal{F}(\delta, E, \|\cdot\|_1) := \sup\{\|x\| : x \in X_1,\ \|Kx\| \leq \delta,\ \|\cdot\|_1 \leq E\}$$

and call $\mathcal{F}(\delta, E, \|\cdot\|_1)$ the worst-case error for the error δ in the data and a priori information $\|x\|_1 \leq E$.

$\mathcal{F}(\delta, E, \|\cdot\|_1)$ depends on the operator K and the norms in X, Y, and X_1. It is desirable that this worst-case error not only converge to zero as $\delta \to 0$ but that it be of order δ. This is certainly true (even without a priori information) for bounded invertible operators, as is readily seen from the inequality $\|x\| \leq \|K^{-1}\|\,\|Kx\|$. For compact operators K, however, and norm $\|\cdot\|_1 = \|\cdot\|$, this worst-case error does not converge.

Thus, we want the approximation x^δ to the exact solution x that is "not much worse" than the worst-case error $\mathcal{F}(\delta, E, \|\cdot\|_1)$.

An additional requirement is that the approximate solution x^δ should depend continuously on the data y^δ. In other words, it is our aim to construct a suitable bounded approximation $R : Y \to X$ of the (unbounded) inverse operator $K^{-1} : \mathcal{R}(K) \to X$.

Definition 6.2. *A regularization strategy is a family of linear and bounded operators*

$$R_a : Y \to X, \quad a > 0,$$

such that

$$\lim_{a \to 0} R_a K x = x, \quad \text{for all} \quad x \in X,$$

that is, the operators $R_a K$ converge pointwise to the identity.

From this definition and the compactness of K, we conclude the following:

Theorem 6.3. *Let R_a be a regularization strategy for a compact operator $K : X \to Y$ where $\dim X = \infty$. Then we have:*

1. *The operators R_a are not uniformly bounded, that is, there exists a sequence (a_j) with $\left\|R_{a_j}\right\| \to \infty$ for $j \to \infty$.*

2. *The sequence $(R_a K x)$ does not converge uniformly on bounded subsets of X, that is, there is no convergence $R_a K$ to the identity I in the operator norm.*

Proof. 1. Assume on the contrary, that there exist $c > 0$ such that $\|R_a\| \leq c$ for all $a > 0$. From $R_a y \to K^{-1}y$ as $a \to 0$ for all $y \in \mathcal{R}(K)$ and $\|R_a y\| \leq c\|y\|$ for $a > 0$ we conclude that $\|K^{-1}y\| \leq c\|y\|$ for every $y \in \mathcal{R}(K)$, that is K^{-1} is bounded. This implies that $I = K^{-1}K : X \to Y$ is compact, a contradiction to $\dim X = \infty$.

2. Assume that $R_a K \to I$. From the compactness of $R_a K$ and theorem (5.5), we conclude that I is also compact, which again would imply that $\dim X < \infty$.

□

The notion of a regularization strategy is based on unperturbed data, that is, the regularizer $R_a y$ converges to x for the exact right-hand side $y = Kx$. Now let $y \in \mathcal{R}(K)$ be the exact right-hand side and $y^\delta \in Y$ be the measured data satisfying (6.1). We define

$$x^{a,\delta} := R_a y^\delta$$

as an approximation of the solution x of $Kx = y$. Then the error splits into two parts by the following obvious application of the triangle inequality:

$$\left\|x^{a,\delta} - x\right\| \leq \left\|R_a y^\delta - R_a y\right\| + \|R_a y - x\| \tag{6.3}$$

$$\leq \|R_a\| \left\|y^\delta - y\right\| + \|R_a K x - x\| \tag{6.4}$$

$$\leq \delta \|R_a\| + \|R_a K x - x\| \tag{6.5}$$

This is our fundamental estimate, which we use often in the following. We observe that the error between the exact and computed solutions consists of two parts: The first term on the right-hand side describes the error in the data multiplied by the "condition number" of the regularized problem. By Theorem (6.3), this term tends to infinity as a tends to zero. The second term denotes the approximation error $\|(R_a - K^{-1})y\|$ at the exact right-hand side $y = Kx$. By the definition of a regularization strategy, this term tends to zero with a.

We need a strategy to choose $a = a(\delta)$ dependent on δ in order to keep the total error as small as possible. This means that we would like to minimize

$$\delta \|R_a\| + \|R_a K x - x\| .$$

Definition 6.4. *A regularization strategy $a = a(\delta)$ is called admissible if $a(\delta) \to 0$ and*

$$\sup\{\|R_{a(\delta)}y^\delta - x\| : y^\delta \in Y, \|Kx - y^\delta\| \le \delta\} \to 0, \quad \delta \to 0$$

for every $x \in X$.

A convenient method to construct classes of admissible regularization strategies is given by filtering singular systems.

Definition 6.5 (Singular Values). *Let X and Y be Hilbert spaces and $K : X \to Y$ be a compact operator with adjoint operator $K^* : Y \to X$,*

$$(Kx, y) = (x, K^*y), \quad \text{for all} \quad x \in X, y \in Y.$$

The square roots $\mu_j = \sqrt{\lambda_j}$, $j \in J$, of the eigenvalues λ_j of the self-adjoint operator $K^*K : X \to X$ are called singular values of K. Here again, $J \subset \mathbb{N}$ could be either finite or $J = \mathbb{N}$.

Note that every eigenvalue λ of K^*K is nonnegative because $K^*Kx = \lambda x$ implies that

$$\lambda(x, x) = (K^*Kx, x) = (Kx, Kx) \ge 0, \quad \text{i.e.,} \quad \lambda \ge 0.$$

Theorem 6.6 (Singular Value Decomposition). *Let $K : X \to Y$ be a linear compact operator, $K^* : Y \to X$ its adjoint operator, and $\mu_1 \ge \mu_2 \ge \mu_3 > ... > 0$ the ordered sequence of the positive singular values of K, counted relative to its multiplicity. Then there exist orthonormal systems $(x_j) \subset X$ and $(y_j) \subset Y$ with the following properties:*

$$Kx_j = \mu_j y_j, \qquad K^*y_j = \mu_j x_j, \quad \text{for all} \quad j \in J.$$

The system (μ_j, x_j, y_j) is called a singular system for K. Every $x \in X$ possesses the singular value decomposition

$$x = x_0 + \sum_{j\in J}(x, x_j)x_j,$$

for some $x_0 \in \mathcal{N}(K)$ and

$$Kx = \sum_{j\in J}\mu_j(x, x_j)y_j$$

The following theorem characterizes the range of a compact operator with the help of a singular system.

Theorem 6.7 (**Picard**). *Let $K : X \to Y$ be a linear compact operator with singular system (μ_j, x_j, y_j). The equation,*

$$Kx = y$$

has a solution of the form

$$x = \sum_{j \in J} \frac{1}{\mu_j}(y, y_j)x_j$$

if and only if

$$y \in \mathcal{N}(K^*)^\perp \quad \text{and} \quad \sum_{j \in J} \frac{1}{\mu_j^2} |(y, y_j)|^2 < \infty$$

A convenient method to construct classes of admissible regularization strategies is given by filtering singular systems. Let $K : X \to Y$ be a linear compact operator, and let (μ_j, x_j, y_j) be a singular system for K. As readily seen, the solution x of $Kx = y$ is given by Picards theorem, provided the series converges, that is, $y \in \mathcal{R}(K)$. This result illustrates again the influence of errors in y. We construct regularization strategies by damping the factors $1/\mu_j$.

Theorem 6.8. *Let $K : X \to Y$ be a linear compact operator with singular system (μ_j, x_j, y_j) and*

$$q : (0, \infty) \times (0, \|K\|) \to \mathbb{R}$$

be a function with the following properties:

1. $|q(a, \mu)| \leq 1$ *for all* $a > 0$ *and* $0 < \mu \leq \|K\|$.
2. *For every* $a > 0$ *there exists* $c(a)$ *such that*

$$|q(a, \mu)| \leq c(a)\mu \quad \text{for all} \quad 0 < \mu \leq \|K\|.$$

3. $\lim_{a \to 0} q(a, \mu) = 1$ *for every* $0 < \mu \leq \|K\|$.

Then the operator $R_a : Y \to X$, $a > 0$, *defined by*

$$R_a y := \sum_{j \in J} \frac{q(a, \mu_j)}{\mu_j} (y, y_j) x_j, \quad y \in Y,$$

is a regularization strategy with $\|R_a\| \le c(a)$. *A choice* $a = a(\delta)$ *is admissible if* $a(\delta) \to 0$ *and* $\delta\, c(a) \to 0$ *as* $\delta \to 0$. *The function* q *is called a regularizing filter for* K.

Proof. Theorem 2.6, page 32 [8]. □

However, we are interested in optimal strategies, that is, those that converge of the same order as the worst-case error. This can be done by a proper replacement of assumption (3), leading to such optimal strategies. There are many examples of functions $q : (0, \infty) \times (0, \|K\|) \to \mathbb{R}$ that satisfy the assumptions of the above theorem. For example:

1. $q(a, \mu) = \dfrac{\mu^2}{a + \mu^2}, \quad \text{with} \quad c(a) = \dfrac{1}{2\sqrt{a}}$

2. $q(a, \mu) = 1 - (1 - \kappa\mu^2)^{1/a}, \quad \text{with} \quad c(a) = \sqrt{\dfrac{\kappa}{a}}, \quad 0 < \kappa < 1/\|K\|^2.$

3. $q(a, \mu) = \begin{cases} 1, & \mu^2 \ge a \\ 0, & \mu^2 < a \end{cases}, \quad \text{with} \quad c(a) = \dfrac{1}{\sqrt{a}}$

All of the functions q are regularizing filters that lead to optimal regularization strategies. We will see later that the regularization methods for the first two choices of q admit a characterization that avoids knowledge of the singular system. The choice (3) of q is called the spectral cut-off. The spectral cut-off solution $x^{a,\delta} \in X$ is therefore defined by

$$x^{a,\delta} = \sum_{\mu_j^2 \ge a} \frac{1}{\mu_j} (y^\delta, y_j) x_j$$

We combine the fundamental estimate (6.3) with the previous theorem and show the following result for the cut-off solution.

Theorem 6.9. *Let* $y^\delta \in Y$ *be such that* (6.1), *where* $y = Kx$ *denotes the exact right-hand side.*

1. *Let $K : X \to Y$ be a linear compact operator with singular system (μ_j, x_j, y_j). The operators*

$$R_a y := \sum_{\mu_j^2 \geq a} \frac{1}{\mu_j}(y^\delta, y_j)x_j, \quad y \in Y,$$

define a regularization strategy with $\|R_a\| \leq 1/\sqrt{a}$. This strategy is admissible if $a(\delta) \to 0$ and $\delta^2/a(\delta) \to 0$ as $\delta \to 0$.

2. *Let $x = K^* z \in \mathcal{R}(K^*)$ with $\|z\| \leq E$ and $c > 0$. For the choice $a(\delta) = c\delta/E$, we have the estimate,*

$$\|x^{a,\delta} - x\| \leq \left(\frac{1}{\sqrt{c}} + \sqrt{c}\right)\sqrt{\delta E}$$

Therefore, the spectral cut-off is optimal for the information $\|K^{*-1}x\| \leq E$ (if K^* is one-to-one).

Proof. Theorem 2.9, page 36 [8]. □

For given concrete integral operators, however, one often wants to avoid the computation of a singular system. In the next section, we give equivalent characterizations for the first two examples without using singular systems.

6.1.1 Tikhonov Regularization

A common method to deal with overdetermined finite linear systems of the form $Kx = y$ is to determine the best fit in the sense that one tries to minimize the defect $\|Kx - y\|$ with respect to $x \in X$ for some norm in Y. If X is infinite-dimensional and K is compact, this minimization problem is also ill-posed by the following lemma.

Lemma 6.10. *Let X and Y be Hilbert spaces, $K : X \to Y$ be linear and bounded, and $y \in Y$. There exists $\hat{x} \in X$ with $\|K\hat{x} - y\| \leq \|Kx - y\|$ for all $x \in X$ if and only if $\hat{x} \in X$ solves the normal equation $K^*K\hat{x} = K^*y$. Here, $K^* : Y \to X$ denotes the adjoint of K.*

Proof. Lemma 2.10, page 36 [8]. □

As a consequence of this lemma we should penalize the defect (in the language of optimization theory) or replace the equation of the first kind $K^*K\hat{x} = K^*y$ with an equation of the second kind (in the language of integral equation theory). Both viewpoints lead to the following minimization problem. Given the linear, bounded operator $K : X \to Y$ and $y \in Y$, determine x^a that minimizes the Tikhonov functional:

$$J_a(x) := \|Kx - y\|^2 + a\,\|x\|^2 \quad \text{for} \quad x \in X$$

We prove the following theorem.

Theorem 6.11. *Let $K : X \to Y$ be a linear and bounded operator between Hilbert spaces and $a > 0$. Then the Tikhonov functional J_a has a unique minimum $x^a \in X$. This minimum x^a is the unique solution of the normal equation*

$$ax^a + K^*Kx^a = K^*y. \tag{6.6}$$

Proof. Theorem 2.11, page 37 [8]. □

The solution x^a of Eq. (6.6) can be written in the form $x^a = R_a y$ with

$$R_a := (aI + K^*K)^{-1}K^* : Y \to X. \tag{6.7}$$

Choosing a singular system (μ_j, x_j, y_j) for the compact operator K, we see that $R_a y$ has the representation

$$R_a y = \sum_{j\in J} \frac{q(a,\mu_j)}{\mu_j}(y, y_j)x_j = \sum_{j\in J} \frac{\mu_j}{a + \mu_j^2}(y, y_j)x_j, \quad y \in Y,$$

with . This function q is exactly the filter function that was studied before.

Theorem 6.12. *Let $K : X \to Y$ be a linear, compact operator and $a > 0$.*

1. *The operator $aI + K^*K$ is bounded invertible. The operators $R_a : Y \to X$ from (6.7) form a regularization strategy with $\|R_a\| \le 1/(2\sqrt{a})$. It is called the Tikhonov regularization method. $R_a y^\delta$ is determined as the unique solution $x^{a,\delta} \in X$ of the equation of the second kind*

$$ax^{a,\delta} + K^*Kx^{a,\delta} = K^*y^\delta. \tag{6.8}$$

   Every choice $a(\delta) \to 0$ and $\delta^2/a(\delta) \to 0$ as $\delta \to 0$ is admissible.

2. *Let $x = K^*z \in \mathcal{R}(K^*)$ with $\|z\| \leq E$. We choose $a(\delta) = c\,\delta/E$ for some $c > 0$. Then the following estimate holds:*

$$\left\|x^{a,\delta} - x\right\| \leq \frac{1}{2}\left(\frac{1}{\sqrt{c}} + \sqrt{c}\right)\sqrt{\delta E}.$$

Proof. Theorem 2.12, page 38 [8]. □

The eigenvalues of K tend to zero, and the eigenvalues of $aI + K^*K$ are bounded away from zero by $a > 0$. From the previous Theorem, we observe that a has to be chosen to depend on δ in such a way that it converges to zero as δ tends to zero but not as fast as δ^2. From part (2), we conclude that the smoother the solution x is the slower a has to tend to zero. On the other hand, the convergence can be arbitrarily slow if no a priori assumption about the solution x is available.

Theorem 6.13. *Let $K : X \to Y$ be linear, compact, and one-to-one such that the range $\mathcal{R}(K)$ is infinite-dimensional. Furthermore, let $x \in X$, and assume that there exists a continuous function $a : [0, \infty) \to [0, \infty)$ with $a(0) = 0$ such that*

$$\lim_{\delta \to 0} \left\|x^{a(\delta),\delta} - x\right\| \delta^{-2/3} = 0 \tag{6.9}$$

for every $y^\delta \in Y$ with $\left\|y^\delta - Kx\right\| \leq \delta$, where $x^{a(\delta),\delta} \in X$ solves (6.8). Then, $x = 0$.

Proof. Theorem 2.13, page 39 [8]. □

Thus, Tikhonov regularization for an ill-posed linear problem with compact operator never yields a convergence rate which is faster than $\mathcal{O}(\delta^{2/3})$. This is in contrast to, e.g., Landweber's method or the conjugate gradient method, i.e., a posteriori parameter choice methods.

If we want to solve $Kx = y$ under the information $\left\|y^\delta - y\right\| \leq \delta$, we have to accept any $x \in X$ with

$$\left\|Kx - y^\delta\right\| \leq \delta \tag{6.10}$$

as an approximate solution, since it is compatible with the only knowledge we have on the data. However, if $\mathcal{R}(K)$ is not closed, the set of all x satisfying the above inequality is unbounded even if $\mathcal{N}(K) = \{0\}$ reflecting the ill-posedness of $Kx = y$. Since we are looking for a solution of $Kx = y$ with minimal norm, it makes sense to use this as a constraint, in order to choose one solution of (6.10), i.e., consider the problem

$$\min \quad \|x\| \quad \text{subject to} \quad \left\|Kx - y^\delta\right\| \le \delta. \tag{6.11}$$

This is a constrained optimization problem and if 0 is not in the feasible region, i.e. $\left\|y^\delta\right\| \le \delta$ (less signal than noise), then the minimum is achieved on the boundary of the feasible set, that is, (6.11) is equivalent to,

$$\min \quad \|x\|^2 \quad \text{subject to} \quad \left\|Kx - y^\delta\right\|^2 = \delta^2. \tag{6.12}$$

Using Lagrange multipliers, we reformulate it to,

$$\min \quad \|x\|^2 + \lambda \left\|Kx - y^\delta\right\|^2. \tag{6.13}$$

This equivalent to the Tikhonov functional with $a = 1/\lambda$. This can be seen as a different motivation for Tikhonov regularization itself, but also provides a rule for choosing the regularization parameter, it should be chosen such that the constraint in (6.12) is satisfied.

6.1.2 The Discrepancy Principle of Morozov

In this section, we study a discrepancy principle based on the Tikhonov regularization method. Throughout this section, we assume again that $K : X \to Y$ is a compact and injective operator between Hilbert spaces X and Y with dense range $\mathcal{R}(K) \subset Y$. Again, we study the equation

$$Kx = y$$

for given $y \in Y$. The Tikhonov regularization, corresponds to the regularization operators (6.7) with $a > 0$ that approximate the unbounded inverse of K on $\mathcal{R}(K)$. We have seen that $x^a = R_a y$ exists and is the unique minimum of the Tikhonov functional. More facts about the dependence on a and y are proven in the following theorem.

Theorem 6.14. *Let $y \in Y$, $a > 0$ and x^a be the unique solution of the equation*

$$ax^a + K^*Kx^a = K^*y. \tag{6.14}$$

Then x^a depends continuously on y and a. The mapping $a \mapsto \|x^a\|$ is monotonously non increasing and

$$\lim_{a\to\infty} x^a = 0.$$

The mapping $a \mapsto \|Kx^a - y\|$ is monotonously non decreasing and

$$\lim_{a\to 0} Kx^a = y.$$

If $K^*y \neq 0$, then strict monotonicity holds in both cases.

Proof. Theorem 2.16, page 46 [8]. □

Now we consider the determination of $a(\delta)$ from the discrepancy principle. We compute $a = a(\delta) > 0$ such that the corresponding Tikhonov solution $x^{a,\delta}$, that is, the solution of the equation

$$ax^{a,\delta} + K^*Kx^{a,\delta} = K^*y^\delta. \tag{6.15}$$

that is, the minimum of

$$J_{a,\delta}(x) := \left\|Kx - y^\delta\right\|^2 + a\left\|x\right\|^2,$$

satisfies the equation

$$\left\|Kx^{a,\delta} - y^\delta\right\| = \delta. \tag{6.16}$$

Note that this choice of a by the discrepancy principle guarantees that, on the one side, the error of the defect is δ and, on the other side, a is not too small. Equation (6.16) is uniquely solvable, provided

$$\left\|y^\delta - y\right\| \leq \delta < \left\|y^\delta\right\|,$$

because by the previous theorem

$$\lim_{a\to\infty} \left\|Kx^{a,\delta} - y^\delta\right\| = \left\|y^\delta\right\| > \delta$$

and

$$\lim_{a\to 0} \left\|Kx^{a,\delta} - y^\delta\right\| = 0 < \delta.$$

Furthermore, $a \mapsto \left\|Kx^{a,\delta} - y^\delta\right\|$ is continuous and strictly increasing.

Theorem 6.15. *Let $K : X \to Y$ be linear, compact and one-to-one with dense range in Y. Let $Kx = y$ with $x \in X$, $y \in Y, y^\delta \in Y$ such that*

$$\left\| y^\delta - y \right\| \le \delta < \left\| y^\delta \right\| .$$

Let the Tikhonov solution x^a satisfy $\left\| Kx^{a,\delta} - y^\delta \right\| = \delta$ for all $\delta \in (0, \delta_0)$. Then

1. *$x^{a(\delta),\delta} \to x$ for $\delta \to 0$, that is, the discrepancy principle is admissible.*
2. *Let $x = K^* z \in K^*(Y)$ with $\|z\| \le E$. Then,*

   $$\left\| x^{a,\delta} - x \right\| \le 2\sqrt{\delta E}.$$

   Therefore, the discrepancy principle is an optimal regularization strategy under the information $\|K^{*-1}x\| \le E$.

Proof. Theorem 2.17, page 48 [8]. □

The condition $\left\| y^\delta \right\| > \delta$ certainly makes sense because otherwise the right-hand side would be less than the error level δ , and $x^\delta = 0$ would be an acceptable approximation to x.

The determination of $a(\delta)$ is thus equivalent to the problem of finding the zero of the monotone function

$$\phi(a) := \left\| Kx^{a,\delta} - y^\delta \right\|^2 - \delta^2$$

(for fixed $\delta > 0$). It is not necessary to satisfy the equation $\left\| Kx^{a,\delta} - y^\delta \right\| = \delta$ exactly. An inclusion of the form

$$c_1\delta \le \left\| Kx^{a,\delta} - y^\delta \right\| \le c_2\delta$$

is sufficient to prove the assertions of the previous theorem.

The computation of $a(\delta)$ can be carried out with Newton's method. The derivative of the mapping $a \mapsto x^{a,\delta}$ is given by the solution of the equation

$$(aI + K^*K)\frac{\partial}{\partial a}x^{a,\delta} = -x^{a,\delta},$$

as is easily seen by differentiating with respect to a.

We remark that the estimate $a(\delta) = \delta \|K\|^2 / (\left\| y^\delta \right\| - \delta)$ can be a starting value for Newton's method to determine $a(\delta)$.

The biggest disadvantage of Tikhonov regularization with Morozov's discrepancy principle is the repeated matrix manipulation done to compute the solution. Another disadvantage of Tikhonov regularization is the over-smoothing effect. In order to reconstruct non-smooth or discontinuous solutions, one has to use a different penalty term.

6.2 Bounded variation penalty methods

Let Ω be a bounded convex region in $\mathbb{R}^n$, $n = 1, 2, 3$ with Lipschitz continuous boundary $\partial\Omega$. We consider the equation,

$$Ku = y,$$

where K is a linear operator from $L^p(\Omega)$ into a Hilbert space H, containing y. We specify the norms,

$$|x| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2},$$

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty$$

and for $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} = \inf \{ c \in \mathbb{R} : |u(x)| < c \text{ for a.e. } x \in X \}$$

and $|\Omega|$ the Lebesgue measure of Ω and χ_S the indicator function of a set $S \subset \Omega$. We define the BV semi-norm, or total variation,

$$TV_0(u) := \sup_{v \in V} \int_\Omega (-u \operatorname{div} v) dx, \tag{6.17}$$

where the test functions v belong to

$$V = \{ v \in C_0^1(\Omega) : |v(x)| \leq 1, \quad \text{for all } \; x \in \Omega \}$$

If $u \in C^1(\Omega)$, then using integration by parts,

$$TV_0(u) = \int_\Omega |\nabla u(x)| dx. \tag{6.18}$$

Then, we consider the problem to minimize the Tikhonov functional, given now by,

$$J_a(x) := \|Ku - y\|^2 + a \, TV_0(u).$$

A more general penalty functional, than the TV semi-norm, can be also considered, for sufficiently smooth u,

$$TV_\beta(u) = \int_\Omega \sqrt{|\nabla u(x)|^2 + \beta} \, dx, \tag{6.19}$$

where $\beta \geq 0$. Of course, when $\beta = 0$ this reduces to the usual TV. The above norm can be defined also for non-smooth functions u as we are going to see later. The advantage of this form is that taking $\beta > 0$ we gain differentiation of the functional TV_β when $\nabla u = 0$.

Going back to (6.18), we state that this holds also for $u \in W^{1,1}(\Omega)$. Now, we are in position to define the space of functions of bounded variations on Ω by,

$$BV(\Omega) = \{u \in L^1(\Omega) : TV_0(u) < \infty\},$$

with BV norm

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + TV_0(u).$$

Remark 6.16. *The space $BV(\Omega)$ is a Banach space with respect to this norm. The Sobolev space $W^{1,1}(\Omega)$ is a proper subset of $BV(\Omega)$. Since Ω is bounded, $L^p(\Omega) \subset L^1(\Omega)$, for $p > 1$ and from the definition, $BV(\Omega) \subset L^1(\Omega)$.*

We define the convex function,

$$f(x) = \sqrt{|x|^2 + \beta},$$

given by

$$f(x) = \sup\{x \cdot y + \sqrt{\beta(1 - |y|^2)} : \ y \in \mathbb{R}^n, \ |y| \leq 1\}.$$

This function attains the supremum for $y = x/\sqrt{|x|^2 + \beta}$. In a similar way, we can define,

$$TV_\beta(u) := \sup_{v \in V} \int_\Omega -u \operatorname{div} v + \sqrt{\beta(1 - |v(x)|^2}\, dx. \tag{6.20}$$

Note that for $\beta > 0, TV_\beta$ is not a semi-norm.

Theorem 6.17. *If $u \in W^{1,1}(\Omega)$, then (6.19) holds.*

Proof. Since $C^1(\Omega)$ is dense in $W^{1,1}(\Omega)$, it is sufficient to show (6.19) for $u \in C^1(\Omega)$. From Green's theorem it follows that,

$$\begin{aligned}\int_\Omega -u \operatorname{div} v + \sqrt{\beta(1 - |v(x)|^2}\, dx &= \int_\Omega \nabla u \cdot v + \sqrt{\beta(1 - |v(x)|^2}\, dx \\ &\leq \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx\end{aligned}$$

where we have used the definition of f. Consequently,

$$TV_\beta(u) \le \int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx.$$

For the reverse inequality, we set $\bar{v} = \nabla u / \sqrt{|\nabla u|^2 + \beta}$, and observe that,

$$\int_\Omega \nabla u \cdot \bar{v} + \sqrt{\beta(1 - |\bar{v}(x)|^2}\, dx = \int_\Omega \sqrt{|\nabla u|^2 + \beta} dx$$

and $\bar{v} \in C(\Omega; R^n)$ with $|\bar{v}| < 1$ for all $x \in \Omega$. By multiplying $\bar{v}$ by a suitable characteristic function compactly supported in Ω, we can deduce $v \in V \cap C_0^\infty(\Omega)$ for which the left-hand side of the above inequality is arbitrary close to $\int_\Omega \sqrt{|\nabla u|^2 + \beta}\, dx$. □

The next theorem shows that both TV_0 and TV_β have the set $BV(\Omega)$ as their feasible region, and that TV_0 is the pointwise limit of TV_β.

Theorem 6.18. *1. For any $\beta > 0$ and $u \in L^1(\Omega)$,*

$$TV_0(u) < \infty \quad \textit{if and only if} \quad TV_\beta(u) < \infty.$$

2. For any $u \in BV(\Omega)$,

$$\lim_{\beta \to 0} TV_\beta(u) = TV_0(u).$$

Proof. For any $v \in V$ and $u \in L^1(\Omega)$,

$$\begin{aligned} \int_\Omega -u \operatorname{div} v\, dx &\le \int_\Omega -u \operatorname{div} v + \sqrt{\beta(1 - |v(x)|^2}\, dx \\ &\le \int_\Omega -u \operatorname{div} v + \sqrt{\beta}\, dx \end{aligned}$$

Taking the sup over $v \in V$,

$$TV_0(u) \le TV_\beta(u) \le TV_0(u) + \sqrt{\beta}|\Omega|.$$

The results follow from the boundedness of Ω. □

Theorem 6.19. *For any $\beta \ge 0$, TV_β is convex.*

Proof. Let $0 \leq \lambda \leq 1$, and u_1, $u_2 \in L^p(\Omega)$. For any $v \in V$,

$$\begin{aligned} TV_\beta(\lambda u_1 + (1-\lambda)u_2) &= \sup_{v\in V} \int_\Omega -(\lambda u_1 + (1-\lambda)u_2)\operatorname{div} v + \sqrt{\beta(1-|v(x)|^2}\, dx \\ &= \lambda \sup_{v\in V} \int_\Omega -u_1 \operatorname{div} v + \sqrt{\beta(1-|v(x)|^2}\, dx \\ &+ (1-\lambda) \sup_{v\in V} \int_\Omega -u_2 \operatorname{div} v + \sqrt{\beta(1-|v(x)|^2}\, dx \\ &\leq \lambda TV_\beta(u_1) + (1-\lambda) TV_\beta(u_2). \end{aligned}$$

□

Theorem 6.20. *For any $\beta \geq 0$, TV_β is weakly lower semicontinuous with respect to the L^p topology for $1 \leq p < \infty$.*

Proof. Let u_n converges weakly in $L^p(\Omega)$, i,e, $u_n \rightharpoonup \bar{u}$. For any $v \in V$, div$v \in C(\Omega)$ and hence,

$$\begin{aligned} \int_\Omega -\bar{u} \operatorname{div} v + \sqrt{\beta(1-|v|^2}\, dx &= \lim_{n\to\infty} \int_\Omega -u_n \operatorname{div} v + \sqrt{\beta(1-|v|^2}\, dx \\ &= \lim_{n\to\infty} \inf \int_\Omega -u_n \operatorname{div} v + \sqrt{\beta(1-|v|^2}\, dx \\ &\leq \lim_{n\to\infty} \inf TV_\beta(u_n) \end{aligned}$$

Taking the supremum over $v \in V$, gives $TV_\beta(\bar{u}) \leq \lim_{n\to\infty} \inf TV_\beta(u_n)$. □

A set of functions S is defined to be BV-bounded if there exists a constant $b > 0$ for which $\|u\|_{BV} \leq b$ for all $u \in S$. The relative compactness of $BV-$bounded sets in $L^p(\Omega)$ follows from the next lemma.

Lemma 6.21. *If $u \in BV(\Omega)$, then there exists a sequence $\{u_n\} \in C^\infty(\Omega)$ such that*

$$\lim_{n\to\infty} \|u_n - u\|_{L^p(\Omega)} = 0, \quad \lim_{n\to\infty} TV_0(u_n) = TV_0(u).$$

Theorem 6.22. *Let S be a BV-bounded set of functions. Then S is relatively compact in $L^p(\Omega)$ for $1 \leq p < d/(d-1)$. S is bounded, and hence relatively weakly compact for dimensions $d \geq 2$, in $L^p(\Omega)$ for $p = d/(d-1)$.*

Proof. Note that $d/(d-1)$ is the Sobolev conjugate of 1 in dimension d, the Sobolev conjugate of p, where $1 \leq p < d$, being defined by $l/p^* = l/p - l/d$. For $1 \leq p < d/(d-l)$, the Rellich-Kondrachov compact embedding theorem holds. A sequence u_n in S may then be approximated by a sequence of functions in $C^\infty(\Omega)$, themselves uniformly bounded in $BV(\Omega)$ and in $L^p(\Omega)$, so that their sequence must have a subsequence converging in $L^p(\Omega)$ to some u. By semicontinuity of TV_0 and the above lemma, $u \in BV(\Omega)$ and is the limit (in L^p) of a subsequence extracted from u_n.

For $p = d/(d-1)$, one can similarly extend to BV-functions the Poincark-Wirtinger inequality: if

$$\mu = \frac{1}{|\Omega|} \int_\Omega u(x)dx,$$

then there exists c such that

$$\|u - \mu\|_{L^p(\Omega)} \leq c\,TV_0(u - \mu) = c\,TV_0(u).$$

Hence, if, say, $\|u\|_{BV} \leq M$, then $TV_0(u - \mu)$ is also bounded by M, and, by the Poincare-Wirtinger inequality, $\|u - \mu\|_{L^p(\Omega)} \leq cM$. Consequently,

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &\leq \|u\chi_\Omega\|_{L^p(\Omega)} + \|u - \mu\|_{L^p(\Omega)} \\
&\leq |\mu||\Omega|^{1/p} + cM \\
&\leq \|u\|_{L^1(\Omega)} |\Omega|^{1/p-1} + cM \\
&\leq (|\Omega|^{1/p-1} + c)M \\
&=\leq (|\Omega|^{-1/d} + c)M
\end{aligned}$$

Relative weak compactness in dimensions $d \geq 2$ follows from the Banach-Alaoglu theorem. □

6.2.1 Well-posehess of minimization problems

Let $T : L^p(\Omega) \to [-\infty, \infty]$. The following theorems guarantee the well-posedness of the unconstrained minimization problem

$$\min_{u \in L^p(\Omega)} T(u). \tag{6.21}$$

In order to use the compactness results while still dealing with unconstrained minimization problems, we introduce the following property. We define T to be BV-coercive if

$$T(u) \to +\infty \quad \text{when} \quad \|u\|_{BV} \to +\infty. \tag{6.22}$$

Note that the sets $\{u \in L^p(\Omega) : T(u) \leq a\}$, where $a \geq 0$ are BV-bounded.

Theorem 6.23 (**Existence ad uniqueness of minimizers**). *Suppose that T is BV-coercive. If $1 \leq p < d(d-1)$ and T is lower semicontinuous, then problem* (6.21) *has a solution. If in addition $p = d/(d-l)$, dimension $d \geq 2$, and T is weakly lower semicontinuous, then a solution also exists. In either case, the solution is unique if T is strictly convex.*

Proof. The following argument is standard. Let u_n, be a minimizing sequence for T, in other words,

$$T(u_n) \to \inf_{u \in L^p(\Omega)} T(u) := T_{min}.$$

By hypothesis (6.22), the u_n are BV-bounded. As a consequence of theorem (6.22), there exists a subsequence u_{n_j} which converges to some $\bar{u} \in L^p(\Omega)$. Convergence is weak if $p = d/(d-1)$. By the (weak) lower semicontinuity of T,

$$T(u) \leq \lim_{j\to\infty} \inf T(u_{n_j}) = T_{min}.$$

Uniqueness of minimizers follows immediately from strict convexity. □

Next consider a sequence of perturbed problems

$$\min_{u \in L^p(\Omega)} T_n(u). \tag{6.23}$$

Theorem 6.24 (**Stability of minimizers**). *Assume that $1 \leq ped/(d-1)$ and that T and each of the T_n are BV-coercive, lower semicontinuous, and have a unique minimizer. Assume in addition:*

1. *Uniform BV-Coercivity: For any sequence $v_n \in L^p(\Omega)$,*

$$\lim_{n\to\infty} T_n(v_n) = +\infty \quad \textit{when} \quad \lim_{n\to\infty} \|v_n\|_{BV} = +\infty \tag{6.24}$$

2. *Consistency:* $T_n \to T$ *uniformly on BV-bounded sets, i.e. given* $b > 0$ *and* $\epsilon > 0$*, there exists* N *such that*

$$|T_n(u) - T(u)| < \epsilon \quad \text{when} \quad n \geq N,\ \|u\|_{BV} \leq b. \tag{6.25}$$

Then problem (6.21) *is stable with respect to the perturbations* (6.23)*, i.e. if* $\bar{u}$ *minimizes* T *and* u_n *minimizes* T_n*, then*

$$\|u_n - \bar{u}\|_{L^p(\Omega)} \to 0. \tag{6.26}$$

If $p = d/(d-1)$, $d \geq 2$, *and one replaces the lower semicontinuity assumption on* T *and each* T_n*, by weak lower semicontinuity, then convergence is weak,*

$$u_n \rightharpoonup \bar{u}. \tag{6.27}$$

Proof. Note that $T_n(u_n) \leq T_n(\bar{u})$. From this and equation (6.25),

$$\lim_{n\to\infty} \inf T_n(u_n) \leq \lim_{n\to\infty} \sup T_n(u_n) \leq T(\bar{u}) < \infty \tag{6.28}$$

and hence by (6.24) the u_n are BV-hounded. Now suppose (6.26) (or (6.27) if $p = d/(d-1)$) does not hold. By Theorem (6.22) there exists a subsequence u_{n_j}, which converges in $L^p(\Omega)$ to some $\hat{u} \neq \bar{u}$. By the (weak) lower semicontinuity of T, (6.28), and (6.25),

$$\begin{aligned} T(\hat{u}) &\leq \lim_{n\to\infty} \inf T(u_{n_j}) \\ &= \lim (T(u_{n_j}) - T_{n_j}(u_{n_j})) + \lim_{n\to\infty} \inf T_{n_j}(u_{n_j}) \\ &\leq T(\bar{u}) \end{aligned}$$

But this contradicts the uniqueness of the minimizer $\bar{u}$ of T. □

Example 6.25 (Existence-uniqueness). *Consider the problem of minimizing*

$$T(u) = \|Au - z\|_Z^2 + a\,\|u\|_{BV}$$

for $u \in L^p(\Omega)$*, where the restrictions on* p *in theorem* (6.23) *apply. Here* ($a > 0$ *and* $z \in Z$ *are fixed, and* $A : L^p(\Omega) \to Z$ *is hounded and linear. Then*

$$\|u\|_{BV} \leq \frac{1}{a} T(u)$$

and hence, the coercivity condition (6.22) *holds. Weak lower semicontinuity of T follows from the houndedness of A, the weak lower semicontinuity of the norms on Banach spaces, and theorem* (6.20)*. By theorem* (6.19)*, the linearity of A, and convexity of norms, T is convex. By theorem* (6.23) *a minimizer exists. T is strictly convex if A is injective, in which case the minimizer is unique.*

The following examples deal with stability. In the next example, assume again that the restrictions on p of theorem (6.23) apply.

Example 6.26 (**Perturbations in the data**). *Let*

$$T_n(u) = \|Au - z_n\|_Z^2 + a\,\|u\|_{BV}$$

where $z_n = z + \delta_n$, where $\|\eta_n\|_Z \to 0$ as $n \to \infty$. Then

$$|T_n(u) - T(u)| = |\,\|\delta_n\|_Z^2 + 2\,\langle Au - z, \delta_n\rangle_Z\,| \tag{6.29}$$

$$\leq \|\delta_n\|_Z\,(\|\delta_n\|_Z + 2\,\|A\|\,\|u\|_{L^p(\Omega)} + 2\,\|z\|_Z) \tag{6.30}$$

Here $\langle\cdot,\cdot\rangle_Z$ denotes the inner product on the Hilbert space Z, and the above inequality follows from Cauchy-Schwarz. Note that if u is BV-hounded, then it is norm bounded in $L^p(\Omega)$ by theorem (6.22)*, and hence* (6.25) *holds. Equation* (6.24) *holds because for each n,*

$$\|u\|_{BV} \leq \frac{T_n(u)}{a}. \tag{6.31}$$

Example 6.27 (**Perturbations of the operator A**). *Assume $1 \leq p < d/(d-1)$, and let*

$$T_n(u) = \|A_n u - z\|_Z^2 + a\,\|u\|_{BV}$$

where the A_n converge strongly (i.e. pointwise) in $L^p(\Omega)$ to A. Note that strong operator convergence is a reasonable assumption. It holds for consistent Galerkin approximations, e.g. finite element approximations as the mesh spacing $h \to 0$. Then

$$|T_n(u) - T(u)| = |\,\|A_n u\|_Z^2 - \|Au\|_Z^2 - 2\,\langle (A_n - A)u, z\rangle\,| \tag{6.32}$$

$$\leq (\|A_n u\|_Z + \|Au\|_Z + 2\,\|z\|_Z)\,\|(A_n - A)u\|_Z \tag{6.33}$$

Note that pointwise convergence of bounded linear operators becomes uniform on compact sets. Since BV-boundedness implies relative compactness in $L^p(\Omega)$, (6.25) *holds. Uniform coercivity* (6.24) *again holds because of* (6.31).

6.2.2 Convergence of minimizers

Assume an exact problem

$$Au = z \tag{6.34}$$

which has a unique solution $u_{ex} \in BV(\Omega)$. Assume a sequence of perturbed problems

$$A_n u = z_n \tag{6.35}$$

having approximate solutions u_n (not necessarily unique) obtained by minimizing the functionals

$$T_n(u) = \|A_n u - z_n\|_Z^2 + a_n \|u\|_{BV}.$$

The following theorem provides conditions which guarantee convergence of the u_n to u_{ex}.

Theorem 6.28. *Let $1 \le p \le d/(d-1)$. Suppose $\|z_n - z\|_Z \to 0$, $A_n \to A$ pointwise in $L^p(\Omega)$ and $a_n \to 0$ at a rate for which $\|A_n u_{ex} - z_n\|^2 / a_n$, remains bounded. Then $u_n \to u_{ex}$ strongly in $L^p(\Omega)$ if $1 \le p < d/(d-1)$. Convergence is weak in $L^p(\Omega)$ if $p = d/(d-1)$.*

Proof. Note that

$$\begin{aligned}\|A_n u_n - z_n\|_Z^2 &\le T_n(u_n) \le T_n(u_{ex}) \\ &= \|A_n u_{ex} - z_n\|_Z^2 + a_n \|u_{ex}\|_{BV}\end{aligned}$$

Thus from the assumption that $\|A_n u_{ex} - z_n\|_Z^2 / a_n$, remains bounded and the fact that $a_n \to 0$,

$$\|A_n u_n - z_n\|_Z^2 \to 0. \tag{6.36}$$

Similarly,

$$\|u_n\|_{BV} \le \frac{T_n(u_n)}{a_n} \le \frac{T_n(u_{ex})}{a_n} = \frac{\|A_n u_{ex} - z_n\|_Z^2}{a_n} + \|u_{ex}\|_{BV}$$

and hence, the u_n are BV-bounded. Suppose they do not converge strongly (weakly, if $p = d/(d-1)$) to u_{ex}. By theorem (6.22) there is a subsequence u_{n_j}, which converges strongly (weakly, respectively) in $L^p(\Omega)$ to some $\hat{u} \ne u_{ex}$. For any $v \in Z$,

$$\begin{aligned}|\langle A\hat{u} - z, v\rangle_Z| \le\ & |\langle A(\hat{u} - u_{n_j}), v\rangle_Z| + |\langle (A - A_{n_j}) u_{n_j}, v\rangle_Z| \\ & + |\langle A_{n_j} u_{n_j} - z_{n_j}, v\rangle_Z| + |\langle z_{n_j} - z, v\rangle_Z|\end{aligned}$$

The third and fourth terms on the right-hand side vanish as $j \to \infty$ because of (6.36) and the assumption $z_n \to z$. The second term also vanishes, since

$$| \left\langle (A - A_{n_j}) u_{n_j}, v \right\rangle_Z | \leq \left\| u_{n_j} \right\|_{L^p} \left\| (A^* - A^*_{n_j}) v \right\|_{L^p} \to 0$$

by the pointwise convergence of the A_n (and hence, their adjoints) and the norm boundedness of the u_n in $L^p(\Omega)$. The first term vanishes as well, taking adjoints and using the (weak) convergence of u_{n_j} to $\hat{u}$. Consequently, for $\langle A\hat{u} - z, v \rangle_Z = 0$ for any $v \in Z$, and hence $A\hat{u} = z$,. But this violates the uniqueness of the solution u_{ex} of (6.34). □

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