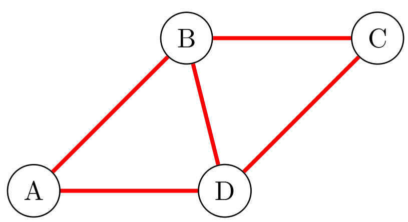
Exercise Sheet 2

1. Prove that the following propositions are equivalent:
   (a) A is total unimodular.
   (b) A^T is total unimodular.
   (c) (A, I) is total unimodular.

2. Show that the matrix
$$A := \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
is not total unimodular, but the solution of the linear system $Ax = b$, for every integral vector b, is integral.

3. Consider the following undirected graph $G = (V, E)$,



Construct the incidence matrix $A \in \{0,1\}^{4\times 5}$ of G and check if it is total unimodular. If it is not, give an example of a graph $G_t = (V, E_t)$, $E_t \subset E$, with $|E_t| = 3$, such that the incidence matrix A_t of G_t is total unimodular.

4. Consider the following two systems of inequalities:
$$(a) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$(b) \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Show that both of them describe the same polyhedron and check if they are total dual integral.

5. Consider the family of inequalities $A_k x \leq b_k$, $k = 1, 2, \ldots$ where

$$A_k = \begin{pmatrix} -1 & 0 \\ 1 & 2k \\ 1 & -2k \end{pmatrix} \quad \text{and} \quad b_k = \begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}$$

(a) Determine the polyhedron $P_k := \{x \in \mathbb{R}^2 : A_k x \leq b_k\}$ and find its integer hull P_I.

(b) Show that $P_{k-1} \subseteq P_k'$, where P' denotes the 0-th Gomory–Chvátal truncation of P.

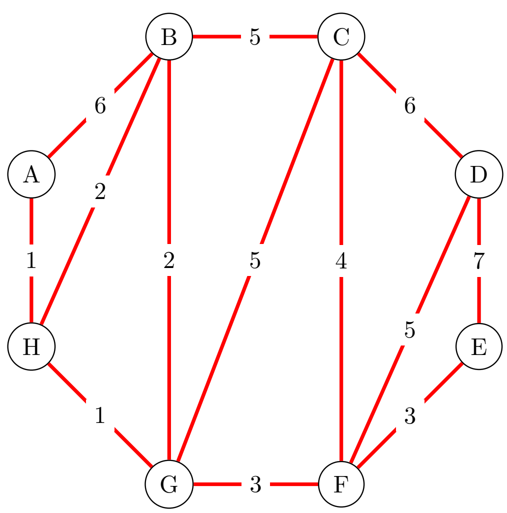
(c) Consider higher order truncations $P_k^{(t)}$ for $t < k$, then show that $P_k^{(t)} \neq P_I$

6. Solve the following Integer Linear Programs using the Gomory's cutting plane algorithm,

$$\begin{aligned} (a) \quad & \max\,(5x_1 + 2x_2) \\ & s.t. \quad 2x_1 + x_2 \leq 3 \\ & \qquad -2x_1 + x_2 \leq 0 \\ & \qquad x_1 \in \mathbb{Z}_+,\ x_2 \in \mathbb{Z}_+. \end{aligned}$$

$$\begin{aligned} (b) \quad & \max\,(x_1 + 2x_2) \\ & s.t. \quad x_1 + 4x_2 \leq 8 \\ & \qquad x_1 + \ \ x_2 \leq 12 \\ & \qquad x_1 \in \mathbb{Z}_+,\ x_2 \in \mathbb{Z}_+. \end{aligned}$$

7. Consider the following graph:



Apply the greedy algorithm to find the minimum spanning tree and it's cost. Is it unique? Write the sequence in which the edges are added to the minimum spanning tree.

8. The Best-in Greedy algorithm can result in different minimum spanning trees for the same input graph G, depending on the sorting of the edges with same weight. Modify the way to sort the edges of G, in Greedy algorithm, such that for each minimum spanning tree T of G, the algorithm returns T. Write a pseudo-code.