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Abstract. In this paper we establish a regularization method for Radon measures. Motivated from sparse L^1 regularization we introduce a new regularization functional for the Radon norm, whose properties are then analyzed. We, furthermore, show well-posedness of Radon measure based sparsity regularization. Finally we present numerical examples along with the underlying algorithmic and implementation details. We shall, here, see that the number of iterations turn out of utmost importance when it comes to obtain reliable reconstructions of sparse data with varying intensities.

1 Introduction

In this paper we consider the solution of the abstract equation

$$Fu' = v \text{ subject to } u' \in \operatorname{dom} F \,. \tag{1}$$

The operator F is linear and bounded between Hilbert spaces W' and V. We assume that $\operatorname{dom} F$ is a subset of Radon measures on a bounded domain $\Omega \subseteq \mathbb{R}^n$.

We consider solving the operator equation (1) approximately by a variational regularization method, which consists in minimizing the functional

$$\hat{\mathcal{T}}_{\alpha,v^\delta}(u') := \left\|Fu' - v^\delta\right\|_V^2 + \alpha \left\|u'\right\|_{\mathcal{RM}} \tag{2}$$

on $\operatorname{dom} F \subseteq W'$. Here $\|u'\|_{\mathcal{RM}}$ is the norm of the Radon measure u'.

In order to see the relation to *sparsity* we note that if u' is absolutely continuous with density U, i.e., $Udx = du'$, then we have that

$$\|u'\|_{\mathcal{RM}} = \sup\left\{\int_\Omega Uv dx : v \in C_0(\Omega), \|v\|_{L^\infty} \le 1\right\} = \|U\|_{L^1} \,.$$

The regularization method with $\hat{\mathcal{T}}_{\alpha,v^\delta}$, where the Radon measure is replaced by the L^1-norm, has been analyzed in [13]. There, however, different assumptions have been made that guarantee existence of a minimizer in $L^1(\Omega)$, while in this work we consider minimizers, which are Radon measures. The notion of sparsity appears in a variety of settings. In the context of regularization it is mostly used in connection with regularization terms

$$\mathcal{R}_S(u') := \sum \omega_i \left|\langle u', \phi_i \rangle\right| ,$$

where ϕ_i is a set of appropriate functions, typically forming a basis or frame. The inner product is on a Hilbert space and ω_i are positive coefficients. We refer to a few papers, which are related to this topic [7, 2–5, 8–12, 14]. Some researchers even call total variation minimization sparsity regularization. We study the reconstruction of sparse functions and measures. In contrast to total variation regularization we focus on reconstructing sparse measures and not gradient measures. There is a fundamental difference between regularization terms $\mathcal{R}_S$ and L^1, respectively Radon measure regularization. To see this, take (ϕ_i) an orthonormal basis and $\omega_i = 1$ in the definition of $\mathcal{R}_S(u')$ and note that standard convex analysis in the Hilbert space l^2 is applicable. Note that $l^1 \subseteq l^2$ and therefore we can consider minimization of $u' \to \left\|Fu' - v^\delta\right\|^2 + \alpha\mathcal{R}_S(u')$ over $l^2 \equiv L^2(\Omega)$. That is, there is a proper extension of the functional from l^1 to l^2 if the operator F can be extended on l^2. However, convex analysis in the Hilbert spaces L^2 is not applicable for $\|\cdot\|_{L^1}$ Regularization, since on domains with finite measure, $L^2(\Omega) \subset L^1(\Omega)$, and minimization of $u' \to \left\|Fu' - v^\delta\right\|^2 + \alpha \left\|u'\right\|_{L^1}$ over $L^2(\Omega)$ is a real restriction of the proper domain of the regularization functional, which is $L^1(\Omega)$. The curiosity is that after discretization with piecewise constant functions of the later a truncated expansion of the former is revealed.

The outline of this paper is as follows: In Section 2 we give a review on the analysis of regularization methods. In Section 4 we review some basic facts on Radon measures and duals of Sobolev spaces. Having specified the ingredients we apply the general results of the review sections to $\hat{\mathcal{T}}_{\alpha,v^\delta}$ in Section 3 and show well–posedness, and regularizing properties. Section 5 shows the analogy in the analysis to total variation minimization. Section 6 presents an example for sparse recovery and shows some reconstructions.

2 Review on Convergence Properties of Variational Regularization Methods

In this section we make the following general assumptions, where we stick to the notation of [13]. Afterwards, we apply the results to the setting already used in the introduction.

Assumption 1.

1. Let U and V be Hilbert spaces.
2. $L : U \to V$ is a bounded linear operator.
3. $F := L|_{\operatorname{dom} F}$, where $\emptyset \neq \operatorname{dom} F$ is closed and convex in U.
4. τ_U and τ_V are the weak topologies on U and V, respectively.

We consider now the solution of the abstract equation

$$Fu = v \text{ subject to } u \in \operatorname{dom} F \ . \tag{3}$$

We consider solving this operator equation by variational regularization methods, which consist in minimizing the functional

$$\mathcal{T}_{\alpha,v^\delta}(u) := \left\|Fu - v^\delta\right\|_V^2 + \alpha\mathcal{R}(u)$$

where $v^\delta \in y$. For most applications it will be considered a noisy approximation of v as in equation 3.

In order to have regularization properties of the family $(\mathcal{T}_{\alpha,v^\delta})$ it is required that $\mathcal{R}$, $\|\cdot\|_V$, and L satisfy:

Assumption 2.

1. The norm $\|\cdot\|_V$ is sequentially lower semi-continuous with respect to τ_V.
2. The functional $\mathcal{R} : U \to [0, \infty]$ is convex and sequentially lower semi-continuous with respect to τ_U. $\operatorname{dom}\mathcal{R} = \{u : \mathcal{R}(u) \neq \infty\}$ is the domain of $\mathcal{R}$.
3. $\mathcal{D} := \operatorname{dom} F \cap \operatorname{dom}\mathcal{R} \neq \emptyset$ (which, in particular, implies that $\mathcal{R}$ is proper).
4. For every $\alpha > 0$ and $M > 0$, the level sets

   $$\mathcal{M}_\alpha(M) := \operatorname{level}_M(\mathcal{T}_{\alpha,v}) := \{u \in U : \mathcal{T}_{\alpha,v}(u) \leq M\}$$

   are sequentially pre-compact with respect to τ_U.
5. For every $M > 0$ the set $\mathcal{M}_\alpha(M)$ is sequentially closed with respect to τ_U and the restriction of F to $\mathcal{M}_\alpha(M)$ is sequentially continuous with respect to the topologies τ_U and τ_V.

We stress that the sets $\mathcal{M}_\alpha(M)$ are defined based on the Tikhonov functional for unperturbed data v and we do not a-priori exclude the case that $\mathcal{M}_\alpha(M) = \emptyset$.

We refer to the following theorems from [13], which guarantee the existence of a minimizer, stability of the regularized solutions, and convergence:

Theorem 3 (Existence). *Let F, $\mathcal{R}$, $\mathcal{D}$, U, and V satisfy Assumption 2. Assume that $\alpha > 0$ and $v^\delta \in V$. Then, there exists a minimizer of $\mathcal{T}_{\alpha,v^\delta}$.*

It has been shown by several authors that information on the noise level

$$\left\|v^\delta - v\right\| \leq \delta \tag{4}$$

is essential for an analysis of regularization methods. In fact without this information the regularization cannot be chosen such that convergence of u_α^δ to a solution of equation 1 can be guaranteed.

Theorem 4 (Stability). *Let F, $\operatorname{dom} F$, U, and V satisfy Assumption 2. Assume that $\alpha > 0$ and $v_k \to v^\delta$. Moreover, let*

$$u_k \in \arg\min \mathcal{T}_{\alpha,v_k} \ , \qquad k \in \mathbb{N} \ .$$

Then, (u_k) has a convergent subsequence. Every convergent subsequence converges to a minimizer of $\mathcal{T}_{\alpha,v^\delta}$.

The following theorem clarifies the role of the regularization parameter α. It has to be chosen in dependence of the noise level to guarantee approximation of the solution of (3).

Theorem 5 (Convergence). *Let F, $\operatorname{dom} F$, U, and V satisfy Assumption 2. Assume that* (3) *has a solution in* $\operatorname{dom} F$ *and that* $\alpha : (0,\infty) \to (0,\infty)$ *satisfies*

$$\alpha(\delta) \to 0 \text{ and } \frac{\delta^2}{\alpha(\delta)} \to 0\,, \text{ as } \delta \to 0\,.$$

Moreover, let the sequence (δ_k) of positive numbers converge to 0, and assume that the data $v_k := v^{\delta_k}$ satisfy $\|v - v_k\| \le \delta_k$.

Let $u_k \in \arg\min \mathcal{T}_{\alpha(\delta_k),v_k}$. Then (u_k) has a convergent subsequence to a solution of (1).

3 Regularization on the Space of Radon Measures

We assume that $\Omega \subseteq \mathbb{R}^n$ and $\Omega' \subseteq \mathbb{R}^m$ are bounded, open and connected with Lipschitz boundary, respectively.

For the sake of simplicity of presentation we take $V = L^2(\Omega')$. Other spaces can be considered but then the notation is not that transparent anymore.

We consider and study minimization of the functional

$$\hat{\mathcal{T}}_{\alpha,v^\delta}(u') := \int_{\Omega'} (Fu' - v^\delta)^2 + \alpha \, \|u'\|_{\mathcal{RM}} \tag{5}$$

over the set of Radon measures on Ω. Here, $\|u'\|_{\mathcal{RM}}$ denotes the norm of the Radon measure of u'.

Radon Measures

Below we shortly review some facts about Radon measures, and specify the according properties.

The set of *Radon measures* is the dual of $C_0(\Omega)$. Here, $C_0(\Omega)$ is the space of continuous functions from Ω into $\mathbb{R}$ with compact support in Ω. We always consider $C_0(\Omega)$ equipped with the supremum norm. We denote the dual by $\mathcal{M} := (C_0(\Omega))'$ and for $u' \in \mathcal{M}$ the Radon measure is defined by

$$\|u'\|_{\mathcal{RM}} := \sup \left\{ \int_\Omega v du' : v \in C_0(\Omega), \|v\|_{L^\infty} \le 1 \right\} .$$

We recall the definition of weak* convergence in $\mathcal{M}$, i.e., a bounded sequence $(u'_k)_k$ in $\mathcal{M}$ is weakly* convergent to $u' \in \mathcal{M}$ if

$$\lim_{k\to\infty} \int_\Omega f du'_k = \int_\Omega f du' \text{ for all } f \in C_0(\Omega)\,.$$

Below we show that $\|\cdot\|_{\mathcal{RM}}$ is lower semi-continuous with respect to the weak* convergence on $\mathcal{M}$.

Lemma 1. $\|\cdot\|_{\mathcal{RM}}$ *is lower semi-continuous with respect to the weak* convergence on* $\mathcal{M}$.

Proof. Let a sequence of Radon measures $(u'_k)_k$ be weakly* convergent to some measure u'. Then,

$$\begin{aligned}\|u'\|_{\mathcal{RM}} &= \sup\left\{\int_\Omega v du' : v \in C_0(\Omega), \|v\|_{L^\infty} \le 1\right\}\\ &= \sup \lim_{k\to\infty}\left\{\int_\Omega v du'_k : v \in C_0(\Omega), \|v\|_{L^\infty} \le 1\right\}\\ &\le \liminf_{k\to\infty} \|u'_k\|_{\mathcal{RM}} \;.\end{aligned}$$

Dual of a Sobolev Space

Let $s \in \mathbb{N}$ be fixed. In the following we investigate the dual of the Sobolev space $\mathcal{W} := W_0^{s,2}(\Omega)$, which is a Hilbert space with the inner product

$$\langle w_1, w_2\rangle_s := \int_\Omega \nabla^s w_1 \cdot \nabla^s w_2 \,,$$

where ∇^s is the tensor containing all s-th derivatives. The associated norm is denoted by $\|w'\|_s$. For $w' \in \mathcal{W}'$, the dual of $W_0^{s,2}(\Omega)$, we have $\|w'\|_{-s} := \sup\{w'\tilde{w} : \tilde{w} \in \mathcal{W}, \|\tilde{w}\|_s \le 1\}$. $\mathcal{W}'$ satisfies the following properties:

1. From the Riesz representation theorem (see e.g. [6, Theorem 3.4]) it follows that for every $w' \in \mathcal{W}'$ there exists $w \in \mathcal{W}$ such that
$$w'\tilde{w} = \langle w, \tilde{w}\rangle_s$$
for all $\tilde{w} \in \mathcal{W}$.
We define the Riesz mapping
$$\mathcal{I}w' = w \,, \tag{6}$$
and note that $\mathcal{I}$ is an isomorphism between $\mathcal{W}'$ and $\mathcal{W}$, i.e., $\|\mathcal{I}w'\|_s = \|w'\|_{-s}$. In particular, we have that $(w'_k)_k \to w'$ with respect to the topology $\tau_{\mathcal{W}'}$ if and only if $(w_k)_k = (\mathcal{I}w'_k)_k \to \mathcal{I}w' = w$ with respect to the topology $\tau_{\mathcal{W}}$.
2. The inner product on the Hilbert space $\mathcal{W}'$ can be defined by $\langle w'_1, w'_2\rangle_{-s} = \langle w_1, w_2\rangle_s$, where w_1, w'_1 and w_2, w'_2 are related by the Riesz representation theorem, respectively.

Now, we state a lemma, which is central for our further considerations:

Lemma 2. *Let* $2s > n$*; Recall that* s *is the order of differentiation in the definition of* $\mathcal{W}$ *and* n *is the dimension of* Ω*. Then*

1. $\|\cdot\|_{\mathcal{RM}}$ *is convex and lower semi-continuous on* $\mathcal{W}'$.
2. $\mathcal{M}$ *is closed in* $\mathcal{W}'$.

3. *There exists a constant C such that $\|w'\|_{-s} \leq C \|w'\|_{\mathcal{RM}}$ for all $w' \in \mathcal{M}$.*

Proof. We make some general statements first. Since, by assumption $2s > n$, the Sobolev embedding theorem (see [1, Thm. 5.4]) guarantees that the embedding from $\mathcal{W}$ into $C_0(\Omega)$ is bounded, i.e., there exists a constant C such that

$$\|u\|_{L^\infty} \leq C \|u\|_s \text{ for all } u \in \mathcal{W} . \tag{7}$$

Since $C_0^\infty(\Omega)$ is dense in $\mathcal{W}$ and $C_0(\Omega)$ (with respect to the topologies of $\mathcal{W}$ and $C_0(\Omega)$, respectively), we have

$$\begin{aligned} \|u'\|_{\mathcal{RM}} &= \sup\left\{u'v : v \in C_0(\Omega), \|v\|_{L^\infty} \leq 1\right\} \\ &= \sup\left\{u'v : v \in C_0^\infty(\Omega), \|v\|_{L^\infty} \leq 1\right\} \\ &= \frac{1}{C} \sup\left\{u'v : v \in C_0^\infty(\Omega), \|v\|_{L^\infty} \leq C\right\} \\ &\geq \frac{1}{C} \sup\left\{u'v : v \in C_0^\infty(\Omega), \|v\|_s \leq 1\right\} \\ &= \frac{1}{C} \sup\left\{u'v : v \in W_0^{s,2}(\Omega), \|v\|_s \leq 1\right\} \\ &= \|u'\|_{-s} . \end{aligned}$$

Thus, $\mathcal{M} \subseteq \mathcal{W}'$.

1. Let $(u'_k)_k$ be a sequence of Radon measures, which is convergent to u' in $\mathcal{W}'$ (i.e., with respect to $\tau_{\mathcal{W}'}$). It remains to prove that u' is a Radon measure. Since $(u'_k)_k$ is bounded in $\mathcal{W}'$, it is also weakly* convergent in $\mathcal{W}'$, meaning that $u'_k v \to u'v$ for all $v \in \mathcal{W}$. Then, in particular, we have $u'_k v \to u'v$ for all $v \in C_0^\infty(\Omega)$. Now, let $v \in C_0^\infty(\Omega)$ satisfy $\|v\|_{L^\infty} \leq 1$, then

$$\begin{aligned} u'v &= \lim_{k\to\infty} u'_k v \\ &\leq \lim_{k\to\infty} \sup\left\{u'_k \tilde{v} : \tilde{v} \in C_0(\Omega), \|\tilde{v}\|_{L^\infty} \leq 1\right\} \\ &\leq \liminf_{k\to\infty} \|u'_k\|_{\mathcal{RM}} . \end{aligned} \tag{8}$$

   Since $C_0^\infty(\Omega)$ is dense in $C_0(\Omega)$, the last inequality shows that $\|u'\|_{\mathcal{RM}} \leq \liminf_{k\to\infty} \|u'_k\|_{\mathcal{RM}}$ and, thus, u' is a Radon measure.
2. From (8) it also follows that $\|.\|_{\mathcal{RM}}$ is lower semi-continuous on $\mathcal{W}'$. The convexity is trivial.
3. Using (7) it follows that

$$\begin{aligned} \|w'\|_{-s} &= \sup\left\{w'\tilde{w} : \tilde{w} \in \mathcal{W}, \|w\|_s \leq 1\right\} \\ &\leq \sup\left\{w'\tilde{w} : \tilde{w} \in \mathcal{M}, \|w\|_{L^\infty} \leq C\right\} \\ &= C \|w'\|_{\mathcal{RM}} . \end{aligned}$$

   This gives the third assertion.

4 Application to Variational Regularization on Radon Measures

We consider minimization of $\hat{\mathcal{T}}_{\alpha,v^\delta}$ on $\mathcal{W}'$, the dual of the Sobolev space $W_0^{s,2}(\Omega)$, with $\operatorname{dom} F := \mathcal{M}$, the space of Radon measures, and $L : \mathcal{W}' \to L^2(\Omega')$ as in Assumption 1 bounded. Here $\mathcal{W}'$, $L^2(\Omega')$ play the role of U and V in Assumption 1; i.e., we consider the weak topologies on $\mathcal{W}'$ (not that since $\mathcal{W}'$ is a Hilbert space, weak and weak* convergence can be identified) and $L^2(\Omega')$. Note that in our notation of Assumption 1 we use here $F := L|_{\operatorname{dom} F}$.

In order to apply the general results stated in Section 1 we have to verify Assumption 2. The requirement in Assumption 1 that $\operatorname{dom} F = \mathcal{M}$ is closed in $\mathcal{W}'$, has already been shown in Lemma 2.

1. We recall that every norm on a Hilbert space is continuous and convex with respect to the weak topology. Therefore, $\|\cdot\|_{\mathcal{W}'}$ is sequentially weakly lower semi-continuous with respect to $\tau_{\mathcal{W}'}$.
2. The functional $\mathcal{R}(\cdot) := \|\cdot\|_{\mathcal{RM}}$ is convex and lower semi-continuous, which has already been shown in Lemma 2.
3. The set of Radon measures, which equals the domain $\mathcal{D}$, is not empty.
4. Let $\alpha > 0$, $M > 0$, and let $(u_k')_k$ be a sequence in $\mathcal{M}_\alpha(M)$. We show that $(u_k')_k$ has a convergent subsequence with respect to $\tau_{\mathcal{W}'}$. From the definition of $\hat{\mathcal{T}}_{\alpha,v^\delta}$ it follows that $(\|u_k\|_{\mathcal{RM}})_k$ is bounded and, therefore, from Lemma 2 it follows that $(u_k')_k$ is bounded with respect to $\|\cdot\|_{-s}$. Thus, $(u_k')_k$ has a subsequence which weakly converges in $\mathcal{W}'$. This shows that the sequence is sequentially precompact with respect to $\tau_{\mathcal{W}'}$.
5. Let us follow up on the proof of the previous item.
   - Let us denote the weak limit of $(u_k')_k$ by u' in $\mathcal{W}'$. We prove that $u' \in \mathcal{M}_\alpha(M)$. We use that $\|.\|_{\mathcal{RM}}$ is lower semi-continuous with respect to $\mathcal{W}'$. Moreover, since $L : \mathcal{W}' \to L^2(\Omega')$ is bounded, the functional $w' \to \left\|Lw' - v^\delta\right\|^2$ is lower semicontinuous with respect to $\mathcal{W}'$. Thus, the sum of both terms is lower semi-continuous and thus $u' \in \mathcal{M}_\alpha(M)$. Thus $\mathcal{M}_\alpha(M)$ is sequentially closed.
   - The operator $L|_{\operatorname{dom} F}$ is weakly continuous and $\operatorname{dom} F$ is weakly sequentially closed, which follows from Lemma 2, which states that $\operatorname{dom} F = \mathcal{M}$ is closed and convex, and since L is bounded on $\mathcal{W}'$.

Therefore, Assumption 2 is satisfied and the assertions follow.

Theorem 5 requires the existence of a solution of (3) in $\mathcal{D}$. Thus, for the application of this result the existence of a solution with finite Radon measure is required.

5 Methodological comparison with finite total variation regularization

The method which we are proposing is methodologically related to *total variation minimization*, which can be viewed as the relaxation of $W^{1,1}$–regularization, which in turn consists in minimization of the functional

$$u \to \int_{\Omega'} (Fu - v^\delta)^2 + \alpha \int_\Omega |\nabla u| \ .$$

Total variation minimization consists in minimization of $u \to \int_{\Omega'} (Fu - v^\delta)^2 + \alpha\,|Du|$, where $|Du|$ is the total variation of u, which is the norm of the finite, vector valued, Radon measure Du. In our context the regularization is with respect to Radon measures, which is a relaxation of L^1–regularization. Thus, total variation regularization can be considered as a regularization method on Radon measures for the first derivatives of the function, while according to our theory, L^1-regularization is for the distributions in $W^{-2,2}(\Omega)$. The derived analogy is not completely satisfactory and certainly subject to further research. The analogy to total variation minimization suggests that the smallest Sobolev space, which is a Hilbert space and contains the Radon measures, is $W^{-1,2}(\Omega)$. However, based on our analysis so far, this space is slightly too small to perform analytical studies. Our analysis is based on using the standard Sobolev embedding theorem and as a consequence, slightly more regularity properties on the linear operator F have to be imposed, than expected from the comparison with the total variation analysis.

6 Application in Nuclear Medicine

Apart from a purely theoretical background the concept of sparse data also proves relevant to a variety of real-world applications. As far as the imaging point of view is concerned we consider the field of nuclear medicine one major area of interest. Basically, however, any type of peaky (clustered) data on an otherwise relatively homogeneous background appears suitable for *sparsity reconstruction*. In the following we give a short description of the above research topic in order to provide a short introduction to the practical part of sparsity regularization: The two most popular techniques in nuclear medicine, PET (**P**ositron **E**mission **T**omography) respectively SPECT (**S**ingle **P**hoton **E**mission **T**omography), both rely on nuclear disintegration. Here, a tomographic scanner measures the decay of a radioactive tracer substance which has previously been injected into the patients body. Such a procedure, e.g., often appears in cancer diagnosis.

As far as the field of imaging is concerned we consider the related isotopes our sparse data. Based on the respective measurements we obtain a so-called *sinogram*, plotting the number of radioactive disintegrations against the different scanner angles. The actual image is, then reconstructed according to the given sinogram. In the medical imaging context sparse variational reconstructions have already been used for *MRI RF* excitation pulse design in [15].

6.1 Algorithm Characteristics

The current section focuses on the most important implementation characteristics of the main reconstruction algorithms involved in sparsity reconstruction.

Firstly, we have decided to apply our sample data (see Paragraph 6.2) to the following *Daubechies, Defrise, DeMol* [7] (DDD)-*type* implementation

$$u^{k+1} := u^k - \lambda F^*(Fu^k - v^\delta) - \alpha \operatorname{sgn}(u^{k+1}) \tag{9}$$

where the last term represents the *sign* (denoted by the sgn) operator, applied to the next step reconstruction, and may also be expressed by $\frac{u^{k+1}}{|u^{k+1}|}$. We, thus, obtain an alternative formulation

$$S^{-1}(u^{k+1}) := \left(1 + \frac{\alpha}{|u^{k+1}|}\right) u^{k+1} = u^k - \lambda F^*(Fu^k - v^\delta) \ . \tag{10}$$

As indicated by the notation the set valued operator S^{-1} contains a univariate inverse and therefore, we get an implementable scheme by applying the inverse of S^{-1}:

$$u^{k+1} = S(u^k - \lambda F^*(Fu^k - v^\delta)) \ . \tag{11}$$

where

$$S(t) := \begin{cases} t + \alpha & \text{if } t \leq -\alpha \\ t - \alpha & \text{if } t \geq +\alpha \\ 0 & \text{else.} \end{cases} \tag{12}$$

We refer to this implementation as of DDD-*type*, since the implementation is for function (actually measures) and not basis coefficients, as the original sparsity is devoted to. Aside from this difference it is the algorithm suggested in [7]. The numerical implementation is for piecewise constant functions approximating Radon measures. The situation is analogous as in the case of total variation regularization with finite elements where derivative (which are Radon measures) are approximated by derivatives of finite element.

6.2 Experimental Results

In order to test the practical relevance of the above method we have created test data with a constant background exhibiting (clusters of) peaks as we consider them the most realistic scenario.

Most practical acquisition devices, however, rarely yield noise free data, which has lead to the decision of adding to our sample data v different types of noise. I. e., in order to achieve a proper real-world scenario we restrict the input to our reconstruction algorithms (see Paragraph 6.1) to noisy sinograms v^δ only. Since the tested algorithms are mainly intended for medical use we have decided to adapt the sample framework to the nature of nuclear medical data acquisition. Most underlying processes in this field exhibit a clear *Poisson* nature, which has motivated the decision to overlay the clear sinogram data with typical Poisson noise. From a programming point of view we have decided to allow for the specification of four different parameters, each of which may have a certain influence on the outcome of the reconstruction process. The weighting parameters λ and α from Equations (9) to (12) appear an obvious choice in this case. Furthermore, we have added one algorithm-independent input parameter, i.e., the number of iteration cycles. With the above implementation details specified, we have, finally, submitted the DDD-*type* algorithm from Paragraph 6.1 to different test cases.

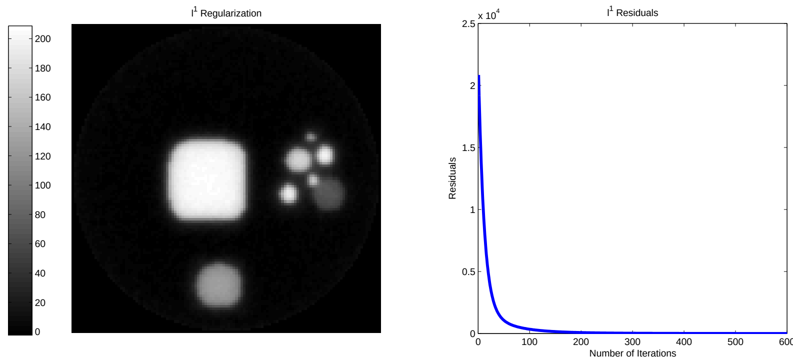

Fig. 1. The above figure is to illustrate the convergence behavior of our proposed regularization scheme from a practical point of view. The right hand side plot shows the declining residuals obtained during the computation process yielding the reconstruction image to the left.

Number of Iterations: As obvious from the problem statement in Equations (9) to (12) the final reconstruction is created from iteratively updating the current reconstruction image. In most cases the starting image will be of random nature. The number of iterations may, thus, have a certain impact on the outcome of the reconstruction process. For our algorithm we have created test cycles within the range of $[25, 1600]$, with the remaining parameters fixed. In this respect we have determined 50 cycles as the minimum value for obtaining a relatively reliable result. Note, however, that here, object boundaries appear blurred on an otherwise constant background. With an increasing number of iterations the different objects become sharper, while on the other hand we are faced with the problem of an ever more inhomogeneous background.

Weighting Parameters: As described in paragraph 6.1 the implementation includes two weighting parameters λ and α closely related to each other. Since we consider the role of the first one to be of higher importance we have decided for a ratio-based test environment. I. e., setting λ with the range of $[0.016, 0.16]$ we have evaluated the quality of the reconstructions with α at $\frac{\lambda}{10^n}$, where $1 \leq n \leq 4$. The described test framework has, furthermore, helped in limiting the computational power involved to a reasonable extent. Interestingly our experiments have shown that the ratio between λ

and α turns out less important provided the first parameter is selected 'correctly'. There were no obvious differences between images with $\alpha = \frac{\lambda}{10^2}$ or $\alpha = \frac{\lambda}{10^3}$. On the other side, we have noticed lower values of λ producing a more homogeneous background while higher ones resulted in sharper object boundaries. In this respect the effects appear similar to those described for varying numbers of iterations.

Fig. 2. Decreasing values of α tend to sharpen even smaller object boundaries but at the same time also produce more background noise. Increasing the parameter, however, results in a quite homogeneous background while blurring and sometimes even removing smaller objects.

Finally we may conclude that there exists a certain relation between the number of iterations and the choice of λ. The higher we set the weighting parameter the sooner we have to stop the iterative cycle in order to limit the background inhomogeneities to a certain extent.

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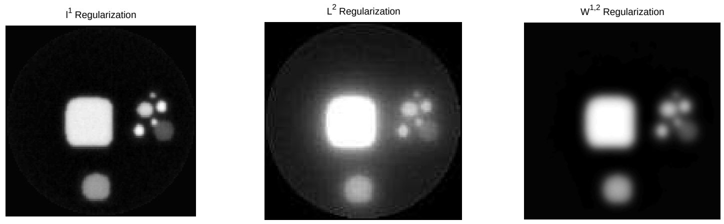

Fig. 3. The above figures are intended to compare our benchmark results to those of other popular methods, e. g., L^2 and $W^{1,2}$ regularization. Here, l^1, as depicted to the left tends to yield the clearest approximations of the original objects. We have however noticed, that in some cases small peaks may not be preserved during the regularization process. On the other hand, L^2 appears not only slightly more blurred but also fails to remove the circle object caused by the Radon Transform which we consider a major drawback. Finally, $W^{1,2}$ regularization tends to produce strong object blurs which may be a problem not only for small and dense peaks but also deteriorate the overall reconstruction quality.

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