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CHARACTERIZATION OF MINIMIZERS OF CONVEX REGULARIZATION FUNCTIONALS

CHRISTIANE PÖSCHL AND OTMAR SCHERZER

Abstract. We study variational methods of bounded variation type for the data analysis. Y. Meyer characterized minimizers of the Rudin-Osher-Fatemi functional in dependence of the G-norm of the data. These results and the follow up work on this topic are generalized to functionals defined on spaces of functions with derivatives of finite bounded variation. In order to derive a characterization of minimizers of convex regularization functionals we use the concept of generalized directional derivatives and duality. Finally we present some examples where the minimizers of convex regularization functionals are calculated analytically, repeating some recent results from the literature and adding some novel results with penalization of higher order derivatives of bounded variation.

1. Introduction

This paper is concerned with variational methods, consisting in minimization of the functional

$$\mathcal{F}(u) := \mathcal{S}(u) + \alpha \left| D^k u \right| , \quad k = 1, 2, \ldots, \alpha > 0$$

for the analysis of data u^δ. Here

(1) $\left| D^k u \right|$ denotes the total variation of the $(k-1)$-th derivative of u and
(2) $\mathcal{S}(u)$ is a similarity measure. Typical examples are $\mathcal{S}(u) = \frac{1}{p} \int_\Omega \left| u - u^\delta \right|^p$.

Y. Meyer [Mey01] characterized minimizers of the ROF-functional (introduced in [RudOshFat92]), where $\mathcal{S}(u) = \frac{1}{2} \int_\Omega (u - u^\delta)^2$ and $k = 1$, in dependence of the G-norm of u^δ. This research has significant impact on the research in image analysis.

In this paper we use an alternative characterization based on *Fenchel's duality theorem* and *generalized directional derivatives* to generalize the results of Y. Meyer and the follow up work [OshSch04, SchYinOsh05]. Moreover, the results can also be applied to characterize minimizers of regularization functionals penalizing for derivatives with finite total variation. This generalizes the ideas in [ObeOshSch05]. Non-differentiable regularization functionals for higher order derivatives have attracted several research (see for instance [ChaLio95, Sch98, ChaMarMul00, SteDidNeu05, SteDidNeu, Ste06, HinSch06]). The abstract results in this paper also allow to characterize minimizers of *metrical regularization* functionals, such as L^1-$BV(\Omega)$ regularization (see for instance Chan & Esedoglu [ChaEse05], Nikolova [Nik03a, Nik03b]) and showing a structure in regularization methods of this type.

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Moreover, exploiting the Fenchel duality concept we exemplarily derive explicit solutions for minimizers of the ROF-functional for denoising one-dimensional data (repeating the results of Strong & Chan [StrCha96] and Y. Meyer [Mey01]), the L^1-$BV(\Omega)$ regularization (repeating the results of Chan & Esedoglu [ChaEse05]), and also for novel metrical regularization techniques as well as regularization techniques with higher order penalization.

In Section 2 and the Appendix A we recall some basic facts on G-norms and bounded variation regularization. In Section 3 we recall the definition of the Fenchel dual of a functional and quote some important theorems from convex analysis. With this we can give a characterization of minimizers of convex regularization functionals in 4.

Finally in Section 5 we present some analytical examples of minimizers of regularization functionals.

Prerequisites for this paper: All along this paper we assume that Ω is a bounded, open, connected domain with Lipschitz boundary (bocL) or that $\Omega = \mathbb{R}^n$. $\vec{n}$ denotes the normal vector to the boundary of Ω. We denote by $|\cdot|$ the Euclidean norm. If $1 < p < \infty$, we denote by p_* the number $p/(p-1)$ so that $1/p + 1/p_* = 1$. For $p = 1$ we set $p_* = \infty$.

2. G-Norm

Y. Meyer [Mey01] characterized minimizers of the ROF-functional

$$\mathcal{F}_{2,1}(u) := \frac{1}{2}\int_{\mathbb{R}^n} (u - u^\delta)^2 + \alpha\,|Du| \quad (\alpha > 0)$$

using the dual norm of $W^{1,1}(\mathbb{R}^n)$, which he called the G-norm. Aubert & Aujol [AubAuj05] derived a characterization of minimizers of the ROF-functional defined on $\Omega \subseteq \mathbb{R}^2$ being bocL. Chan & Shen [ChaShe05] used a characterization of dual functions which applies both for bounded and unbounded domains. In [ObeOshSch05] we derived a characterization of minimizers of ROF-like functionals with penalization by the total variation of second order derivatives. In [OshSch04] we characterized minimizers of regularization functionals with anisotropic total variation regularization penalization term.

Here we aim for a unified analysis. We rely on fundamental results in Adams [Ada75], which characterize the duals of $W^{k,1}(\Omega)$ for every $k \geq 1$ and Ω in any space dimension. Due to some structural properties of regularization functionals the results in Adams [Ada75] have to be slightly adapted.

Theorem 2.1. *Let $\emptyset \neq N$ be a closed subspace of the Sobolev space $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, $m = 1, 2, \ldots$. With N we associate $\|\cdot\|_N$, which is equivalent to the $W^{m,p}(\Omega)$-norm on N. Moreover, let $\mathcal{N} := \{\gamma : 0 \leq |\gamma| \leq m\}$ be a subset of multi-indices with $|\mathcal{N}| = N$.*

For $1 \leq p_\gamma < \infty$, $\gamma \in \mathcal{N}$ let $L_N := \prod_{\gamma\in\mathcal{N}} L^{p_\gamma}(\Omega)$ with dual $L_N^* := \prod_{\gamma\in\mathcal{N}} L^{p_\gamma^*}(\Omega)$. We define $P : N \to L_N$ by

$$P(u) = (D^\gamma u)_{\gamma\in\mathcal{N}}$$

and assume that $\|P(u)\| = \|u\|_N$. That is, P is an isometric isomorphism of N onto a subspace of L_N.

Then for every $L \in N^*$, there exists $\vec{v} = (v_\gamma)_{\gamma\in\mathcal{N}} \in L_N^*$ such that

$$Lu = \sum_{\gamma\in\mathcal{N}} \langle D^\gamma u, v_\gamma \rangle\,, \quad u \in N\,. \tag{2.1}$$

Moreover,

$$\|L\|_{N^*} = \min\left\{\|\vec{v}\|_{L^*_N} : \vec{v} \textit{ satisfies } (2.1)\right\} . \tag{2.2}$$

Proof. N associated with $\|u\|_N$ is a Banach space (this follows from the fact that a closed subspace is again a Banach space), and therefore has a dual N^*. P is an isometric isomorphism of N onto a subspace $W \subseteq L_N$. Since N is complete, W is a closed subspace of L_N. A linear functional L^* is defined as follows:

$$L^*(Pu) = L(u)\,, \quad u \in N.$$

Since P is an isometric isomorphism, $L^* \in W^*$ and

$$\|L^*\|_{W^*} = \|L\|_{N^*} \ .$$

From the Hahn-Banach Theorem it follows that there exists a norm preserving extension $\tilde{L}$ of L^* on L_N and therefore, from [Ada75, Lemma 3.7] it follows that

$$\tilde{L}(u) = \sum_{\gamma\in\mathcal{N}} \langle u_\gamma, v_\gamma\rangle\,.$$

Note that the dual pairing can have different meaning for different γ.

Thus for $u \in N$

$$L(u) = L^*(Pu) = \tilde{L}(Pu) = \sum_{\gamma\in\mathcal{N}} \langle D^\gamma u, v_\gamma\rangle \ .$$

Moreover,

$$\|L^*\|_{W^*} = \left\|\tilde{L}\right\|_{(L_\mathcal{N})^*} = \|\vec{v}\|_{L^*_\mathcal{N}} \ .$$

Since this equation holds for all functions $\vec{v}$ satisfying (2.1), (2.2) follows. □

From this Theorem we can derive the characterization of the duals and dual norms of

$$N := W^{1,1}_\diamond(\Omega) = \left\{w \in W^{1,1}(\Omega) : \int_\Omega w = 0\right\}$$

if Ω is bocL and for $W^{1,1}(\Omega)$ if $\Omega = \mathbb{R}^n$.

Definition 2.2.

- *Assume that Ω is bocL and $G_\diamond = W^{1,1}_\diamond(\Omega)^*$ where $W^{1,1}_\diamond(\Omega)$ is associated with the norm $\int_\Omega |\nabla u|$. According to Theorem 2.1 every $L \in W^{1,1}_\diamond(\Omega)^*$ can be represented as*

$$Lu = \int_\Omega \nabla u \cdot \vec{v} \textit{ with } \vec{v} \in L^\infty(\Omega;\mathbb{R}^n)\ .$$

We call

$$\|L\|_{G_\diamond} := \inf\left\{\||\vec{v}|\|_{L^\infty(\Omega)} : Lu = \int_\Omega \nabla u\cdot\vec{v}\right\}$$

the $G_\diamond$-norm of L.

- *Assume that Ω is bocL or $\Omega = \mathbb{R}^n$ and $G_0 = W^{1,1}_0(\Omega)^*$, where as norm on $W^{1,1}_0(\Omega)$ the total variation is used. Every $L \in W^{1,1}_0(\Omega)^*$ can be identified with a distribution of order 1 in*

$$G_0 := \{v : v = (\nabla\cdot\vec{v}) : \vec{v} \in L^\infty(\Omega;\mathbb{R}^n)\}\ .$$

We call

$$\|v\|_{G_0} := \inf\left\{\||\vec{v}|\|_{L^\infty(\Omega)} : v = (\nabla\cdot\vec{v})\right\}$$

the G_0-norm.

Remark 2.3. *Y. Meyer considered the G-norm for generalized functions defined on $\mathbb{R}^n$ (see* [Mey01, p. 30]*). Using that $W_0^{1,1}(\mathbb{R}^n) = W^{1,1}(\mathbb{R}^n)$, it follows from Theorem 3.2 that for every $v \in G_0$ there exists $\vec{v} \in L^\infty(\Omega;\mathbb{R}^n)$ such that*

$$\|v\|_{G_0} := \||\vec{v}|\|_{L^\infty(\mathbb{R}^n)} \text{ and } v = (\nabla \cdot \vec{v})$$

In [OshSch04] *we generalized the definition of the G-norm on $\mathbb{R}^n$ by defining*

$$\|v\|_{G_0^s} := \inf\left\{\||\vec{v}|_s\|_{L^\infty(\mathbb{R}^n)} : v = (\nabla \cdot \vec{v})\right\} \quad (1 \le s \le \infty)\ .$$

Note that in the above definition instead of the Euclidean norm, the s-norm of the vector valued function $\vec{v}$ is used. This definition can be used to characterize minimizers of regularization functionals with anisotropic total variation regularization penalization term. Note that for every $w \in W_0^{1,1}(\mathbb{R}^n)$ and all $v \in G^s$

$$\int_{\mathbb{R}^n} vw = \int_{\mathbb{R}^n} (\nabla \cdot \vec{v})\, w = -\int_{\mathbb{R}^n} \vec{v}\nabla w \le \||\vec{v}|_s\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla w|_{s_*}\ ,$$

where $1/s_* + 1/s = 1$ with $1 \le s_*, s \le \infty$.

If $\Omega \subseteq \mathbb{R}^2$ is bocL, Aubert & Aujol [AubAuj05] *gave the following definition of the G-norm: let*

$$G_{AA} := \left\{v = (\nabla \cdot \vec{v}) \in L^2(\Omega) : \vec{v} \in L^\infty(\Omega)\,, \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega\right\}$$

and

$$\|v\|_{G_{AA}} := \inf\left\{\||\vec{v}|\|_{L^\infty(\Omega)} : v = (\nabla \cdot \vec{v})\right\}\ .$$

In the following we extend the definition of the G-norm for higher order derivatives. We use

$$W_\diamond^{k,1}(\Omega) = \left\{w \in W^{k,1}(\Omega) : \int_\Omega x^{\vec{l}} w = 0, \quad \left|\vec{l}\right| = 0,1,\dots,k-1\right\}$$

and $W_0^{k,1}(\Omega)$. $W_\diamond^{k,1}(\Omega)$ is the subspace of functions in $W^{k,1}(\Omega)$ with k orders of vanishing moments.

Definition 2.4. *Let $k = 1,2,\dots$.*

- *Assume that Ω is bocL and $G_\diamond^k = W_\diamond^{k,1}(\Omega)^*$ where $W_\diamond^{k,1}(\Omega)$ is associated with the norm $\int_\Omega \left|\nabla^k u\right|$.*

  According to Theorem 2.1 every $L \in W_\diamond^{k,1}(\Omega)^*$ can be represented as

  $$Lu = \int_\Omega \nabla^k u \cdot \vec{v} \text{ with } \vec{v} \in L^\infty(\Omega;\mathbb{R}^{n^k})\ .$$

  We call

  $$\|L\|_{G_\diamond^k} := \inf\left\{\||\vec{v}|\|_{L^\infty(\Omega)} : Lu = \int_\Omega \nabla^k u \cdot \vec{v}\right\}$$

  the $G_\diamond^k$-norm of L.
- *Assume that Ω is bocL or $\Omega = \mathbb{R}^n$ and $G_0 = W_0^{k,1}(\Omega)^*$, where as norm on $W_0^{k,1}(\Omega)$ the total variation of the k-th derivative is taken. Every $L \in W_0^{k,1}(\Omega)^*$ can be identified with a distribution of order k in*

  $$G_0^k := \left\{v : v = \left(\nabla^k \cdot \vec{v}\right) : \vec{v} \in L^\infty(\Omega;\mathbb{R}^{n^k})\right\}\ .$$

We call

$$\|v\|_{G_0^k} := \inf\left\{\||\vec{v}|\|_{L^\infty(\Omega)} : v = \left(\nabla^k \cdot \vec{v}\right)\right\}$$

the G_0^k*-norm.*

Remark 2.5. *In the definition above* $v = \left(\nabla^k \cdot \vec{v}\right)$ *has to be considered a distributional derivative, that is* v *is a linear operator on* $C_0^\infty(\Omega;\mathbb{R}^{n^k})$ *satisfying*

$$v[\vec{\phi}] = (-1)^k \int_\Omega v \nabla^k \vec{\phi} \text{ for all } \vec{\phi} \in C_0^\infty(\Omega;\mathbb{R}^{n^k})\ .$$

Theorem 2.1 gives a characterization of dualnorms. We used this characterization to define G-norms on bounded and unbounded domains. If we only write $\|\cdot\|_{G^k}$ *we mean* $\|\cdot\|_{G_\diamond^k}$ *for* $\Omega \subset \mathbb{R}^n$ *bocL and* $\|\cdot\|_{G_0^k}$ *for* $\Omega = \mathbb{R}^n$.

3. Fenchel Duality

In this Section we use Fenchel's duality theorem to characterize minimizers of convex regularization functionals. Below we review basic concepts from functional analysis (see for instance Ekeland & Temam [EkeTem76] and Aubin [Aub79]).

Definition 3.1. *Assume that* X *is a locally convex space (for instance a Banach space). The* Fenchel transform *of a functional*

$$\mathcal{S} : X \to \mathbb{R} \cup \{+\infty\}\,, \quad u \mapsto \mathcal{S}(u)$$

is defined by

$$\mathcal{S}^* : X^* \to \mathbb{R} \cup \{+\infty\}\,, \quad u^* \mapsto \mathcal{S}^*(u^*) := \sup_{u \in X} \left(\langle u^*, u\rangle - \mathcal{S}(u)\right)$$

where $\langle\cdot,\cdot\rangle$ *denotes the bilinear pairing with respect to* X^* *and* X.

For a definition of the Fenchel transform in a finite dimensional space setting we refer to Rockafellar [Roc70] and for the infinite dimensional setting we refer to Ekeland & Temam [EkeTem76] and Aubin [Aub79].

Theorem 3.2. *Let* $\mathcal{S}, \mathcal{R}$ *be convex and lower semi continuous functionals from a locally convex space* X *into* $\mathbb{R} \cup \{+\infty\}$.

If $\tilde{u}$ *is a solution of*

$$\inf_{u \in X} \{\mathcal{S}(u) + \mathcal{R}(u)\}\ , \tag{3.1}$$

$\tilde{u}^*$ *is a solution of*

$$\sup_{u^* \in X^*} \{-\mathcal{S}^*(u^*) - \mathcal{R}^*(-u^*)\} \tag{3.2}$$

and

$$\inf_{u \in X} \{\mathcal{S}(u) + \mathcal{R}(u)\} = \sup_{u^* \in X^*} \{-\mathcal{S}^*(u^*) - \mathcal{R}^*(-u^*)\} < +\infty \tag{3.3}$$

then $\tilde{u} \in X$ *and* $\tilde{u}^* \in X^*$ *satisfy the extremality relation*

$$\mathcal{S}(\tilde{u}) + \mathcal{R}(\tilde{u}) + \mathcal{S}^*(\tilde{u}^*) + \mathcal{R}^*(-\tilde{u}^*) = 0 \tag{3.4}$$

which is equivalent to

$$\tilde{u}^* \in \partial\mathcal{S}(\tilde{u}) \text{ and } -\tilde{u}^* \in \partial\mathcal{R}(\tilde{u}) \tag{3.5}$$

or

$$\tilde{u} \in \partial\mathcal{S}^*(\tilde{u}^*) \text{ and } -\tilde{u} \in \partial\mathcal{R}^*(\tilde{u}^*).$$

Conversely, if $u \in X$ and $u^* \in X^*$ satisfy (3.4), then u, u^* satisfy (3.1) and (3.2), respectively.

Proof. Follows from [EkeTem76, Proposition 2.4, Proposition 4.1, Remark 4.2 in Chapter 3] □

Below we use the following basic results for convex analysis:

Theorem 3.3. *(see for instance* [Aub91]*).*

(1) *Let $\mathcal{S}(u) := \mathcal{T}(u - u_0) + \langle u_0^*, u\rangle + a$, then*
$$\mathcal{S}^*(u^*) := \mathcal{T}^*(u^* - u_0^*) + \langle u^*, u_0\rangle - (a + \langle u_0^*, u_0\rangle)$$
(2) *Let $\mathcal{S}(u) := \mathcal{T}(\lambda u)$, then $\mathcal{S}^*(u^*) := \mathcal{T}^*\left(\frac{u^*}{\lambda}\right)$.*

Example 3.4. *Let $1 \le p < \infty$ and denote by p_* the dual of p; that is the p_* satisfies $1/p_* + 1/p = 1$ (for $p = 1$, $p_* = \infty$). We assume that Ω is bocL or $\Omega = \mathbb{R}^n$.*

We use Theorem 3.3 to calculate the Fenchel transform of
$$\mathcal{S} : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}\,, u \to \frac{1}{p}\int_\Omega \left|u - u^\delta\right|^p =: \mathcal{T}(u - u^\delta)\,, \quad p \ge 1\,.$$
For $p > 1$ and $u^* \in L^{p_*}(\Omega)$
$$\mathcal{T}^*(u^*) := \sup\left\{\int_\Omega uu^* - \frac{1}{p}\int_\Omega |u|^p : u \in L^p(\Omega)\right\}$$
The supremum is attained for $u_\alpha \in L^p(\Omega)$ satisfying
$$u_\alpha^* = |u_\alpha|^{p-1}\, sgn(u_\alpha) \in L^{p_*}(\Omega)\,.$$
Therefore, the Fenchel transform of $\mathcal{T}$ is given by
$$\mathcal{T}^*(u^*) = \frac{1}{p_*}\int_\Omega |u^*|^{p_*}$$
and consequently
$$\mathcal{S}^*(u^*) = \mathcal{T}^*(u^*) + \left\langle u^*, u^\delta\right\rangle_{L^{p_*}, L^p} = \frac{1}{p_*}\int_\Omega |u^*|^{p_*} + \int_\Omega u^* u^\delta\,.$$
For $p = 1$,
$$\mathcal{S} : L^1(\Omega) \to \mathbb{R}\,, \quad u \to \left|u - u^\delta\right|_{L^1(\Omega)}$$
we have
$$\mathcal{S}^*(u^*) = \begin{cases} +\infty & \text{if } meas\ \{x \in \Omega : u^* \notin [-1, 1]\} > 0, \\ \int_\Omega u^\delta u^* & else\,. \end{cases}$$

Theorem 3.5. *Let X be a non trivial Banach space and*
$$\tilde{X} = M \oplus M^\perp \subseteq X$$
be the direct sum (for a definition of the direct sum see for instance Zeidler [Zei93, p 766]*) of a normed space $M \ne \emptyset$ with norm $\|\cdot\|_M$ and its complement.*

The extension of $\|\cdot\|_M$ on $\tilde{X}$ is defined by
$$\|u + v\|_M = \|u\|_M\,, \quad u \in M\,, v \in M^\perp\,. \tag{3.6}$$
Moreover,
$$\mathcal{R} : X \to \mathbb{R} \cup \{+\infty\}\,, \quad x \to \left\{ \begin{array}{ll} \|u\|_M & for\ u \in \tilde{X} \\ +\infty & else \end{array}\right.$$

with

$$\mathcal{D} := \{u \in X : \mathcal{R}(u) \neq +\infty\} \neq \emptyset \,.$$

For $u^* \in X^*$ we define

$$\|u^*\|_{M^*} = \sup_{\{v \in M : \|v\|_M = 1\}} \langle u^*, v \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing with respect to X^*, X. Note that the dual pairing is well-defined since by definition every $v \in M$ satisfies $v \in X$.

Then the Fenchel transform of $\mathcal{R}$ is given by

$$\mathcal{R}^* : X^* \to \mathbb{R} \cup \{+\infty\} \,, \quad u^* \to \begin{cases} 0 \text{ if } \|u^*\|_{M^*} \leq 1 \text{ and } M^\perp \subseteq \mathcal{N}(u^*) \,, \\ +\infty \text{ else.} \end{cases} \tag{3.7}$$

where $\mathcal{N}$ denotes the nullspace of the operator $v \to \langle u^*, v \rangle$.

Proof. Let $u^* \in X^*$, then from the definitions of $\mathcal{R}$ and $\tilde{X}$ it follows that

$$\begin{aligned} \mathcal{R}^*(u^*) &= \sup_{\tilde{u} \in \tilde{X}} \{\langle u^*, \tilde{u} \rangle - \|\tilde{u}\|_M\} \\ &= \sup_{\{u \in M, v \in M^\perp\}} \{\langle u^*, u \rangle - \|u\|_M + \langle u^*, v \rangle\} \,. \end{aligned}$$

Taking into account that $M, M^\perp \subseteq X$, it follows that

$$\mathcal{R}^*(u^*) = \sup_{u \in M \cap X} \{\langle u^*, u \rangle - \|u\|_M\} + \sup_{v \in M^\perp} \langle u^*, v \rangle \,.$$

For all $u \in M$ we have

$$\langle u^*, u \rangle - \|u\|_M \leq \|u^*\|_{M^*} \|u\|_M - \|u\|_M \leq \|u\|_M \left(\|u^*\|_{M^*} - 1\right) \,,$$

which shows that $\mathcal{R}^*(u^*) = 0$ if $\|u^*\|_{M^*} < 1$ and $+\infty$ else.

Moreover,

$$\sup_{v \in M^\perp} \langle u^*, v \rangle = \begin{cases} 0 & \text{if } M^\perp \subseteq \mathcal{N}(u^*) \\ +\infty & \text{else} \end{cases}$$

Thus in total $\mathcal{R}^*(u^*) = 0$ if and only if $\|u^*\|_{M^*} \leq 1$ and $M^\perp \subseteq \mathcal{N}(u^*)$ and $\mathcal{R}^*(u^*) = +\infty$ else. □

Example 3.6. *Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^n$ is bocL and $X = L^p(\Omega)$. Take $\tilde{X} = X \cap TV^k$ (see Definition A.4) for some $k = 1, 2, \ldots$ and*

$$\alpha\mathcal{R} : X \to \mathbb{R} \cup \{+\infty\} \,, \quad u \mapsto \begin{cases} \alpha \left|D^k u\right| & \text{if } u \in \tilde{X} \,, \\ +\infty & \text{else .} \end{cases}$$

In the sequel, for the sake of simplicity of notation we set $|Du| = +\infty$ if $u \in X$ but not in $TV(\Omega)$.

We take $M_k^\perp := \mathcal{P}_k$, the set of polynomials of order less than k and the mapping $\Pi_{\mathcal{P}_k}$ from $\tilde{X}$ to $\mathcal{P}_k$ which maps $u \in \tilde{X}$ onto the unique polynomial p satisfying

$$\left\langle x^{\vec{l}}, p \right\rangle = \left\langle x^{\vec{l}}, u \right\rangle \,, \quad \left|\vec{l}\right| = 0, 1, \ldots, k-1.$$

We define M_k as the range of the operator $I - \Pi_{\mathcal{P}_k}$. Then,

$$M_k \oplus M_k^\perp = \tilde{X} \,.$$

With the linear space M_k we associate the norm $\|\cdot\|_{M_k} := \left|D^k \cdot\right|$, which is extended by zero on $M_k^\perp$.

Let $u^* \in X^*$, then the condition $M_k^\perp \subseteq \mathcal{N}(u^*)$ in (3.7) is equivalent to

$$\int_\Omega u^* p = 0\,, \quad p \in \mathcal{P}_k\,. \tag{3.8}$$

Moreover, $\|v^*\|_{M_k^*} \leq \alpha$ is equivalent to

$$\sup_{\{u \in M_k \subseteq L^p(\Omega) : |D^k u| \leq 1\}} \int_\Omega v^* u = \sup_{\left\{u \in \tilde{X} : |D^k u| \leq 1\right\}} \int_\Omega v^* u \leq \alpha\,.$$

For $n \leq k$ and $k < n\,, 1 \leq p \leq \frac{n}{n-k}$ it follows from the Sobolev embedding theorem [Ada75] *that $X^* = (L^p(\Omega))^* \subset \left(W_\diamond^{k,1}\Omega\right)^*$. Therefore, with every $v^* \in L^{p_*}(\Omega)$ there can be associated a linear bounded functional*

$$\begin{aligned} L_{v^*} : W_\diamond^{k,1}(\Omega) &\to \mathbb{R} \\ u &\mapsto \langle v^*, u \rangle_{L^{p_*}, L^p}\,. \end{aligned}$$

Now, we show that under either one of the assumptions

- $n \leq k$,
- $k < n\,, 1 \leq p_* \leq \frac{n}{n-k}$,

respectively, we have

$$\|L_{v^*}\|_{G_\diamond^k} = \|v^*\|_{M_k^*}\,.$$

The set

$$S_\diamond^k(\Omega) := \left\{ \phi \in C^k(\overline{\Omega}) : \int_\Omega x^{\vec{l}} \phi = 0, \left|\vec{l}\right| = 0 \dots k-1, \left\|\nabla^k \phi\right\|_{L^1(\Omega)} < \infty \right\}$$

is contained in $W_\diamond^{k,1}(\Omega)$ and M_k. Moreover, it is dense in both spaces with respect to the topologies $\left\|\nabla^k \cdot\right\|_{L^1(\Omega)}, \left|D^k \cdot\right|$ respectively. Therefore,

$$\begin{aligned} \|L_{v^*}\|_{G_\diamond^k} &= \sup\left\{ \int v^* u : u \in W_\diamond^{k,1}(\Omega), \left\|\nabla^k u\right\|_{L^1(\Omega)} \leq 1 \right\} \\ &= \sup\left\{ \int v^* u : u \in S_\diamond^k(\Omega), \left\|\nabla^k u\right\|_{L^1(\Omega)} \leq 1 \right\} \\ &= \sup\left\{ \int v^* u : u \in M_k, \left|D^k u\right| \leq 1 \right\} = \|v^*\|_{M_k^*}\,. \end{aligned}$$

Thus $(\alpha\mathcal{R})^*(v^*) = 0$ if $\|L_{v^*}\|_{G^k} \leq \alpha$ and the first $k-1$ moments of v^* vanish. Otherwise $(\alpha\mathcal{R})^* = +\infty$. Thus $(\alpha\mathcal{R})^*$ is again a Barrier functional. The proof of this result follows immediately from Theorem 3.5.

Example 3.7. *We consider $\alpha\mathcal{R}$ as in Example 3.6 with $\Omega = (-1, 1) \subset \mathbb{R}$ and derive a characterization of $\|v^*\|_{M_k^*} = \|L_{v^*}\|_{G_\diamond^k}$.*

From Theorem 2.1 it follows that for every L_{v^*} there exists $\rho^* \in L^\infty(\Omega)$ such that

$$L_{v^*} u = \int_\Omega v^* u = \int_\Omega \rho^* u^{(k)}\,,$$

Here we denote by $u^{(k)}$ the k-th derivative of a function u defined on a one-dimensional domain Ω.

ρ^*, is the k-th primitive of v^* times $(-1)^k$, that is

$$\rho^*(x) = (-1)^k \int_{-1}^x \cdots \int_{-1}^{t_3} \left(\int_{-1}^{t_2} v^*(t_1) dt_1 \right) dt_2 \cdots dt_k \quad \text{for } x \in (-1, 1).$$

Since we only consider v^* *that fulfill* (3.8), *by partial integration we can show that* $\rho^*(l)(\pm 1) = 0$ *for* $l = 0, \ldots, k-1$.

Moreover, from Theorem 2.1 it follows that

$$\|L_{v^*}\|_{G_\diamond^k} = \inf\left\{ \||\tilde{\rho}^*|\|_{L^\infty(\Omega)} : L_{v^*}u = \int_\Omega u^{(k)}\tilde{\rho}^* \right\} = \|\rho^*\|_{L^\infty(\Omega)} \ .$$

Therefore, the condition $\|v^*\|_{M_k^*} \le \alpha$ *is equivalent to*

$$\left| \int_{-1}^{x} \cdots \int_{-1}^{t_3} \left(\int_{-1}^{t_2} v^*(t_1)dt_1 \right) dt_2 \cdots dt_k \right| \le \alpha \quad \text{for all } x \in (-1,1).$$

Moreover we have $\|v^*\|_{M_k^*} = \|(\rho^*)'\|_{M_1^*}$.

Example 3.8. *We consider* $\alpha\mathcal{R}$ *as in Example 3.6 with* $p = 2$, $k = 1$, $\Omega \subset \mathbb{R}^2$ *bocL and* $X = L^2(\Omega)$. *From Theorem 2.1 it follows that there exists* $\vec{v} \in L^\infty(\Omega)$ *such that*

$$L_{v^*}u = \int_\Omega v^* u = \int_\Omega \vec{v} \cdot \nabla u$$

From integration by parts we see that $\nabla \cdot \vec{v} = -v^*$ *and* $\vec{v} \cdot \vec{n} = 0$ *on* $\partial\Omega$. *Moreover*

$$\begin{aligned} \|L_{v^*}\|_{G_\diamond} &= \inf\left\{ \||\vec{v}|\|_{L^\infty(\Omega)} : L_{v^*}u = \int_\Omega \nabla u \cdot \vec{v} \right\} \\ &= \inf\left\{ \|\vec{v}\|_{L^\infty(\Omega)} : v^* = (\nabla \cdot \vec{v}) \in L^2(\Omega), \vec{v} \in L^\infty(\Omega)\,, \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega \right\} \\ &= \|v^*\|_{G_{AA}} \,. \end{aligned}$$

Example 3.9. *For* $\Omega = \mathbb{R}^n$ *the dual of*

$$\alpha\mathcal{R} : L^p(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}\,, \quad u \mapsto \alpha \left|D^k u\right|$$

is given by

$$(\alpha\mathcal{R})^* : L^{p_*}(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}\,, \quad u^* \mapsto \begin{cases} 0 & \text{if } \|u^*\|_{G_0^k} \le \alpha \\ +\infty & \text{else} \end{cases}$$

4. Characterization of Minimizers with Convex Regularization Functionals

In the following we characterize properties of minimizers of the family of functionals

$$\mathcal{F}(u) := \mathcal{S}(u) + \alpha\mathcal{R}(u) \quad \alpha > 0.$$

For this purpose we use special differentiabilty concepts:

Definition 4.1. *Let* $F : \mathcal{D} \subseteq X \to Y$ *be an operator between Banach spaces* X *and* Y. *The* one-sided *directional derivative of* F *at* u *in the direction* h *is defined by*

$$F'(u;h) := \lim_{t \to 0+} \frac{F(u+th) - F(u)}{t} \ .$$

F *is said to admit a* Gâteaux derivative F' *at* u *provided that* $F'(u;h)$ *exists and can be written as a linear operator*

$$F'(u)h = F'(u,h) \ .$$

In the sequel we make the following assumptions:

Assumption 4.2. *Let $\emptyset \neq U$ be a subspace of a real Banach space X.*

Assume that $\mathcal{R} : X \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{S}$ satisfy:

1. *$\mathcal{R}$ and $\mathcal{S}$ are convex on X and uniformly bounded from below.*
2. *For $u, h \in X$, $\mathcal{R}$ and $\mathcal{S}$ attain directional derivatives at u in direction h: $\mathcal{R}'(u;h)$ and $\mathcal{S}'(u;h)$.*
3. *If $u \in X \backslash U$, then either $\mathcal{S}(u) = +\infty$ or $\mathcal{R}(u) = +\infty$.*
4. *There exists a point $u_0 \in U$ such that $\mathcal{R}(u_0) < +\infty$ and $\mathcal{S}(u_0) < +\infty$.*

Theorem 4.3. *Let $\mathcal{R}$ and $\mathcal{S}$ satisfy Assumption 4.2. Moreover, we assume that $\mathcal{F}$ attains a minimizer u_α.*

Then $u = u_\alpha$ minimizes $\mathcal{F}$ if and only if $u \in U$ satisfies

$$-\mathcal{S}'(u;h) \leq \alpha \mathcal{R}'(u;h) \quad h \in X\ . \tag{4.1}$$

Proof. If $u \in X \backslash U$ by assumption either $\mathcal{R}(u) = +\infty$ or $\mathcal{S}(u) = +\infty$, showing that a minimizer must be an element of U.

Moreover, from the definition of u_α and the convexity of $\mathcal{R}$ and $\mathcal{S}$ it follows that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \to 0^+} \left(\frac{\mathcal{S}(u_\alpha + \varepsilon h) - \mathcal{S}(u_\alpha)}{\varepsilon} + \alpha \frac{\mathcal{R}(u_\alpha + \varepsilon h) - \mathcal{R}(u_\alpha)}{\varepsilon} \right) \\ &\leq \lim_{\varepsilon \to 0^+} \left(\frac{\mathcal{S}(u_\alpha + \varepsilon h) - \mathcal{S}(u_\alpha)}{\varepsilon} \right) + \alpha \lim_{\varepsilon \to 0^+} \left(\frac{\mathcal{R}(u_\alpha + \varepsilon h) - \mathcal{R}(u_\alpha)}{\varepsilon} \right) \\ &= \mathcal{S}'(u_\alpha;h) + \alpha \mathcal{R}'(u_\alpha;h)\,, \quad h \in X\,, \end{aligned}$$

showing (4.1).

To prove the converse direction we note that from the convexity of $\mathcal{S}$ and $\mathcal{R}$ and (4.1) it follows that

$$(\mathcal{S}(u+h) - \mathcal{S}(u)) + \alpha\,(\mathcal{R}(u+h) - \mathcal{R}(u)) \geq \mathcal{S}'(u;h) + \alpha \mathcal{R}'(u;h) \geq 0 \quad u, h \in X\ .$$

Thus u is a global minimizer. □

Corollary 4.4. *Let Assumption 4.2 hold. Assume that $\mathcal{F}$ attains a minimizer u_α. Then*

$$-\mathcal{S}'(0;h) \leq \alpha \mathcal{R}'(0;h)\,, \quad h \in X$$

if and only if

$$u_\alpha \equiv 0\ .$$

Corollary 4.4 follows immediately from Theorem 4.3.

Remark 4.5. *The definition of u_α shows that if $\mathcal{R}$ is a seminorm, then*

$$\mathcal{S}(u_\alpha) + \alpha \mathcal{R}(u_\alpha) \leq \mathcal{S}(u_\alpha + \varepsilon(\pm u_\alpha)) + \alpha(1 \pm \varepsilon)\mathcal{R}(u_\alpha)\,, \quad 0 < \varepsilon < 1\,,$$

and therefore

$$\mp \alpha \mathcal{R}(u_\alpha) \leq \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left(\mathcal{S}(u_\alpha + \varepsilon(\pm u_\alpha)) - \mathcal{S}(u_\alpha) \right)\ .$$

Taking $\varepsilon \to 0$ gives

$$-\mathcal{S}'(u_\alpha;u_\alpha) \leq \alpha \mathcal{R}(u_\alpha) \leq \mathcal{S}'(u_\alpha;-u_\alpha)\ . \tag{4.2}$$

In particular if $\mathcal{S}$ is Gâteaux-differentiable then

$$-\mathcal{S}'(u_\alpha)u_\alpha = \alpha \mathcal{R}(u_\alpha)\ .$$

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Example 4.6. *We consider regularization functionals of the form*

$$\mathcal{R} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}, \quad u \to \alpha \, |Du|^p \, .$$

The definition of u_α *shows that*

$$\mathcal{S}(u_\alpha) + \alpha \, |Du_\alpha|^p \leq \mathcal{S}(u_\alpha + \varepsilon(\pm u_\alpha)) + \alpha(1 \pm \varepsilon)^p \, |Du_\alpha|^p \ , \quad 0 < \varepsilon < 1 \, ,$$

and therefore

$$\mathcal{S}(u_\alpha) - \mathcal{S}(u_\alpha + \varepsilon(\pm u_\alpha)) \leq (\pm p\epsilon + \sum_{k=2}^{p} \binom{p}{k} (\pm\epsilon)^{p-k}) \alpha \, |Du_\alpha|^p \, .$$

Dividing by ϵ *and taking* $\epsilon \to 0+$ *gives*

$$-\mathcal{S}'(u_\alpha; u_\alpha) \leq \alpha \, p \mathcal{R}(u_\alpha) \leq \mathcal{S}'(u_\alpha; -u_\alpha) \, . \tag{4.3}$$

In particular if $\mathcal{S}$ *is Gâteaux-differentiable then*

$$-\mathcal{S}'(u_\alpha) u_\alpha = \alpha \, p \mathcal{R}(u_\alpha) \, .$$

Example 4.7. *For the ROF-functional we have*

$$\mathcal{S} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}, \quad u \to \frac{1}{2} \int_\Omega (u - u^\delta)^2 \, ,$$

and

$$\mathcal{R} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}, \quad u \to \alpha \, |Du| \ .$$

Let $U = L^2(\Omega) \cap TV(\Omega)$*, then*

$$\mathcal{S}'(0; h) = -\int_\Omega u^\delta h \text{ and } \mathcal{R}'(0; h) = \alpha \, |Dh| \ , \quad h \in X \, .$$

Thus Corollary 4.4 implies that $u_\alpha \equiv 0$ *if and only if*

$$\left| \int_\Omega u^\delta h \right| \leq \alpha \, |Dh| \ , \quad h \in X \, ,$$

or in other words if $\left\| u^\delta \right\|_G \leq \alpha$*. This result has been stated for the first time in* [Mey01]*. From (4.3) it follows that*

$$\alpha \, |Du_\alpha| = -\int_\Omega (u_\alpha - u^\delta) u_\alpha \, . \tag{4.4}$$

Moreover, it follows from (4.1), the convexity of $\mathcal{R}$ *and the triangle inequality that*

$$-\int_\Omega (u_\alpha - u^\delta) h \leq \alpha(|D(u_\alpha + h)| - |D(u_\alpha)|) \leq \alpha \, |Dh| \ . \tag{4.5}$$

(4.5) implies that $\left\| u_\alpha - u^\delta \right\|_G \leq \alpha$*, which together with (4.4) shows that* $\left\| u_\alpha - u^\delta \right\|_G = \alpha$ *for* $u_\alpha \neq 0$ *or* $\alpha \leq \left\| u^\delta \right\|_G$*. It has been shown for the first time in* [Mey01] *that* $\left\| u_\alpha - u^\delta \right\|_G = \alpha$ *and (4.5) characterize the minimizer of the ROF-functional.*

Example 4.8. *In* [OshSch04] *the result of Y. Meyer has been generalized to arbitrary Gâteaux-differentiable functionals* $\mathcal{S}(u) = \int_\Omega f(x, u(x)) dx$ *in which case we can write*

$$\mathcal{S}'(u; h) = \mathcal{S}'(u) h = \int_\Omega \frac{\partial f}{\partial u}(x, u(x)) h(x) \, dx \, .$$

Example 4.9. *Let* $S(u) = \int_\Omega |u-u^\delta|$ *and* $\mathcal{R}(u) = |Du|$*, then*

$$S'(u;h) = \int_{\{u \neq u^\delta\}} \operatorname{sgn}(u-u^\delta) h + \int_{\{u = u^\delta\}} |h| \ .$$

Thus $u \in X = BV(\Omega)$ *minimizes* $\mathcal{F}$ *if and only if*

$$-\int_{\{u \neq u^\delta\}} \operatorname{sgn}(u-u^\delta) h - \int_{\{u = u^\delta\}} |h| \leq \alpha \mathcal{R}'(u;h) \ . \tag{4.6}$$

Since $\mathcal{R}$ *is convex,* $\mathcal{R}'(u;h) \leq \mathcal{R}(u+h) - \mathcal{R}(u)$ *and thus from (4.6) it follows that*

$$-\int_{\{u \neq u^\delta\}} \operatorname{sgn}(u-u^\delta) h - \int_{\{u = u^\delta\}} |h| \leq \alpha \left(|D(u+h)| - |Du|\right) \ . \tag{4.7}$$

Replacing h *by* εh*, with* $\varepsilon > 0$*, it follows from (4.7) that*

$$\begin{aligned} -\int_{\{u \neq u^\delta\}} \operatorname{sgn}(u-u^\delta) h - \int_{\{u = u^\delta\}} |h| &\leq \limsup_{\varepsilon \to 0+} \alpha \frac{|D(u+\varepsilon h)| - |Du|}{\varepsilon} \\ &\leq \alpha \mathcal{R}'(u;h) \ . \end{aligned}$$

Thus (4.6) and (4.7) are equivalent.

In particular, $u_\alpha \equiv 0$ *if and only if*

$$\int_{\{0 \neq u^\delta\}} \operatorname{sgn}(u^\delta) h - \int_{\{0 = u^\delta\}} |h| \leq \alpha\, |Dh| \ .$$

By using this estimate both with h *and* $-h$ *it follows that*

$$\left(\left| \int_{\{0 \neq u^\delta\}} \operatorname{sgn}(u^\delta) h \right| - \int_{\{0 = u^\delta\}} |h| \right)^+ \leq \alpha\, |Dh| \ .$$

These results have been derived in [SchYinOsh05] *using a different mathematical methodology.*

In [ChaEse05] *minimizers of the functional*

$$\mathcal{F}_{1,1}(u) = \int_{\mathbb{R}^n} \left|u - u^\delta\right| + \alpha\, |Du| =: \mathcal{S}(u) + \alpha \mathcal{R}(u)$$

with $u^\delta = \chi_\Omega$ *have been calculated analytically for special parameters* $\alpha > 0$*. Some of the results follow from the general considerations above. From Corollary 4.4 it follows that* $u_\alpha = 0$ *if and only if*

$$\left(\left| \int_\Omega h \right| - \int_{\mathbb{R}^n \setminus \Omega} |h| \right)^+ \leq \alpha\, |Dh| \ . \tag{4.8}$$

Taking $h = \chi_\Omega$ *it follows from (4.8) that*

$$\frac{meas(\Omega)}{Per(\Omega)} \leq \alpha \ .$$

If $u_\alpha = u^\delta$*, then* $\mathcal{S}'(u_\alpha; h) = -\|h\|_{L^1(\Omega)}$ *and therefore* $u_\alpha = u^\delta$ *if and only if* $-\|h\|_{L^1(\Omega)} \leq \alpha \mathcal{R}'(u^\delta; h)$*, which is equivalent to*

$$-\|h\|_{L^1(\Omega)} \leq \alpha \left(\left|D(u^\delta + h)\right| - \left|Du^\delta\right| \right) \ .$$

Taking $h = -\chi_\Omega$ *shows that*

$$\alpha \leq \frac{meas(\Omega)}{Per(\Omega)} \ .$$

Example 4.10. *Assume that Ω is bocL. Theorem 4.3 also applies to regularization methods*

$$\mathcal{F}(u) := \frac{1}{2}\left\|Au - u^\delta\right\|^2_{L^2(\Omega)} + \alpha\,|Du|\;,$$

where $A : L^2_\diamond(\Omega) \cap TV(\Omega) \to L^2(\Omega)$ is a linear, compact operator.

Note, that if A is linear, then the functional $\left\|Au - u^\delta\right\|^2_{L^2(\Omega)}$ is convex.

Therefore, from Theorem 4.3 it follows that $u_\alpha = 0$ if and only if $\left\|A^* u^\delta\right\|_{G_\diamond} \leq \alpha$. For quadratic Tikhonov regularization

$$\mathcal{F}(u) := \left\|Au - u^\delta\right\|^2_{L^2(\Omega)} + \alpha \int_\Omega |\nabla u|^2$$

it follows from Theorem 4.3 that $u_\alpha = 0$ if and only if

$$\int_\Omega A^* u^\delta h \leq \alpha \int_\Omega \nabla 0 \nabla h = 0\,, \quad h \in W^{1,2}_0(\Omega)$$

which shows that $A^* u^\delta = 0$. If we had that $u^\delta = Au^\dagger$ then this means that $u^\dagger$ is an element of the nullspace of A.

Therefore, aside from trivial situtations, it is **not** *possible to remove data errors completely as for total variation regularization.*

We consider minimization of

$$\mathcal{F}(u) := \frac{1}{2}\left\|Au - u^\delta\right\|^2_{L^2(\Omega)} + \alpha\sqrt{\int_\Omega |\nabla u|^2}$$

over $X = W^{1,2}_0(\Omega)$, which is associated with the H^1-semi norm $\sqrt{\int_\Omega |\nabla u|^2}$.

The directional derivative of $\mathcal{R}$ is given by

$$\mathcal{R}'(0;h) = \sqrt{\int_\Omega |\nabla h|^2},$$
$$\mathcal{R}'(u;h) = \frac{1}{\sqrt{\int_\Omega |\nabla u|^2}} \int_\Omega \nabla u \nabla h \text{ if } 0 \neq u\;.$$

Thus from (4.1) it follows that $u_\alpha = 0$ if and only if

$$\int_\Omega A^* u^\delta h \leq \alpha\sqrt{\int_\Omega |\nabla h|^2}.$$

Or in other words $u_\alpha = 0$ if $\left\|A^* u^\delta\right\|_{W^{-1,2}(\Omega)} \leq \alpha$. Here $W^{-1,2}(\Omega)$ is the dual of $W^{1,2}_0(\Omega)$.

Example 4.11. *Assume that Ω is bocL or $\Omega = \mathbb{R}^n$. Let $1 < p < \infty$, $u_\alpha \in L^p(\Omega)$ and $u^*_\alpha \in L^{p_*}(\Omega)$. We consider the functional*

$$\mathcal{F}(u) = \frac{1}{p}\int \left|u - u^\delta\right|^p + \alpha\left|D^k u\right|\;.$$

From Example 3.4 and Example 3.6 it follows that

$$\mathcal{F}^*(u^*) = \begin{cases} \frac{1}{p_*}\int_\Omega |u^*|^{p_*} + \int_\Omega u^* u^\delta & \text{if } \|u^*\|_{G^k} \leq \alpha \\ +\infty & \text{else} \end{cases}$$

Let u_α and u_α^* be extrema of $\mathcal{F}$, $\mathcal{F}^*$, respectively, then from (3.3) it follows that

$$\int \frac{1}{p}\left|u_\alpha - u^\delta\right|^p + \frac{1}{p_*}\left|u_\alpha^*\right|^{p_*} + u_\alpha^* u^\delta = -\alpha \left|D^k u_\alpha\right|$$

Since u_α and u_α^* are related by (3.5) it follows that

$$u_\alpha^* = \left|u_\alpha - u^\delta\right|^{p-2}(u_\alpha - u^\delta). \tag{4.9}$$

This shows that

$$\begin{aligned} & -\alpha\left|D^k u_\alpha\right| \\ &= \int_\Omega \frac{1}{p}\left|u_\alpha - u^\delta\right|^p + \frac{1}{p_*}\left|(u_\alpha - u^\delta)^{p-1}\right|^{p_*} + \left|u_\alpha - u^\delta\right|^{p-2}(u_\alpha - u^\delta)u^\delta \\ &= \int_\Omega \left|u_\alpha - u^\delta\right|^{p-2}(u_\alpha - u^\delta)\left(\frac{1}{p}(u_\alpha - u^\delta) + \frac{1}{p_*}(u_\alpha - u^\delta) + u^\delta\right) \\ &= \int_\Omega u_\alpha^* u_\alpha\ . \end{aligned} \tag{4.10}$$

This is a generalization of the results in [Mey01, p. 33] *and* [OshSch04, Theorem 7] *for arbitrary $k = 1, 2, \ldots$.*

5. Analytical Examples

We apply duality arguments to analytically calculate minimizers of the functionals

$$\mathcal{F}_{p,k} : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}\,, \quad u \mapsto \underbrace{\frac{1}{p}\int_\Omega \left|u - u^\delta\right|^p}_{\mathcal{S}(u)} + \alpha \underbrace{\left|D^k u\right|}_{\mathcal{R}(u)}$$

when $\Omega = (-1, 1)$, $p = 1, 2$, and $k = 1, 2$.

By u_α, u_α^* we denote minimizers of $\mathcal{F}_{p,k}$ and $\mathcal{F}_{p,k}^*$.

To analytically calculate u_α, we show below that either u_α is piecewise a polynomial of order $k-1$ or equals u^δ.

Moreover, as we show below, ρ_α^*, the k-th primitive of $(-1)^k u_\alpha^*$, shows structural behavior of u_α: if $\left|\rho_\alpha^*\right|(x_1) = \alpha$ and $\left|\rho_\alpha^*\right| < \alpha$ in a sourrounding of x_1, then for $k = 1$, u_α is discontinuous and for $k = 2$, u_α bends (that is, the derivative has a discontinuity) at $x = x_1$. Compare Figure 1.

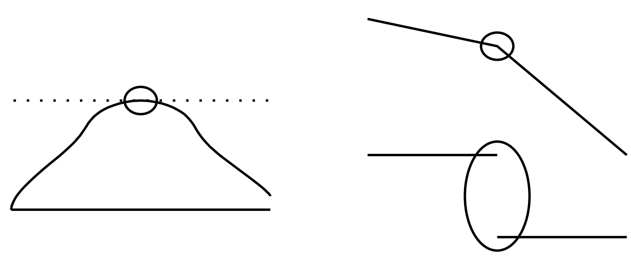

Figure 1. *Left:* ρ_α^* touching the α-tube. *Right:* u_α is continuous (case $\mathcal{F}_{p,1}$) or bends (case $\mathcal{F}_{p,2}$) at that x-value where ρ touches the $\alpha-$tube.

Theorem 5.1. *Let $\Omega = (-1,1)$. Assume that u_α and u_α^* are minimizers of $\mathcal{F}_{p,k}$, $\mathcal{F}_{p,k}^*, p = 1,2$. Then u_α and*

$$\rho_\alpha^*(x) := (-1)^k \int_{-1}^{x} \left(\cdots \int_{-1}^{t_2} u_\alpha^*(t_1) dt_1 \cdots \right) dt_k \text{ for } x \in (-1,1) \tag{5.1}$$

satisfy the following relations:

(a) *If $(a,b) \subset \Omega$ such that $|\rho_\alpha^*(x)| < \alpha$ for all $x \in (a,b)$, then u_α is a polynomial of order $k-1$ in (a,b).*

(b) *If $(a,b) \subset \Omega$ such that $|\rho_\alpha^*(x)| = \alpha$ for all $x \in (a,b)$, then $u_\alpha = u^\delta$ in (a,b).*

(c) *If x_1, x_2, x_3, x_4 such that $\rho_\alpha^*(x_1) = \rho_\alpha^*(x_4) = 0$, $\rho_\alpha^*(x) = \alpha$ for all $x \in [x_2, x_3]$ and $|\rho_\alpha^*(x)| < \alpha$ for $x \in (x_1,x_4) \backslash [x_2,x_3]$ then*

$$u_\alpha = \begin{cases} \sum_{i=0}^{k-1} c_i x^i & \text{in } (x_1,x_2), \\ u^\delta & \text{in } (x_2,x_3), \\ \sum_{i=0}^{k-1} d_i x^i & \text{in } (x_3,x_4). \end{cases}$$

and additionally $(-1)^{(k-1)}$ hinzugefügt.

$$\begin{aligned} (-1)^{(k-1)} d_{k-1} &\le (-1)^{(k-1)} \inf_{x \in (x_2,x_3)} (u^\delta)^{(k-1)}(x) \\ &\le (-1)^{(k-1)} \sup_{x \in (x_2,x_3)} (u^\delta)^{(k-1)}(x) \le (-1)^{(k-1)} c_{k-1}. \end{aligned}$$

(c') *For $x_2 = x_3$ we have*

$$u_\alpha(x) = \begin{cases} \sum_{i=0}^{k-1} c_i x^i & \text{for } x \in (x_1,x_2) \\ \sum_{i=0}^{k-1} d_i x^i & \text{for } x \in (x_2,x_4) \end{cases}$$

and

$$d_{k-1} \le c_{k-1}.$$

Proof. (a) We first prove the case $k=1$. Assume that $\psi \in C_0^1(\Omega)$ with $\|\psi\|_{L^\infty(\Omega)} \le \alpha$ satisfies

$$\rho_\alpha^*(x) = \psi(x) \qquad \text{for } x \notin (a,b). \tag{5.2}$$

Then

$$\int_a^b (\psi' - {\rho_\alpha^*}') = (\psi - \rho_\alpha^*)\,|_a^b = 0 \qquad \text{and } \psi'(x) = {\rho_\alpha^*}'(x) \qquad \text{for } x \notin (a,b).$$

Since ψ has compact support and $\|\psi\|_{L^\infty(\Omega)} \le \alpha$, from the definition of $|Du_\alpha|$ it follows that

$$\int_\Omega \psi u_\alpha \le \alpha \, |Du_\alpha| \, .$$

From (4.10) it follows $\alpha\,|Du_\alpha| = -\int_\Omega u_\alpha^* u_\alpha = \int_\Omega {\rho_\alpha^*}' u_\alpha$ and consequently

$$\int_\Omega (\psi' - {\rho_\alpha^*}') u_\alpha = \int_a^b (\psi' - {\rho_\alpha^*}') u_\alpha \le 0$$

for all $\psi \in C_0^1(\Omega)$ with $\|\psi\|_{L^\infty(\Omega)} \le \alpha$ that satisfy (5.2). Since $C_0^1(\Omega)$ is dense in $C_0(\Omega)$, u_α has to be constant in (a,b).

For $k > 1$ from (4.10) and the definition of ρ_α^* in (5.1) it follows that

$$-\left|D^k u_\alpha\right| = \int_\Omega u_\alpha^* u_\alpha = (-1)^k \int_\Omega {\rho_\alpha^*}^{(k)} u_\alpha.$$

For $u \in TV^k(\Omega)$, $u^{(k-1)} \in TV(\Omega)$ (see Appendix). Define $w^* = -{\rho_\alpha^*}'$ and $w := u_\alpha^{(k-1)}$. Then

$$-\alpha\,|Dw| = -\alpha\left|D^k u_\alpha\right| = \int_\Omega u_\alpha^* u_\alpha = (-1)^k \int_\Omega {\rho_\alpha^*}^{(k)} u_\alpha = (-1)\int_\Omega {\rho_\alpha^*}' u_\alpha^{(k-1)} \quad = \int_\Omega w^* w$$

If $|\rho_\alpha^*| < \alpha$, then $w = const$. Since $w = u_\alpha^{(k-1)}$, we conclude that u_α is a polynomial of order $(k-1)$.

(b) If $|\rho_\alpha^*| = \alpha$ in (a,b), then

$${\rho_\alpha^*}^{(k)}(x) = u_\alpha^*(x) = 0 \text{ for } x \in (a,b).$$

The Kuhn-Tucker condition $u_\alpha^* \in \partial\mathcal{S}(u_\alpha)$ reads as follows

$$\begin{aligned} u_\alpha^*(x) = -1 &\Leftrightarrow u^\delta(x) \geq u_\alpha(x),\\ u_\alpha^*(x) = 1 &\Leftrightarrow u^\delta(x) \leq u_\alpha(x),\\ u_\alpha^*(x) \in (-1,1) &\Leftrightarrow u^\delta(x) = u_\alpha(x). \end{aligned} \tag{5.3}$$

for $p=1$ and (4.9) for $p=2$. Thus it follows in both cases that $u_\alpha = u^\delta$.

(c) Again we start with $k=1$. From the Kuhn-Tucker condition $-u_\alpha \in \partial\mathcal{R}^*\left(\frac{u_\alpha^*}{\alpha}\right)$, it follows that for all $v^* \in L^{p_*}(\Omega)$

$$\mathcal{R}^*\left(\frac{v^*}{\alpha}\right) - \mathcal{R}^*\left(\frac{u_\alpha^*}{\alpha}\right) + \int_\Omega u_\alpha(v^* - u_\alpha^*) \geq 0. \tag{5.4}$$

Since by assumption $|\rho_\alpha^*(x)| < \alpha$ in $(x_1,x_4)\backslash[x_2,x_3]$ it follows from (a) and (b) that

$$u_\alpha(x) = \begin{cases} c_0 & \text{for } x \in (x_1,x_2)\\ u^\delta(x) & \text{for } x \in (x_2,x_3)\\ d_0 & \text{for } x \in (x_3,x_4)\end{cases}$$

with $c_0, d_0 \in \mathbb{R}$.

For $x_2 < x_3$, $0 \leq \epsilon \leq \min\{x_2 - x_1, x_4 - x_3\}$ and $\delta > 0$ define

$$w^* = \begin{cases} +\delta & \text{for } x \in (x_2-\epsilon, x_2)\\ -\delta & \text{for } x \in (x_0, x_0+\epsilon) \subset (x_2,x_3)\\ 0 & \text{else}\end{cases}$$

Since $w^* \in L^{p_*}(\Omega)$ with $M_1^\perp \subset \mathcal{N}(w^*)$, $v^* = w^* + u_\alpha^* \in L^{p_*}(\Omega)$ with $M_1^\perp \subset \mathcal{N}(v^*)$. With this choice of v^* and the fact that $\mathcal{R}^*\left(\frac{u_\alpha^*}{\alpha}\right) = \mathcal{R}^*\left(\frac{v^*}{\alpha}\right) = 0$ (see (3.7)), it follows from (5.4) that

$$\int_\Omega u_\alpha(v^* - u_\alpha^*) = \int_{x_2-\epsilon}^{x_2} \delta u_\alpha - \int_{x_0}^{x_0+\epsilon} \delta u_\alpha = c_0\delta\epsilon - \delta\int_{x_0}^{x_0+\epsilon} u^\delta \geq 0.$$

Therefore we have

$$\sup_{x \in (x_2,x_3)} u^\delta(x) \leq c_0.$$

Analogously as above it can be shown that

$$d_0 \leq \inf_{x \in (x_2,x_3)} u^\delta(x).$$

For higher k we can argue as before by choosing $w^* = (\psi^*)^{(k)}$ such that

- $M_k^\perp \subset \mathcal{N}(w^*)$,

- $(\psi^*)'(x) = 0$ for $x \in \Omega \setminus ([x_2-\epsilon, x_2] \cup [x_0, x_0+\epsilon])$
- and
$$\int_{x_2-\epsilon}^{x_2} (\psi^*)' = -\int_{x_0}^{x_0+\epsilon} (\psi^*)' = -\delta.$$

We set $v^* = w^* + u^*_\alpha$, then it follows from (5.4) that

$$\begin{aligned}\int_\Omega u_\alpha(v^* - u^*_\alpha) &= (-1)^{(k-1)} \int_\Omega u_\alpha^{(k-1)} (\psi^*)' \\ &= (-1)^{k-1} \left(\int_{x_2-\epsilon}^{x_2} u_\alpha^{(k-1)}(\psi^*)' - \int_{x_0}^{x_0+\epsilon} u_\alpha^{(k-1)}(\psi^*)' \right) \\ &= (-1)^{k-1} \left(c_{k-1}\delta - \int_{x_0}^{x_0+\epsilon} (u^\delta)^{(k-1)}(\psi^*)' \right) \geq 0.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(-1)^{(k-1)} d_{k-1} &\leq (-1)^{(k-1)} \inf_{x \in (x_2,x_3)} u^{\delta\,(k-1)}(x) \\ &\leq (-1)^{(k-1)} \sup_{x \in (x_2,x_3)} u^{\delta\,(k-1)}(x) \leq (-1)^{(k-1)} c_{k-1}.\end{aligned}$$

(c') We have to show also in the cases $x_2 = x_3$, $d_0 \leq c_0$. In this case we set

$$v^*(x) := \begin{cases} 0 & \text{for } x \in (x_1, x_4), \\ u^*_\alpha(x) & \text{else.} \end{cases}$$

Since $M_k^\perp \subset \mathcal{N}(v^*)$ it follows from (5.4) that

$$\int_\Omega u_\alpha(v^* - u^*_\alpha) = -\int_{x_1}^{x_4} u_\alpha u^*_\alpha \geq 0 \tag{5.5}$$

Since $\int_{x_1}^{x_2} u^*_\alpha = -\alpha = -\int_{x_2}^{x_4} u^*_\alpha$ inequality (5.5) reduces to

$$\int_{x_1}^{x4} u_\alpha u^*_\alpha = c_0 \int_{x_1}^{x_2} u^*_\alpha + d_0 \int_{x_2}^{x_4} u^*_\alpha = -c_0\alpha + d_0\alpha \leq 0$$

Hence $d_0 \leq c_0$. For higher k we can argue as before and obtain

$$(-1)^{(k-1)} d_{k-1} \leq (-1)^{(k-1)} c_{k-1}.$$

□

Example 5.2 (L^1-TV regularization)**.** *We use Examples 3.4 and 3.6 to derive the dual functional $\mathcal{F}^*_{1,1}$. We set $M_1^\perp$, the set of polynomials of order 0 (constant functions on Ω) and*

$$M_1 := \left\{ u \in L^1(\Omega) \cap TV(\Omega) : \int_\Omega u = 0 \right\} .$$

The condition $M_1^\perp \subseteq \mathcal{N}(u^*)$ can be expressed as

$$u^* \in L^\infty_\diamond(\Omega) := \left\{ v^* \in L^\infty(\Omega) : \int_\Omega v^* = 0 \right\} .$$

The dual problem consists in maximization of $-\mathcal{F}^*_{1,1}(u^*) := -\int_\Omega u^\delta u^*$ over the set

$$\Psi_\alpha := \left\{ u^* \in L^\infty_\diamond(\Omega) : \|u^*\|_{L^\infty(\Omega)} \leq 1 \text{ and } \|u^*\|_{M_1^*} \leq \alpha \right\}.$$

From Example 3.7 we know that the condition $\|u^*\|_{M_1^*} \leq \alpha$ *is equivalent to*

$$\left|\int_{-1}^{x} u^*(t)dt\right| \leq \alpha \textit{ for all } x \in (-1,1).$$

Thus

$$\begin{aligned}\Psi_\alpha &= \left\{u^* : \left|\int_{-1}^{x} u^*\right| \leq \alpha\,, \int_{-1}^{1} u^* = 0 \textit{ and } \|u^*\|_{L^\infty(\Omega)} \leq 1\right\} \\ &= \left\{u^* = -\rho^{*\prime} : \rho^*(\pm 1) = 0,\, \|\rho^*\|_{L^\infty(\Omega)} \leq \alpha\,,\ \textit{and } \|u^*\|_{L^\infty(\Omega)} \leq 1\right\}\ .\end{aligned}$$

Using, that for $u^* \in \Psi_\alpha$ *and* ρ^* *the first primitive of* u^*,

$$-\mathcal{F}_{1,1}^*(u^*) = -\int_\Omega u^\delta u^* = -\int_{-1/2}^{1/2} u^* = \rho^*(-1/2) - \rho^*(1/2) \leq \min\{1, 2\alpha\}\,, \tag{5.6}$$

we find that the maximizer of $-\mathcal{F}_{1,1}^*$ *is*

$$u_\alpha^* := \begin{cases} -\min\{1,2\alpha\} & \textit{in } (-1/2,1/2) \\ \in [-1,1] & \textit{in } (-1,1)\backslash(-1/2,1/2) \end{cases} \in \Psi_\alpha\ .$$

Using the Kuhn-Tucker condition $u_\alpha^* \in \partial\mathcal{S}(u_\alpha)$, *we see that the minimizers* u_α *and* u_α^* *of the functional and its dual are related as follows:*

We distinguish between $\alpha < \frac{1}{2}, \alpha = \frac{1}{2}$, *and* $\alpha > \frac{1}{2}$.

$\alpha > 1/2$: *In this case* $u_\alpha^* = -2u^\delta$ *minimizes* (5.6). *Since* ρ_α^* *does not have contact with the* α*-tube, according to Theorem 5.1(a)* u_α *is constant in* $(-1,1)$. *From* (5.3) *it follows that*

$$\begin{aligned} u_\alpha(x) &\geq u^\delta(x) = -1/2 && \textit{for } x \in (-1,-1/2)\cup(1/2,1), \\ u_\alpha(x) &\leq u^\delta(x) = 1/2 && \textit{for } x \in (-1/2,1/2). \end{aligned}$$

Thus $u_\alpha = const$ *with* $const \in [-1/2, 1/2]$ *is a solution.*

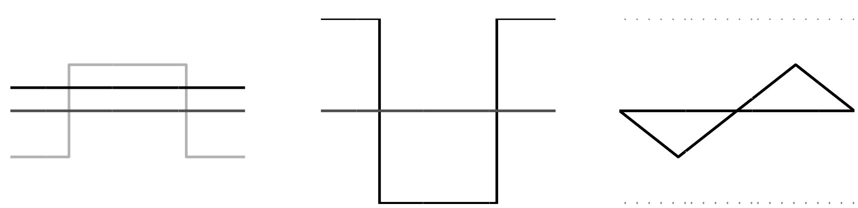

FIGURE 2. $\alpha > 1/2$. *Left:* u_α. *Gray:* u^δ. *Middle:* $u_\alpha^* = -\rho_\alpha^{*\prime}$. *Right:* ρ_α^*/α; ρ_α^* does not touch the $\alpha-$tube.

$\alpha < 1/2$: *The optimility condition in* (5.6) *fixes* ρ_α^* *for* $x = \pm 1/2$. *Let*

$$\begin{aligned} A(u_\alpha^*) &:= \{x \in (-1,-1/2) : u_\alpha^*(x) \in (-1,1)\}\,, \\ B^+(u_\alpha^*) &:= \{x \in (-1,-1/2) : u_\alpha^*(x) = 1, |\rho_\alpha^*(x)| < 1\}\,, \\ B^-(u_\alpha^*) &:= \{x \in (-1,-1/2) : u_\alpha^*(x) = -1, |\rho_\alpha^*(x)| < 1\}\ . \end{aligned}$$

Then $u^\delta = u_\alpha$ *in* $A(u_\alpha^*)$, $-1/2 = u^\delta \leq u_\alpha$ *in* $B^+(u_\alpha^*)$ *and* $u_\alpha \leq u^\delta = -1/2$ *in* $B^-(u_\alpha^*)$. *From Theorem 5.1 we know that* u_α *is constant in* $B^+(u_\alpha^*) \cup B^-(u_\alpha^*)$ *with* $const \geq -1/2$ *in* $B^+(u_\alpha^*)$ *and* $const \leq -1/2$ *in* $B^-(u_\alpha^*)$.

Thus $u_\alpha = u^\delta$ in $(-1/2, 1/2)$. For $(-1/2, 1/2)$ and $(1/2, 1)$ we can argue analogously to show that $u_\alpha = u^\delta$.

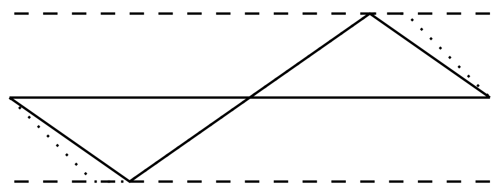

FIGURE 3. For $\alpha < 1/2$, ρ^*_α is not unique. The maximization condition only fixes ρ^*_α for $x = \pm\frac{1}{2}$.

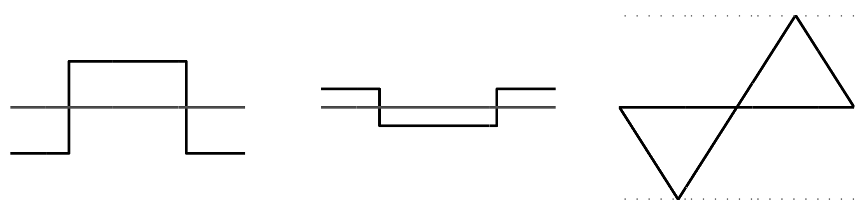

FIGURE 4. $\alpha < 1/2$: *Left:* $u_\alpha = u^\delta$. *Middle:* $u^*_\alpha = -(\rho^*_\alpha)'$. *Right:* ρ^*_α/α; ρ^*_α touches the $\alpha-$tube at $x = \pm 1/2$, these are the positions where u_α is discontinuous.

$\alpha = 1/2$: *Here $u^*_\alpha = -2u^\delta \in \Psi_\alpha$ and $\rho^*_\alpha(-1/2) = -\alpha = -\rho^*_\alpha(1/2)$. From (5.3) it follows that*

$$\begin{aligned} -1/2 = u^\delta &\leq u_\alpha && \textit{in } (-1,-1/2) \cup (1/2,1), \\ 1/2 = u^\delta &\geq u_\alpha && \textit{in } (-1/2,1/2). \end{aligned}$$

According to Theorem 5.1 (b) u_α is constant in the intervals $(-1,-1/2)$, $(-1/2,1/2)$, $(1/2,1)$. From (c) we know that $a_1 = u_\alpha(-3/4) \leq a_2 = u(0)$ and $a_2 = u_\alpha(0) \geq a_3 = u(3/4)$. Thus the solutions of the L^1-TV minimization problem are

$$u_\alpha = \begin{cases} -1/2 \leq a_1 & \textit{in } (-1,-1/2), \\ a_1 \leq a_2 \leq 1/2 & \textit{in } (-1/2,1/2), \\ -1/2 \leq a_3 \leq a_2 & \textit{in } (1/2,1). \end{cases}$$

Example 5.3 (L^2-TV regularization)**.** *To calculate minimizers of $\mathcal{F}_{2,1}$ with $\Omega = (-1,1)$, we use Examples 3.4 and 3.6 to derive the dual functional $\mathcal{F}^*_{2,1}$. We set $M_1^\perp$, the set of polynomials of order 0 (constant functions on Ω) and*

$$M_1 := \left\{ u \in L^2(\Omega) \cap TV(\Omega) : \int_\Omega u = 0 \right\} .$$

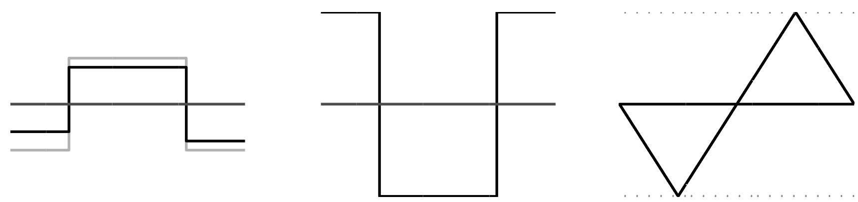

FIGURE 5. $\alpha = 1/2$. *Left:* u_α. *Gray:* u^δ. *Middle:* $u_\alpha^* = -(\rho_\alpha^*)'$. *Right:* ρ_α^*/α; ρ_α^* touches the $\alpha-$tube where u_α jumps.

The condition $M_1^\perp \subseteq \mathcal{N}(u^*)$ *can be expressed as*

$$u^* \in L^2_\diamond(\Omega) := \left\{ v^* \in L^2(\Omega) : \int_\Omega v^* = 0 \right\} .$$

Thus the dual problem consists in maximization of $-\mathcal{F}^*_{2,1}(u^*) := -\int_\Omega \left(\frac{1}{2}(u^*)^2 + u^\delta u^*\right)$ *over the set*

$$\Psi_\alpha := \left\{ u^* \in L^2_\diamond(\Omega) : \|u^*\|_{M_1^*} \leq \alpha \right\} .$$

From Example 3.7 we know that the condition $\|u^*\|_{M_1^*} \leq \alpha$ *is equivalent to the condition*

$$\left| \int_{-1}^x u^*(t)dt \right| \leq \alpha \text{ for all } x \in (-1,1).$$

Hence we have

$$\begin{aligned} \Psi_\alpha &= \left\{ u^* \in L^2_\diamond(\Omega) : \left| \int_{-1}^x u^*(t)dt \right| \leq \alpha \text{ for all } x \in (-1,1) \right\} \\ &= \left\{ u^* \in L^2(\Omega) : u^* = -(\rho^*)',\ \rho(\pm 1) = 0,\ \|\rho^*\|_{L^\infty(\Omega)} \leq \alpha \right\} . \end{aligned}$$

Taking into account that the minimizer u_α^* *of* $\mathcal{F}^*_{2,1}$ *is the same as the minimizer of* $\frac{1}{2}\int_\Omega (u^* + u^\delta)^2$ *we see that* u_α^* *is the* L^2*-projection of* $-u^\delta$ *onto* Ψ_α.

Examplarily we choose $u^\delta = \chi_{(-1/2,1/2)} - \frac{1}{2}$. *Integrating* u^δ *once shows that the minimal* α *for which* $u^\delta \in \Psi_\alpha$ *is* $\alpha = \frac{1}{4}$. *Thus for* $\alpha \geq \frac{1}{4}$ *we have* $u_\alpha^* = -u^\delta$. *If* $\alpha < \frac{1}{4}$ *then* $u_\alpha = -4\alpha\, u^\delta$. *In summary*

$$u_\alpha^* = -u^\delta \min\{1, 4\alpha\}.$$

and hence

$$u_\alpha = u^\delta + u_\alpha^* = \begin{cases} 0 & \text{for } \alpha \geq \frac{1}{4} \\ (1-4\alpha)u^\delta & \text{for } 0 \leq \alpha \leq \frac{1}{4} \end{cases}$$

Example 5.4 (L^1-TV^2 regularization)**.** *We consider the problem of* L^1-TV^2 *regularization, i.e., the minimization of the functional* $\mathcal{F}_{1,2}$ *with* $\Omega = (-1,1)$. *We use Examples 3.4 and 3.6 to derive the dual functional* $\mathcal{F}^*_{1,2}$. *We set* $M_2^\perp$, *the set of polynomials of order* 1 *(affine functions) and*

$$M_2 := \left\{ u \in L^1(\Omega) \cap TV^2(\Omega) : \int_\Omega u = 0 \text{ and } \int_\Omega u\, x = 0 \right\} .$$

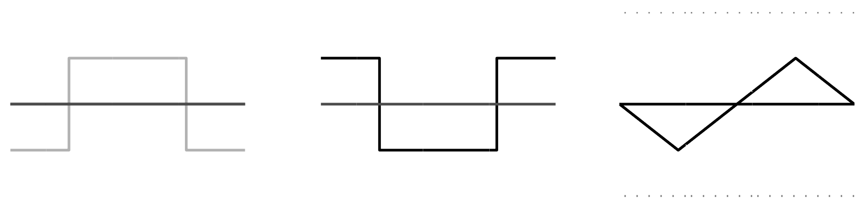

FIGURE 6. $\alpha \geq \frac{1}{4}$ *Left:* $u_\alpha = 0$. *Gray:* u^δ. *Middle:* $u_\alpha^* = -{\rho_\alpha^*}'$. *Right:* ρ_α^*/α; ρ_α^* does not touch the $\alpha-$tube.

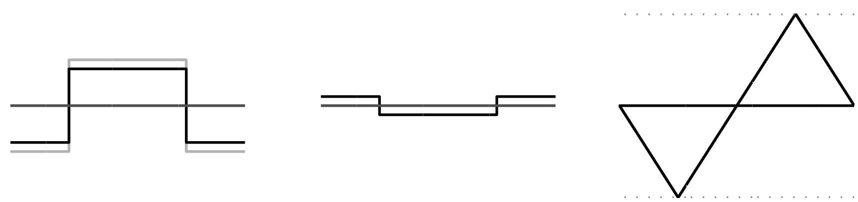

FIGURE 7. $\alpha < \frac{1}{4}$ *Left:* u_α *Gray:* u^δ. *Middle:* $u_\alpha^* = (-\rho_\alpha^*)'$. *Right:* ρ_α^*/α; ρ_α^* touches the $\alpha-$tube at the positions where u_α is discontinuous.

The condition $M_2^\perp \subseteq \mathcal{N}(u^*)$ can be expressed as

$$u^* \in L_\diamond^\infty(\Omega) := \left\{ v^* \in L^\infty(\Omega) : \int_\Omega v^* = 0 \text{ and } \int_\Omega v^* \, x = 0 \right\} .$$

Using 3.4 and 3.6 it follows that the dual problem consists in maximization of

$$-\mathcal{F}_{1,2}^*(u^*) := -\int_\Omega u^\delta u^*$$

over the set

$$\Psi_\alpha := \left\{ u^* \in L_\diamond^\infty(\Omega) : \|u^*\|_{L^\infty(\Omega)} \leq 1 \text{ and } \|u^*\|_{M_2^*} \leq \alpha \right\}.$$

From Example 3.7 we know that the condition $\|u^*\|_{M_2^*} \leq \alpha$ is equivalent to $\|\rho_\alpha^*\|_{L^\infty(\Omega)} \leq \alpha$ in $(-1,1)$. Hence we can write

$$\begin{aligned}\Psi_\alpha : &= \left\{ u^* \in L_\diamond^\infty(\Omega) : \|u^*\|_{L^\infty(\Omega)} \leq 1, \|u^*\|_{M_2^*} \leq \alpha \right\} \\ &= \big\{ u^* \in L^\infty(\Omega) : \|u^*\|_{L^\infty(\Omega)} \leq 1, \; u^* = {\rho^*}'', \|\rho^*\|_{L^\infty(\Omega)} \leq \alpha, \\ &\qquad {\rho^*}'(\pm 1) = 0, \, \rho^*(\pm 1) = 0 \big\}.\end{aligned}$$

For $u^* \in \Psi_\alpha$ and $u^\delta = \chi_{[-1/2,1/2]} - 1/2$ we have

$$-\mathcal{F}_{1,2}^*(u^*) = -\int_\Omega u^\delta u^* = {\rho^*}'\left(-\frac{1}{2}\right) - {\rho^*}'\left(\frac{1}{2}\right) . \tag{5.7}$$

We calculate the minimizer of $\mathcal{F}_{1,2}$ with $u^\delta = \chi_{[-1/2,1/2]} - 1/2$. We distinguish between the three cases $\alpha > 1/4, \alpha \in (\frac{3}{8} - \frac{\sqrt{2}}{4}, \frac{1}{4}]$ and $\alpha \leq \frac{3}{8} - \frac{\sqrt{2}}{4}$.

- *Let $\alpha > 1/4$, then $u^*_\alpha := -2u^\delta$ is an element of Ψ_α which maximizes* (5.7). *Moreover $|\rho^*_\alpha| < \alpha$ and according to Theorem 5.1 (a) u_α is a polynomial of order* 1 *on* $(-1,1)$. *From* (5.3) *it follows that*

$$\begin{aligned} 0 \le u_\alpha \quad & \textit{in } (-1,-1/2)\cup(1/2,1),\\ u_\alpha \le 1 \quad & \textit{in } \in (-1/2,1/2). \end{aligned} \tag{5.8}$$

Thus the minimizers of $\mathcal{F}_{1,2}$ are affine functions satisfying (5.8) *(Figure 8).*

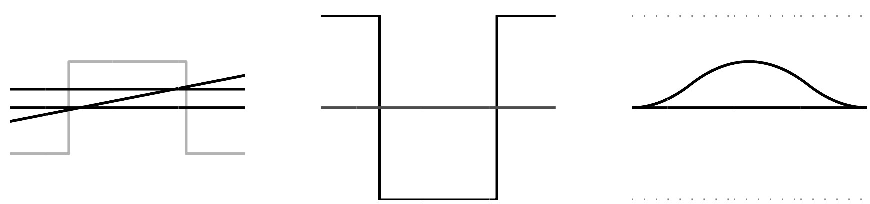

FIGURE 8. $\alpha > \frac{1}{4}$ *Left: bold:* u_α, *gray:* u^δ. Note that u_α is not unique. *Middle:* $u^*_\alpha = {\rho^*_\alpha}''$. *Right:* ρ^*_α/α, ρ^*_α does not touch the $\alpha-$tube.

- *Let $\alpha \in (3/8 - \sqrt{2}/4, 1/4)$. If $\alpha \le 1/4$, ρ^*_α has at least one contact point $0 \le x_1 \le 1/2$ with the α-tube and according to Theorem 5.1 u_α bends at $x = \pm x_1$. From Theorem 5.1 it follows that u_α is affine in $(-1,-x_1)$ and $(x_1,1)$.*

  Since u^δ is symmetric, there exists a symmetric minimizer of $\mathcal{F}_{1,2}$ which satisfies $|\rho^*_\alpha(-x_1)| = \alpha$. In the following we concentrate on calculating symmetric minimizers.

  We calculate the minimizer u_{C_α} of $\mathcal{F}_{1,2}$ in the class

$$\mathcal{C} := \left\{ u_C : C = \{s, x_1, d\} \ \textit{with } s\,x_1 + d = \frac{1}{2} \textit{ and } \frac{1}{2} \le -\frac{s}{d} \le 1 \right\},$$

where

$$u_C(x) = \begin{cases} 1/2 & x \in [-x_1, x_1],\\ s\,|x| + d & x \in (-1,-1)\backslash[-x_1,x_1]\,, \end{cases}$$

and prove afterwards that $u_\alpha = u_{C_\alpha}$.

We set

$$\mathcal{C}^* := \{u^*_C \in \Psi_\alpha : (u^*_C, u_C) \textit{ satisfy } (5.3)\}$$

One possibility of a function $u^*_C \in \mathcal{C}^*$ related to u_C by (5.3) *is as follows:*

$$u^*_C(x) = \begin{cases} -1 & x \in (-1, s/d)\cup(-\frac{1}{2},-x_1)\cup(x_1,\frac{1}{2})\cup(-s/d,1),\\ +1 & x \in (s/d,-\frac{1}{2})\cup(\frac{1}{2},-s/d),\\ 0 & x \in (-x_1,x_1)\,. \end{cases} \tag{5.9}$$

We determine $C_\alpha := \{s_\alpha, x_{1,\alpha}, d_\alpha\}$ as follows:

– *Since* u_α *bends at* $x_{1,\alpha}$, *we aim for* u_{C_α} *which bends at* $x = \pm x_{1,\alpha}$. *Thus*

$$\rho^*_{C_\alpha}(x_{1,\alpha}) = \frac{4s_\alpha^2 - 8s_\alpha x_{1,\alpha} d_\alpha + 6x_{1,\alpha}^2 d_\alpha^2 - 8x_{1,\alpha} d_\alpha^2 - 4d_\alpha^2}{4d_\alpha^2} = \alpha.$$

– $\rho^*_{C_\alpha}$ *is maximal at* $\pm x_{1,\alpha}$

$$\rho^{*\prime}_{C_\alpha}(x_{1,\alpha}) = \frac{x_{1,\alpha} d_\alpha - 2s_\alpha - 2d_\alpha}{d_\alpha} = 0.$$

From this calculations it follows that:

$$x_{1,\alpha} = \frac{1}{s_\alpha}\left(\frac{1}{2} - d_\alpha\right),\ d_\alpha = \frac{4+\sqrt{1+12\alpha}}{3\sqrt{1+12\alpha}},\ s_\alpha = \frac{-2}{\sqrt{1+12\alpha}}\ .$$

Since u_{C_α} *and* $u^*_{C_\alpha}$ *satisfy* (3.4) *they are minimizers of* $\mathcal{F}_{1,2}, \mathcal{F}^*_{1,2}$ *respectively.*

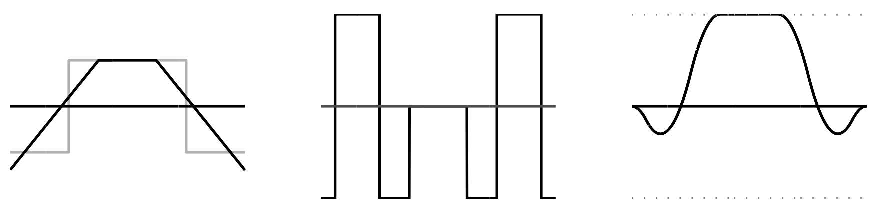

FIGURE 9. *Left:* u_α bends at $x = \pm x_{1,\alpha}$. *Gray:* u^δ. *Middle:* $u^*_\alpha = \rho^{*\prime\prime}_\alpha$. *Right:* ρ^*_α/α; ρ^*_α touches the $\alpha-$tube at $x = \pm x_{1,\alpha}$, where u_α bends.

- *Let* $\alpha \le \frac{3}{8} - \frac{1}{4}\sqrt{2}$. *We calculate the minimizers* u_{C_α} *of* $\mathcal{F}_{1,2}$ *in*

$$\mathcal{C} := \big\{u_C : C := \{s_1, s_2, d_1, d_2, x_{1,\alpha}, x_2\} \text{ with } 0 \le x_1 \le \frac{1}{2} \le x_2 \le 1,\ s_1 \le 0 \text{ and } s_1 \le s_2\big\}$$

with

$$u_C(x) = \begin{cases} s_2\,|x| + d_2 & x \in (-1,1)\backslash(-x_2, x_2) \\ s_1\,|x| + d_1 & x \in (-x_2, -x_1) \cup (x_1, x_2) \\ 1/2 & x \in (-x_1, x_1) \end{cases}. \tag{5.10}$$

Here the functions u_C *can bend at least four times. Afterwards we verify that* $u_\alpha = u_{C_\alpha}$.

Let

$$\mathcal{C}^* := \{u^*_C \in \Psi_\alpha : (u^*_C, u_C) \text{ satisfy } (5.3)\}\ .$$

One possibility of a function $u^*_C \in \mathcal{C}^*$ *related to* u_C *by* (5.3) *is as follows:*

$$u^*_C(x) \mapsto \begin{cases} -c_1 & x \in (-1, -\frac{1}{2} - \frac{1}{2}x_2) \cup (\frac{1}{2} + \frac{1}{2}x_2, 1) \\ c_1 & x \in (-\frac{1}{2} - \frac{1}{2}x_2, -x_2) \cup (x_2, \frac{1}{2} + \frac{1}{2}x_2) \\ c_2 & x \in (-x_2, -\frac{1}{2}) \cup (\frac{1}{2}, x_2) \\ -c_2 & x \in (-\frac{1}{2}, -x_1) \cup (x_1, \frac{1}{2}) \\ 0 & x \in (-x_1, x_1) \end{cases}$$

with $0 \le c_1, c_2 \le 1$. Let ρ_C^* be the second primitive of u_C^*.

Since u_α bends at $x = \pm x_1, \pm x_2$, we aim to find u_{C_α} which bends at x_1, x_2 as well and according to Theorem 5.1 enforcing the properties of ρ_α onto ρ_{C_α} we additionally require that $\rho_{C_\alpha}^*$ is maximal at $\pm x_1$, $\pm x_2$. Thus, by using that by assumption $s_1 \le 0$ and $s_1 \le s_2$ it follows that

$$\begin{aligned} \rho_C{}^*(-x_{2,\alpha}) &= \int_{-1}^{-x_{2,\alpha}} (\rho_{C_\alpha}^*)' = -\alpha, \quad & \rho_C{}^*(-x_{1,\alpha}) &= \int_{-1}^{-x_{1,\alpha}} (\rho_{C_\alpha}^*)' = \alpha, \\ \rho_C{}^*(x_{1,\alpha}) &= \int_{-1}^{x_{1,\alpha}} (\rho_{C_\alpha}^*)' = \alpha, & \rho_C{}^*(x_{2,\alpha}) &= \int_{-1}^{x_{2,\alpha}} (\rho_{C_\alpha}^*)' = -\alpha. \end{aligned} \tag{5.11}$$

Since $\rho_{C_\alpha}^*$ attains an extremum at $\pm x_{1,\alpha}, x_{2,\alpha}$, we have that

$$\rho_{C_\alpha}^*{}'(\pm x_{2,\alpha}) = 0 \text{ and } \rho_{C_\alpha}^*{}'(\pm x_{1,\alpha}) = 0\ .$$

Taking into account that $\rho_C^* \in \Psi_\alpha$ (and thus satisfies boundary conditions), if we see that maximizing $-\mathcal{F}_{2,2}$ on $\mathcal{C}$ is equivalent to maximizing

$$\rho_C^*{}'\left(-\frac{1}{2}\right) = \int_{-1}^{-1/2} u_C^* = \frac{c_2}{2}\left(x_2 - \frac{1}{2}\right)$$

Hence it follows that $c_{2,\alpha} = 1$. Since $\rho_{C_\alpha}^*(-x_{2,\alpha}) = -\alpha$ and $\rho_{C_\alpha}'(-1/2) = 0$, $x_{2,\alpha}$ has to satisfy

$$\int_{-x_{2,\alpha}}^{-1/2} \rho_{C_\alpha}^*{}' = \int_{-x_{2,\alpha}}^{-1/2} c_{2,\alpha}(x + x_{2,\alpha}) = \frac{1}{2}x_{2,\alpha}^2 - \frac{1}{2}x_{2,\alpha} + \frac{1}{8} = \alpha$$

Hence

$$x_{2,\alpha} = \frac{1}{2} + \sqrt{2\alpha}.$$

Analogous we find that

$$x_{1,\alpha} = \frac{1}{2} - \sqrt{2\alpha}.$$

Since $\rho_C^*(-x_{2,\alpha}) = -\alpha$ it follows that

$$c_{1,\alpha} = \frac{16\alpha}{\left(2\sqrt{2\alpha} - 1\right)^2} \le 1.$$

Next we determine the coefficients such that u_{C_α} and $\hat{u}_{C_\alpha}^*$ are connected via (5.3) and get

$$\begin{aligned} s_{2,\alpha} &= 0, & d_{2,\alpha} &= 0, \\ k_{1,\alpha} &= -\frac{1}{2\sqrt{2\alpha}}, & d_{1,\alpha} &= \frac{x_{2,\alpha}}{2\sqrt{2\alpha}}. \end{aligned}$$

Then u_{C_α} minimizes $\mathcal{F}_{1,2}$ as can be shown by testing (3.4).

Example 5.5 (L^2-TV^2 regularization)**.** *We consider the problem of L^2-TV^2 regularization, i.e., the minimization of the functional $\mathcal{F}_{2,2} :\ L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$. We use Examples 3.4 and 3.6 to derive the dual functional $\mathcal{F}_{2,2}^*$. We set $M_2^\perp$, the set of polynomials of order 1 (affine functions on Ω) and*

$$M_2 := \left\{ u \in L^2(\Omega) \cap TV^2(\Omega) : \int_\Omega u = 0 \text{ and } \int_\Omega u\, x = 0 \right\}\ .$$

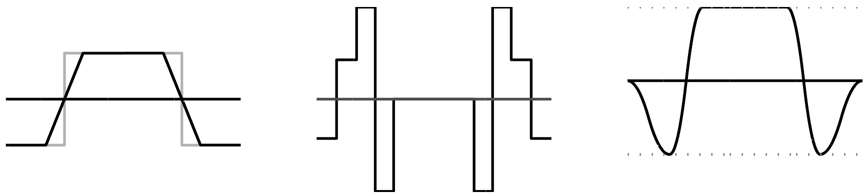

FIGURE 10. $\alpha < \frac{3}{8} - \frac{1}{4}\sqrt{2}$ *Left:* u_α bends at $x = \pm\left(\frac{1}{2} \pm \sqrt{2\alpha}\right)$. *Gray:* u^δ. *Middle:* $u^*_\alpha = (\rho^*_\alpha)''$ *Right:* ρ^*_α/α; ρ^*_α touches the α-tube at $x = \pm\left(\frac{1}{2} \pm \sqrt{2\alpha}\right)$, where u_α bends.

The condition $M_2^\perp \subseteq \mathcal{N}(u^*)$ can be expressed as

$$u^* \in L^2_\diamond(\Omega) := \left\{ v^* \in L^2(\Omega) : \int_\Omega v^* = 0 \text{ and } \int_\Omega v^* x = 0 \right\} .$$

The dual problem consists in maximization of

$$-\mathcal{F}^*_{2,2}(u^*) = -\int_\Omega \left(\frac{1}{2}(u^*)^2 + u^* u^\delta \right) \tag{5.12}$$

over the set

$$\begin{aligned}\Psi_\alpha :&= \left\{ u^* \in L^2_\diamond(\Omega) : \|u^*\|_{M_2^*} \le \alpha \right\}\\ &= \left\{ u^* \in L^2(\Omega) : u^* = \rho^{*\prime\prime}, \|\rho^*\|_{L^\infty(\Omega)} \le \alpha, \rho^{*\prime}(\pm 1) = 0, \rho^{*\prime\prime}(\pm 1) = 0 \right\}.\end{aligned}$$

We consider again as test data $u^\delta = \chi_{(-1/2,1/2)} - \frac{1}{2}$.

We investigate four different cases $\alpha > \frac{1}{8} = \left\|u^\delta\right\|_{M_2^*}$, $\alpha \in (\frac{1}{24}, \frac{1}{8})$, $\alpha \in (\alpha_m, \frac{1}{24})$, and $\alpha < \alpha_m$. Here α_m denotes the largest α-value such that ρ^*_α takes the value $-\alpha$ for some $x \in (-1, 1)$.

- *If $\alpha > \left\|u^\delta\right\|_{M_2^*} = \frac{1}{8}$, then $-u^\delta \in \Psi_\alpha$. Thus $u^*_\alpha = -u^\delta$ is the L^2-projection of $-u^\delta$ onto Ψ_α and (4.9) shows that $u_\alpha = u^*_\alpha + u^\delta = 0$. Since $(u_\alpha, u^*_\alpha = -u^\delta)$ satisfy (3.4) they are minimizers of $\mathcal{F}_{2,2}, \mathcal{F}^*_{2,2}$ respectively.*
- *For $\alpha \in (\frac{1}{24}, \frac{1}{8}]$ we calculate the minimizer u_{C_α} of $\mathcal{F}_{2,2}$ in the class of functions*

  $$\mathcal{C} := \{ u_C : C := \{s, d\} \text{ and } u_C(x) = s\,|x| + d \} ,$$

  and verify afterwards that $u_\alpha = u_{C_\alpha}$.

  Let $u^*_C = u_C - u^\delta$ and ρ^*_C the second primitive of u^*_C. We determine $C_\alpha := \{s_\alpha, d_\alpha\}$ such that

  – *$u^* C_\alpha \in \Psi_\alpha$, that is*

  $$\int_\Omega u^*_{C_\alpha} = s_\alpha + 2d_\alpha = 0 \qquad \to d_\alpha = -\frac{1}{2} s_\alpha$$

  and

  – *u_{C_α} bends at $x = 0$, which requires that*

$$\rho^*_{C_\alpha}(0) = \frac{1}{12} s_\alpha + \frac{1}{8} = \alpha, \qquad \to s_\alpha = \frac{24\alpha - 3}{2}. \tag{5.13}$$

Then $(u^*_{C_\alpha} + u^\delta, u^*_{C_\alpha})$ *satisfy* (3.4) *and thus are minimizers of* $\mathcal{F}_{2,2}, \mathcal{F}^*_{2,2}$, *respectively.*

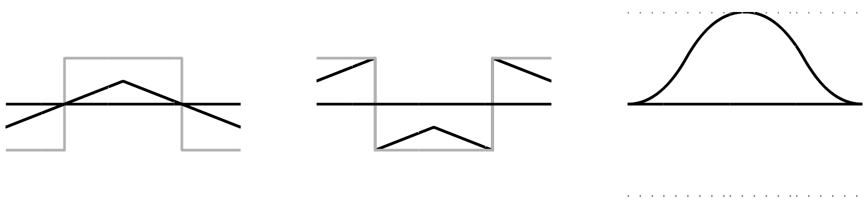

FIGURE 11. *Left:* u_α bends at $x = 0$. *Gray:* u^δ. *Middle:* $u^*_\alpha = (\rho^*_\alpha)''$. *Right:* ρ^*_α/α, ρ^*_α touches the $\alpha-$tube at $x = 0$, where u_α bends.

- *For* $\alpha \in (\alpha_m, \frac{1}{24}]$ *we calculate the minimizer* u_{C_α} *of* $\mathcal{F}_{2,2}$ *in*
$$\mathcal{C} := \{u_C : C := \{s, d, x_1\}\}$$
with
$$u_C(x) \quad = \quad \begin{cases} s\,|x| + d & x \in (-1, -x_1) \cup (x_1, 1) \\ u^\delta & \text{in } (-x_1, x_1) \end{cases} \tag{5.14}$$
We verify afterwards that $u_\alpha = u_{C_\alpha}$. *Let again* $u^*_C = u_C - u^\delta$ *and* ρ^*_C *the second primitive of* u^*_C. *We determine* $C_\alpha = \{s_\alpha, d_\alpha, x_{1,\alpha}\}$ *such that*
  - $u^*_{C_\alpha} \in \Psi_\alpha$, *that is*
$$\int_\Omega u^*_C = s_\alpha + 2d_\alpha + s_\alpha\, x^2_{1,\alpha} = 0\,.$$
This condition implies that
$$d_\alpha = -\frac{1}{2}\left(s_\alpha + s_\alpha\, x^2_{1,\alpha}\right).$$
  - $\rho^*_{C_\alpha}$ *is extremal at* $\pm x_1$
$$\rho^*_{C_\alpha}{}'(x_{1,\alpha}) = s_\alpha\left(x^2_{1,\alpha} - \frac{1}{2}x_{1,\alpha} - \frac{1}{2}x^3_{1,\alpha}\right) - \frac{1}{2}x_{1,\alpha} = 0\,,$$
$$\rho^*_{C_\alpha}(x_{1,\alpha}) = -\frac{1}{6}x_{1,\alpha} + \frac{1}{24} = \alpha.$$
These two conditions are guaranteed if $s_\alpha = -\frac{1}{(1-x_{1,\alpha})^2}$ *and* $x_{1,\alpha} = 6\alpha - \frac{1}{4}$.

Since $u_{C_\alpha} = u^*_{C_\alpha} + u^\delta$ *and* $u^*_{C_\alpha}$ *satisfy* (3.4) *they are minimizers of* $\mathcal{F}_{2,2}, \mathcal{F}^*_{2,2}$ *respectively. See Figure 12.*
- *For* $\alpha \le \alpha_m$ *we minimize* $\mathcal{F}_{2,2}$ *on the set of piecewise affine functions*
$$u_C(x) \quad = \quad \begin{cases} s_2\,|x| + d_2 & x \in (-1, -x_2) \cup (x_2, 1) \\ s_1\,|x| + d_1 & x \in (-x_2, -x_1) \cup (x_1, x_2) \\ s_1 x_1 + d_1 & x \in (-x_1, x_1) \end{cases}$$
with $C := \{x_2, x_1, k_2, k_1, d_1, d_2\}$ *and proceed as above to determine* C_α. *Afterwards we show that* $u_\alpha = u_{C_\alpha}$. *See Figure 13.*

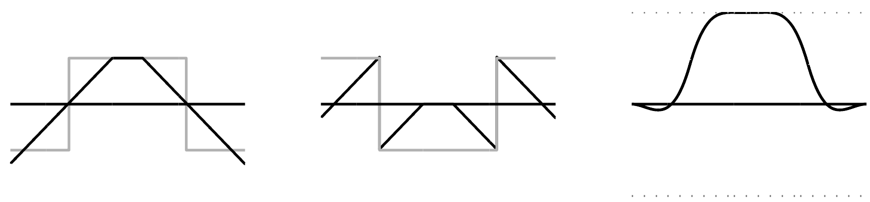

FIGURE 12. *Left:* u_α bends at $x = \pm x_1$, *gray:* u^δ. *Middle:* $u^*_\alpha = {\rho^*_{C_\alpha}}''$, *gray:* $-u^\delta$. *Right:* ρ^*_α/α; $\rho^*_\alpha = \alpha$ in $(-x_1, x_1)$.

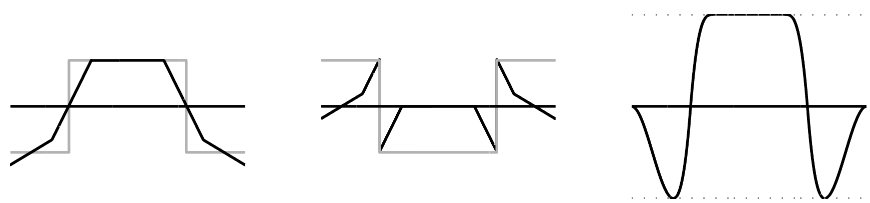

FIGURE 13. *Left:* u_α bends at $x = \pm x_1$ and $x = \pm x_2$, *gray:* u^δ. *Middle:* $u^*_\alpha = (\rho^*_{C_\alpha})''$,*gray:* $-u^\delta$. *Right:*ρ^*_α/α; ρ^*_α touches the $\alpha-$tube at $x = \pm x_2$ and $x = \pm.x_1$, where u_α bends.

Appendix A. Functions of Bounded Variation

In the following we highlight some properties of functions of bounded variation, which are collected from Evans & Gariepy [EvaGar92] and Ambrosio, Fusco & Pallara [AmbFusPal00].

Definition A.1. *For $1 \le p < n$ the Sobolev conjugate is*

$$p_n = \frac{np}{n-p}\,.$$

Definition A.2. *The space of functions of bounded variation (BV) consists of functions $u \in L^{p_n}(\Omega)$ satisfying*

$$|Du| := \left\{ \int u \nabla \cdot \vec{\phi} : \vec{\phi} \in C^1_0(\mathbb{R}^n, \mathbb{R}^n), \left\| \vec{\phi} \right\|_{L^\infty(\Omega)} \le 1 \right\} < \infty$$

The standard definition of BV requires $u \in L^1(\Omega)$ ([EvaGar92]). In this case $u \in L^{p_n}(\Omega)$ (which follows from the Gagliardo-Nirenberg-Sobolev inequality).

Definition A.3. *We define the set of functions with derivatives of bounded variation (BV^k) as functions $u \in L^p(\Omega)$ (for some $p \ge 1$) satisfying*

$$\left|D^k u\right| := \sup \left\{ \int u \nabla^k \cdot \vec{\phi} : \vec{\phi} \in C^k_0(\mathbb{R}^n, \mathbb{R}^{n^k}), \left|\vec{\phi}\right| \le 1 \right\} < \infty$$

where

$$\nabla^k \cdot \vec{\phi} = \sum_{\substack{i_l = 1,\dots,n \\ l = 1,\dots,k}} \frac{\partial^k \phi_{i_1,\dots,i_k}}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

The gradient $Du = (D_1 u, \dots, D_n u)$ of a function of bounded variation is representable by a finite Radon measure in Ω and satisfies

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx \;= -\int_\Omega \phi d D_i u \quad \phi \in C_0^\infty(\Omega)\,, i = 1, \dots, N\ .$$

The following superspaces of $\mathrm{BV}^k(\Omega)$ are of importance for this work:

Definition A.4. *Functions with finite total variation. For $k \in \mathbb{N}$ let*

$$TV^k(\Omega) := \left\{ u \in C_0^k(\Omega)^* : \left| D^k u \right| < \infty \right\}\ .$$

In the definition above u has to be considered a distribution of order k, that is u is a linear operator on $C_0^k(\Omega; \mathbb{R}^{n^k})$ satisfying

$$u[\vec{\phi}] = (-1)^k \int_\Omega u \nabla^k \vec{\phi} \quad \textit{for all } \vec{\phi} \in C_0^\infty(\Omega; \mathbb{R}^{n^k})\ .$$

Theorem A.5. *For $k > 1$ and $u \in TV^k(\Omega)$, $\nabla^{(k-1)} u$ exists and*

$$\left| D^k u \right| = \left| D(\nabla^{(k-1)} u) \right|.$$

Proof. Since $TV^k(\Omega) \subset \left(C_0^k(\Omega) \right)^*$ and $C_0^k(\Omega) \subset C_0^{k-1}(\Omega)$ every $u \in \left(C_0^k(\Omega) \right)^*$ can be extended to a functional $\tilde{u} \in \left(C_0^{(k-1)}(\Omega) \right)^*$ such that

$$u[\phi] = \tilde{u}[\phi] \text{ for all } \phi \in C_0^k(\Omega).$$

Since $C_0^\infty(\Omega) \subset C_0^k(\Omega)$ it follows that

$$\int_\Omega u \nabla^k \cdot \psi = u[\phi] = \int_\Omega \nabla u \nabla^{k-1} \cdot \psi = \tilde{u}[\psi]$$

for all $\psi \in C_0^\infty(\Omega)$. Thus ∇^u exists. According to Hahn-Banach, this extension is norm preserving, so that we have

$$\left| D^k u \right| = \left| D^{k-1} (\nabla u) \right|.$$

□

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