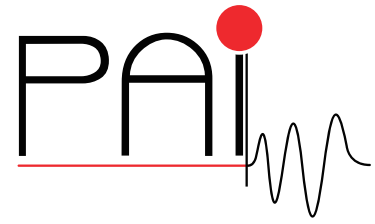# Photoacoustic Imaging in Medicine and Biology

Frequency Dependent Attenuation Revisited

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Frequency Dependent Attenuation Revisited

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Abstract

The work is inspired by thermo-and photoacoustic imaging, where recent efforts are devoted to take into account attenuation and varying wave speed parameters. In this paper we study causal equations describing propagation of attenuated pressure waves. We review standard models like frequency power laws and and the thermo-viscous equation. The lack of causality of standard models in the parameter range relevant for photoacoustic imaging requires to derive novel equations. The main ingredients for deriving causal equations are the Kramers-Kronig relation and the mathematical concept of linear system theory. The theoretical results of this work are underpined by numerical experiments.

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I.   INTRODUCTION

The work is inspired by thermo-and photoacoustic imaging (see e.g. [8, 13, 20, 22, 25] for some articles related to the subject), where the problem is the reconstruction of the *absorption density* from measurements of the pressure outside of the object. This is the *Inverse Problems* according to the forward problem, which maps the absorption density onto the pressure by solving the *standard* wave equation. Various reconstruction methods have been suggested in the literature for photoacoustic imaging, which can for instance be found in the excellent survey [9]. Recent efforts have been made to take into account attenuation [1, 10, 14] and varying wave speed [5]. The standard model of attenuation (which is reviewed in Section III) is formulated in the frequency range and models the physical reality that high frequency components of waves are attenuated more rapidly over time.

In this paper we review standard attenuation models, like *power laws* [16–19, 23] and the *thermo-viscous* wave equation [7]. In this context, we discuss causality, which is the desired feature of attenuation models. The lack of causality of standard models in the parameter range relevant for photoacoustic imaging requires to investigate novel equations, which are derived in Section III and the following.

We base the derivation of causal attenuation models on the Kramers-Kronig relation (see e.g. [23]), as usual, and the mathematical concept of *linear system theory*, which can for instance be found in the book of Hörmander [4]. In Section IV the abstract formulations are translated in equations which are formally similar to the wave equation. However, in general, the novel equations are integro-differential equations. An important issue is that the equations are formulated as inhomogeneous equations with homogeneous initial conditions, which is not standard for attenuated wave equations, where typically the equations are considered homogeneous with inhomogeneous initial conditions. For the standard wave equation these two concepts are equivalent, but only the one considered here, is mathematically sensible for the attenuation model.

The approach leads to some novel causal attenuation models, in particular power law models (valid for a bounded frequency range), which are documented in the literature to be relevant for biological specimen (in the terminology used later on this means that $\gamma \in (1, 2]$ – see [24, Chapter 7]) and also for instance also for castor oil, which satisfies a power law with index $\gamma = 1.66$ [18]. These models are presented in Section VIII. The rotationally symmetric

examples, presented in Section IX, illustrate the unphysical behavior of some existing attenuation models. Aside from unmeaningly physical effects, the stable and convergent numerical implementation of attenuated, non-causal wave equations is an unconsidered problem since these equations lack the Courant-Friedrich-Levy (CFL) condition [2]. The attenuation models considered here have a finite front wave speed and therefore can be implemented in a stable manner. Thus aside from physical considerations also from a point of view of stable numerical solution of wave equations questions of causality are most relevant.

Concerning the presentation of the paper, the basic notation and mathematic results are summarized in the appendix.

II.   LINEAR SYSTEM THEORY

This section surveys *linear system theory* (see e.g. [4, 12]), which provides the link between linear systems and convolution operators. For notational convenience, when we speak about functions they are understood in the most wide meaning of the word, and can for instance be distributions.

In the following, we give a characterization of *causal* functions and operators.

Definition II.1.   1. A function $f := f(\mathbf{x}, t)$ defined on the Euclidean space over time (i.e. in $\mathbb{R}^4$) is said to be *causal* if it satisfies

$$f(\mathbf{x}, t) = 0 \qquad \text{for} \qquad t < 0 .$$

2. In this paper $\mathcal{A}$ (with and without subscripts) denotes a real (that is, it is a mapping between sets of real functions on $\mathbb{R}^4$) and bounded operator.

- $\mathcal{A}$ is *translation invariant* if for every function f and every linear transformation $L := L(\mathbf{x}, t) := (\mathbf{x} - \mathbf{x}_0, t - t_0)$, with $\mathbf{x}_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}$, it holds that

$$\mathcal{A}(f \circ L) = (\mathcal{A}f)L .$$

   Here $\circ$ denotes the decomposition, i.e., $(f \circ L)(\mathbf{x}, t) = f(L(\mathbf{x}, t))$

- $\mathcal{A}$ is called *causal*, if it maps causal functions to causal functions.

- The operator $\mathcal{A}$ has a *causal domain of influence* if the function

$$T(\mathbf{x}) := \sup\{t : \mathcal{A}\delta_{\mathbf{x}, t}(\mathbf{x}, \tau) = 0 \text{ for all } \tau \leq t\} \qquad \text{for all} \qquad \mathbf{x} \in \mathbb{R}^3$$

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is rotationally symmetric and the derivative with respect to the radial component T' satisfies

$$0 < (T'(r))^{-1} \le c_0 < \infty .$$ (1)

For convenience of notation we identify $T(|\mathbf{x}|) = T(\mathbf{x})$.

The function T, presumably it exists, corresponds to the travel time of a wavefront initiated in $\mathbf{0}$ at $t = 0$. (1) guarantees that the wavefront speed is finite.

Remark 1. If the operator $\mathcal{A}$ models a physical process in a *homogeneous and isotropic medium*, then $\mathcal{A}$ is shift invariant and $\mathcal{A}\delta_{\mathbf{x},t}$ is rotationally symmetric.

If T exists, and in addition satisfies (1), then the property of a causal domain of influence guarantees that a wavefront can propagate with a speed of at most c_0.

Now, we recall a fundamental mathematical theorem (see [4, Theorem 4.2.1]) of systems theory, which relates invariant operators with *space–time* convolutions.

Theorem II.1. *[4, Theorem 4.2.1] Every (causal and) translation invariant operator $\mathcal{A}$ can be written as a space–time convolution operator with (causal) kernel G. That is, for arbitrary f from a suitable class of functions we have*

$$\mathcal{A}f = G *_{\mathbf{x},t} f .$$ (2)

In analogy to linear system theory we call the kernel G the *Green function* of $\mathcal{A}$. According to Definition II.1 the considered operators are real and therefore the according Green functions are real valued too. From the definition of the Green function it follows that

$$G = \mathcal{A}\delta_{\mathbf{x},t} .$$

Example II.1. We consider the standard wave equation in an isotropic medium with phase speed $c_0 \in (0, \infty)$:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p = -f,$$ (3)

together with *initial conditions*

$$p|_{t<0} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}p\bigg|_{t<0} = 0 .$$ (4)

With source term $f = \delta_{\mathbf{x},t}$, the according solution G_0 of (3) and (4) is the *Green function*

$$G_0(\mathbf{x}, t) = \frac{\delta_t\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi |\mathbf{x}|} .$$ (5)

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Because of (5) G_0 is commonly denoted *spherical wave* originating from $\mathbf{x} = \mathbf{0}$ at time $t = 0$.

In the space–frequency domain the Green function can be expressed by

$$\mathcal{F}\{G_0\} := \mathcal{F}\{G_0\}(\mathbf{x}, \omega) = \frac{1}{\sqrt{2\pi}} \frac{\exp\left(\mathbf{i}\omega\frac{|\mathbf{x}|}{c_0}\right)}{4\pi|\mathbf{x}|}.$$

It satisfies

$$\nabla\mathcal{F}\{G_0\} = \left[\frac{\mathbf{i}\omega}{c_0} - \frac{1}{|\mathbf{x}|}\right] \cdot \mathcal{F}\{G_0\} \cdot \mathrm{sgn}, \tag{6}$$

and is the solution of the *Helmholtz equation*

$$\nabla^2\mathcal{F}\{G_0\} + \frac{\omega^2}{c_0^2}\mathcal{F}\{G_0\} = -\frac{1}{\sqrt{2\pi}}\delta_{\mathbf{x}}. \tag{7}$$

The operator

$$\mathcal{A}_0 f := G_0 *_{\mathbf{x},t} f$$

is causal and maps a causal function f onto the solution of (3) and (4).

III.   ATTENUATION

In the chapter we investigated causality of attenuation models in a homogeneous, isotropic medium. In mathematical terms, it is common to describe attenuation by a *multiplicative law* in the frequency range:

Definition III.1. A real, bounded, linear, translation invariant operator $\mathcal{A}$ with causal domain of influence is called *attenuation operator* if there exists a complex function $\beta_* := \beta_*(r, \omega)$ such that the associated Green function $G := \mathcal{A}\delta_{\mathbf{x},t}$ satisfies

$$\boxed{\mathcal{F}\{G\}(\mathbf{x}, \omega) = \exp\left(-\beta_*(|\mathbf{x}|, \omega)\right) \cdot \mathcal{F}\{G_0\}(\mathbf{x}, \omega) \qquad \text{for all} \qquad \mathbf{x} \in \mathbb{R}^3, \omega \in \mathbb{R}.} \tag{8}$$

Here, $\mathcal{F}$ is the Fourier transform (see Appendix).

We rewrite (8) in the space–time domain by using

$$K := K(\mathbf{x}, t) := \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(-\beta_*)\}(|\mathbf{x}|, t). \tag{9}$$

Therefore

$$\boxed{G(\mathbf{x}, t) = [K *_t G_0](\mathbf{x}, t) = \frac{1}{4\pi|\mathbf{x}|}K\left(\mathbf{x}, t - \frac{|\mathbf{x}|}{c_0}\right).} \tag{10}$$

Since in the context of this paper the operator $\mathcal{A}$ is real, the associated Green function is real-valued, and consequently $\beta_*(r, \omega)$ has to be even with respect to ω (cf. Property 5 in Appendix).

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Remark 2. In physical terms attenuation is a result of frequency dependent energy dissipation and therefore the ratio of the attenuated and un-attenuated wave amplitude must be smaller or equal to 1. That is

$$\exp\left(-\Re(\beta_*)\right) = \left|\frac{\mathcal{F}\{G\}}{\mathcal{F}\{G_0\}}\right| \le 1 \,.$$

This implies that the *attenuation coefficient* β_* satisfies $\Re(\beta_*) \ge 0$.

In the literature a special form of the attenuation coefficient is assumed:

Definition III.2 (Standard Form)**.** The *standard form* of β_* considered in the literature is (see e.g. [16])

$$\beta_*(r,\omega) = \alpha_*(\omega)r \qquad \text{for} \qquad r > 0, \, \omega \in \mathbb{R}. \tag{11}$$

The function $\alpha := \Re(\alpha_*)$ is called *attenuation law.*

For the standard form β_* several properties for the attenuation operator are at hand. For instance the following results concerning travel time and causality.

Theorem III.1. *Let $\mathcal{A}$ be an attenuation operator with β_* of standard form. Then the travel time satisfies $T(|\mathbf{x}|) = |\mathbf{x}|/c$ for some constant $0 < c \le c_0$.*

Proof. The definition of the travel time T in Definition II.1 states that $T(|\mathbf{x}|)$ is the largest positive number such that for the Green function $G = \mathcal{A}\delta_{\mathbf{x},t}$

$$G(\mathbf{x},t) = 0 \qquad \text{for} \qquad t < T(|\mathbf{x}|) \,.$$

This condition is equivalent to the condition that the function

$$(\mathbf{x},t) \to G(\mathbf{x},t+T(|\mathbf{x}|)) \qquad \text{is causal.} \tag{12}$$

The operator $\mathcal{A}$ is causal and has causal domain of influence, which implies that $T(0) = 0$ and $(T'(r))^{-1} \le c_0$. Consequently

$$T(|\mathbf{x}|) = \int_0^{|\mathbf{x}|} T'(r)dr \ge \frac{|\mathbf{x}|}{c_0} \,. \tag{13}$$

$\tau(r) := T(r) - r/c_0$ denotes the largest number such that $K(\mathbf{x}, \cdot + \tau(|\mathbf{x}|)) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{\exp(-\beta_*)\}$ is causal. From (8) it follows that

$$K(\mathbf{x},t) = K(\mathbf{x}/2,t) *_t K(\mathbf{x}/2,t) \,.$$

This and the Theorem of Supports (cf. [4]) imply that $\tau(r) = 2\tau(r/2)$, and consequently τ is linear in r and after all T is linear as well. $\qquad\square$

In particular, from (10) and Theorem III.1 it follows that $\mathcal{A}$ has a causal domain of influence if and only if K is a causal function.

Remark 3. In the literature (for instance in [23]) causality is aimed to be enforced by the assumption that α_* satisfies the *Kramers-Kronig* relations

$$\Im(\alpha_*) = \mathcal{H}\left\{\Re(\alpha_*)\right\} = \mathcal{H}\left\{\alpha\right\} \qquad \text{and} \qquad \alpha := \Re(\alpha_*) = -\mathcal{H}\left\{\Im(\alpha_*)\right\}, \tag{14}$$

where $\mathcal{H}\{\cdot\}$ is the Hilbert Transform (see Appendix). (14) is equivalent to assuming that

$$\mathcal{F}^{-1}\left\{\alpha_*\right\} \qquad \text{is causal.} \tag{15}$$

(15) follows already from the causality of K: From the definition of K it follows that

$$|\nabla K| = \frac{1}{\sqrt{2\pi}} \left|\mathcal{F}^{-1}\left\{\alpha_* \cdot \exp\left(-\alpha_* |\mathbf{x}|\right)\right\}\right| \ .$$

Using some sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n \neq \mathbf{0}$ and $\mathbf{x}_n \to \mathbf{0}$ shows that

$$\lim |\nabla K|(\mathbf{x}_n, t) = \frac{1}{\sqrt{2\pi}} \left|\mathcal{F}^{-1}\left\{\alpha_*\right\}\right|(t) \ .$$

Due to the causality of K the left hand side is zero for $t < 0$, and thus $\mathcal{F}^{-1}\{\alpha_*\}$ is causal.

However, as we show in Example III.1 below, causality of $\mathcal{F}^{-1}\{\alpha_*\}$ does not imply causality of K. In other words, in general, from the causality of $\mathcal{F}^{-1}\{\alpha_*\}$ it cannot be deduced that $\mathcal{A}$ has a causal domain of influence. As a consequence several attenuation models considered in the literature lack causality.

Example III.1 (Frequency power laws [16, 19]). For the *frequency power law,*

$$\boxed{\alpha(\omega) = \alpha_0 |\omega|^\gamma,} \tag{16}$$

where $\gamma, \alpha_0 \geq 0$ and $\gamma \notin \mathbb{N}$, the complex extension by the Kramers-Kronig relation is given by (cf. [19, 23])

$$\boxed{\alpha_*(\omega) = \frac{\alpha_0}{\cos\left(\frac{\pi}{2}\gamma\right)}(-\mathbf{i}\omega)^\gamma \ .} \tag{17}$$

Therefore, as pointed out in Remark 3, $\mathcal{F}^{-1}\{\alpha_*\}$ is causal. However, according to [4, Theorem 7.4.3] K, as defined in (9), is causal if and only if $\gamma \in [0, 1)$. Consequently, for frequency power law with $\gamma \geq 1$ the according operator $\mathcal{A}$, defined in III.1 does **not** have a causal domain of influence.

IV. EQUATIONS FOR ATTENUATED PRESSURE WAVES

In this section we formulate a causal wave equation which takes into account attenuation and review the literature (cf. [10, 14, 17–19]).

Let $\mathcal{A}$ denote a translation invariant operator with causal domain of influence with travel time function T and c_0 as in Definition II.1. The Green function G satisfies (10) and (9) and therefore the according attenuation coefficient is given by

$$\beta_*(\mathbf{x}, \omega) = -\log\left\{2\sqrt{(2\pi)^3}\,|\mathbf{x}|\,\mathcal{F}\left\{G\left(\mathbf{x}, \cdot + \frac{|\mathbf{x}|}{c_0}\right)\right\}(\omega)\right\}. \tag{18}$$

In the following we rewrite the term $\nabla^2 \mathcal{F}\{G\}$ from which we derive the Helmholtz equation for $\mathcal{F}\{G\}$. Using (10), which states that $G = K *_t G_0 = \mathcal{A}\delta_{\mathbf{x},t}$, and the product rule yields

$$\nabla^2 \mathcal{F}\{G\} = \nabla^2 \mathcal{F}\{K\} \cdot \mathcal{F}\{G_0\} + 2\nabla\mathcal{F}\{K\} \cdot \nabla\mathcal{F}\{G_0\} + \mathcal{F}\{K\} \cdot \nabla^2\mathcal{F}\{G_0\}. \tag{19}$$

To evaluate this expression, we calculate $\nabla\mathcal{F}\{K\}$ and $\nabla^2\mathcal{F}\{K\}$. From (9) it follows that

$$\nabla\mathcal{F}\{K\} = -\beta'_* \cdot \mathcal{F}\{K\} \cdot \mathrm{sgn}, \tag{20}$$

where β'_* denotes the derivative of $\beta_*(r, \omega)$ with respect to r. This together with the formula (61) in the Appendix implies that

$$\begin{aligned}
\nabla^2 \mathcal{F}\{K\} &= -\nabla \cdot (\beta'_* \cdot \mathcal{F}\{K\} \cdot \mathrm{sgn}) \\
&= -(\nabla \cdot \mathrm{sgn}) \cdot \beta'_* \cdot \mathcal{F}\{K\} - (\mathrm{sgn} \cdot \nabla\beta'_*) \cdot \mathcal{F}\{K\} - (\mathrm{sgn} \cdot \nabla\mathcal{F}\{K\}) \cdot \beta'_* \\
&= \left[-\frac{2}{|\mathbf{x}|} \cdot \beta'_* - \beta''_* + (\beta'_*)^2\right] \cdot \mathcal{F}\{K\}.
\end{aligned} \tag{21}$$

Inserting (20) and (21) into (19) and using the identity $G = K *_t G_0$ (cf. (10)), shows that

$$\begin{aligned}
\nabla^2 \mathcal{F}\{G\} &= \left[-\frac{2}{|\mathbf{x}|} \cdot \beta'_* - \beta''_* + (\beta'_*)^2\right] \cdot \mathcal{F}\{G\} \\
&\quad - 2\beta'_* \cdot \mathcal{F}\{K\} \cdot (\mathrm{sgn} \cdot \nabla\mathcal{F}\{G_0\}) + \mathcal{F}\{K\} \cdot \nabla^2\mathcal{F}\{G_0\}.
\end{aligned} \tag{22}$$

Together with (6) and (7), the last identity simplifies to

$$\begin{aligned}
\nabla^2 \mathcal{F}\{G\} &= \left[-\frac{2}{|\mathbf{x}|} \cdot \beta'_* - \beta''_* + (\beta'_*)^2\right] \cdot \mathcal{F}\{G\} - 2\left[\frac{i\omega}{c_0} - \frac{1}{|\mathbf{x}|}\right] \cdot \beta'_* \cdot \mathcal{F}\{G\} \\
&\quad - \frac{\omega^2}{c_0^2} \cdot \mathcal{F}\{G\} - \frac{1}{\sqrt{2\pi}} \cdot \mathcal{F}\{K\} \cdot \delta_{\mathbf{x}}.
\end{aligned} \tag{23}$$

Since $\mathcal{F}\{K\}(\mathbf{x},\omega)\cdot\delta_{\mathbf{x}}=\mathcal{F}\{K\}(\mathbf{0},\omega)\cdot\delta_{\mathbf{x}}$, we obtain from (23) the Helmholtz equation

$$
\boxed{
\begin{aligned}
&\nabla^2\mathcal{F}\{G\}-\left[\beta_*'+\frac{(-\mathbf{i}\omega)}{c_0}\right]^2\cdot\mathcal{F}\{G\}\\
&=-\beta_*''\cdot\mathcal{F}\{G\}-\frac{1}{\sqrt{2\pi}}\mathcal{F}\{K\}(\mathbf{0},\omega)\cdot\delta_{\mathbf{x}}\\
&=-\beta_*''\cdot\mathcal{F}\{G\}-\frac{1}{2\pi}\exp\left(-\beta_*(\mathbf{0},\omega)\right)\cdot\delta_{\mathbf{x}}.
\end{aligned}
}
\tag{24}
$$

To reformulate (24) in space–time coordinates, we introduce two convolution operators:

$$
D_*f:=K_**_{\mathbf{x},t}f\qquad\text{and}\qquad D_*'f:=K_*'*_{\mathbf{x},t}f,
\tag{25}
$$

where the kernels K_* and K_*' are given by

$$
K_*:=K_*(\mathbf{x},t):=K_*(|\mathbf{x}|,t)\qquad\text{and}\qquad K_*(r,t):=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta_*'\}(r,t)
\tag{26}
$$

and

$$
K_*':=K_*'(\mathbf{x},t):=K_*'(|\mathbf{x}|,t)\qquad\text{and}\qquad K_*'(r,t)=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta_*''\}(r,t).
\tag{27}
$$

Using these operators and applying the inverse Fourier transform to (24) gives

$$
\boxed{\nabla^2 G-\left[D_*+\frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 G=-D_*'G-K(\mathbf{0},t)\delta_{\mathbf{x}}.}
\tag{28}
$$

For a general source term f, $p_{\text{att}}:=\mathcal{A}f=G*_{\mathbf{x},t}f$ solves the equation

$$
\boxed{\nabla^2 p_{\text{att}}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_{\text{att}}=-\mathcal{A}_s f\,,}
\tag{29}
$$

where $\mathcal{A}_s$ denotes the space–time convolution operator with kernel

$$
K_s:=K_s(\mathbf{x},t):=-\mathcal{B}G+D_*'G+K(\mathbf{0},t)\cdot\delta_{\mathbf{x}}
\tag{30}
$$

where

$$
\mathcal{B}:=D_*^2+\frac{2}{c_0}D_*\frac{\partial}{\partial t}\,.
\tag{31}
$$

Equation (29) is the pressure wave equation that obeys attenuation with attenuation coefficient (18).

Remark 4. In this remark we consider again the standard model, as in Definition III.2. For this case the wave equation (29) can be casted in a form that resembles the standard attenuation wave equation (cf. Example IV.1). Since K is causal, it follows that K_* is

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causal too (the argumentation is analogous to Remark 3) and therefore the operator D_* is well-defined for all causal functions. Moreover, since $K'_* = 0$, it follows that $D'_* \equiv 0$. Using that K_* depends only on t it follows that

$$(D_* G) *_{\mathbf{x},t} f = [K_* *_t G] *_{\mathbf{x},t} f = K_* *_t [G *_{\mathbf{x},t} f] = D_*(G *_{\mathbf{x},t} f).$$

Convolving each term in (28) with a function f, using the previous identity and that $D'_* \equiv 0$, it follows that

$$\nabla^2 p_{\text{att}} - \left[D_* + \frac{1}{c_0} \frac{\partial}{\partial t} \right]^2 p_{\text{att}} = -f \; . \tag{32}$$

In the following we review some wave equations obeying attenuation, which are frequently considered in the literature:

Example IV.1. • For $\gamma > 0$ and $\gamma \notin \mathbb{N}$, denote by D_t^γ be the *Riemann-Liouville fractional derivative* (see [6, 15])) with respect to time. It is defined in the Fourier domain by

$$\mathcal{F}\{D_t^\gamma f\} = (-\mathbf{i}\omega)^\gamma \mathcal{F}\{f\} \tag{33}$$

and satisfies

$$D_t^{2\gamma} f = D_t^\gamma D_t^\gamma f \qquad \text{and} \qquad \frac{\partial}{\partial t} D_t^\gamma f = D_t^\gamma \frac{\partial}{\partial t} f = D_t^{\gamma+1} f. \tag{34}$$

Now, we consider the attenuation coefficient

$$\beta_*(r,\omega) := \tilde{\alpha}_0 (-\mathbf{i}\omega)^\gamma r \qquad \text{with} \qquad \tilde{\alpha}_0 := \alpha_0 / \cos(\gamma\pi/2) \,, \tag{35}$$

which satisfies the attenuation law

$$\Re(\beta_*)(r,\omega) = \alpha(\omega) r \qquad \text{and} \qquad \alpha(\omega) = \alpha_0 |\omega|^\gamma$$

(cf. Example III.1 and [19, 23]). Let D_* denote the time-convolution operator with kernel K_* defined by (26) and (35). Then form (33) and $K_* = \mathcal{F}^{-1}\{\tilde{\alpha}_0(-\mathbf{i}\omega)^\gamma\}/\sqrt{2\pi}$ it follows that $D_* = \tilde{\alpha}_0 D_t^\gamma$. In [10, 17] (see also [18, 19]) the following equation for the pressure function p_{att} of attenuated waves is investigated:

$$\boxed{\nabla^2 p_{\text{att}} - \left[\tilde{\alpha}_0 D_t^\gamma + \frac{1}{c_0} \frac{\partial}{\partial t} \right]^2 p_{\text{att}} = -f \; ,} \tag{36}$$

which is equivalent to equation (32) with operator $D_* = \tilde{\alpha}_0 D_t^\gamma$. Let $\mathcal{A}$ denote the solution operator of (36), then from [4, Theorem 7.4.3] it follows that $\mathcal{A}$ has a causal domain of influence only for $\gamma \in [0, 1)$.

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- Let $\gamma > 0, \gamma \notin \mathbb{N}$. Neglecting in (36) the operator $\tilde{\alpha}_0^2 D_t^{2\gamma}$ (which one finds after expanding the decomposition operator) one finds Szabo's equation [18]

$$\nabla^2 p_{\text{att}} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p_{\text{att}} - \frac{2\tilde{\alpha}_0}{c_0} D_t^{\gamma+1}(p_{\text{att}}) = -f . \tag{37}$$

This equation is equivalent to equation (32) if we define the kernel of D_* by (26) with

$$\beta_*(r, \omega) := \tilde{\alpha}_0(-\mathbf{i}\omega)^\gamma r \quad \text{and} \quad \alpha_*(\omega) = \frac{\mathbf{i}\omega}{c_0} + \frac{1}{c_0}\sqrt{(-\mathbf{i}\omega)^2 + 2\tilde{\alpha}_0 c_0(-\mathbf{i}\omega)^{\gamma+1}} . \tag{38}$$

Again, if $\mathcal{A}$ denotes the solution operator of (37), then [4, Theorem 7.4.3] implies that $\mathcal{A}$ has a causal domain of influence only for $\gamma \in [0, 1)$.

In the literature, the standard attenuation models (36) and (37) are considered as *homogeneous Cauchy problems* with *inhomogeneous* initial conditions. In contrast, in our setting, we consider *inhomogeneous Cauchy problems* with *homogeneous* initial conditions. In the following section we show that the two concepts can be equivalent. However, in general, only the concept suggested here leads to a rigorous framework, in which we can define solution operators for attenuated wave equations.

For the readers convenience, we summarize some important notation and facts in the following table. Note the difference between K, K_* and K'_*, respectively, with respect to the involved exponential function.

Kernel	General	Standard Form	Convolution Operator
K (9)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\exp\left(-\alpha_*|\mathbf{x}|\right)\right\}$	
K_* (26)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\beta'_*\right\}$	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\alpha_*\right\}$	D_* (25)
K'_* (26)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\beta''_*\right\}$	0	D'_* (25)

V. THE HOMOGENEOUS CAUCHY PROBLEM WITH MEMORY

We consider the standard attenuation model $\beta_*(r, \omega) = \alpha_*(\omega)r$. Let $\mathcal{A}$ denote a translation invariant operator with causal domain of influence and and let the operator D_* be as defined as in (25).

In this section we investigate under which conditions the inhomogeneous wave equation (32) with homogeneous initial conditions (4) (where p is replaced by p_{att}) and the homogeneous equation

$$\nabla^2 q_{\text{att}} - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 q_{\text{att}} = 0 \tag{39}$$

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with the inhomogeneous initial condition

$$q_{\text{att}} = q_0 \qquad \text{for} \qquad t \leq 0 \qquad \text{and} \qquad \frac{\partial}{\partial t} q_{\text{att}}\bigg|_{t=0+} = \frac{\partial}{\partial t} q_0\bigg|_{t=0-} \tag{40}$$

are equivalent. That is, both equations have the same solution for $t > 0$.

Theorem V.1. *Assume that (39), (40) and (32), (4) have unique solutions, respectively. Then $q_{att} = p_{att}$ for $t > 0$ if and only if q_0 and p_{att} are related by the following conditions*

$$\varphi := \lim_{t\to 0-} q_0 = \lim_{t\to 0+} p_{att}, \qquad \psi := \lim_{t\to 0-} \frac{\partial}{\partial t} q_0 = \lim_{t\to 0+} \frac{\partial}{\partial t} p_{att} \tag{41}$$

and

$$\boxed{H \cdot \mathcal{B}q_0 = -f + \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right),} \tag{42}$$

with $\mathcal{B}$ is as in (31) and H is the Heaviside function.

Proof. $\Rightarrow$: Assume that $q_{\text{att}} = p_{\text{att}}$ for $t > 0$. Then, using that $p_{\text{att}} = 0$ for $t < 0$, implies that

$$p_{\text{att}} = H \cdot q_{\text{att}} \qquad \text{and} \qquad q_{\text{att}} = p_{\text{att}} + q_0. \tag{43}$$

In particular, property (41) holds. Moreover, (43) implies

$$\nabla^2 p_{\text{att}} = H \cdot \nabla^2 q_{\text{att}} \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2} p_{\text{att}} = H \cdot \frac{\partial^2}{\partial t^2} q_{\text{att}} + \psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t. \tag{44}$$

Since

$$\mathcal{B} + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} = \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 \tag{45}$$

it follows from (44), (45) and (43) that

$$\nabla^2 p_{\text{att}} - \mathcal{B}p_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} p_{\text{att}} + f = H \cdot \left[\nabla^2 q_{\text{att}} - \mathcal{B}q_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} q_{\text{att}}\right]$$

$$- \mathcal{B}p_{\text{att}} + H \cdot \mathcal{B}q_{\text{att}} - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f. \tag{46}$$

Using the definitions of q_{att} and p_{att}, (46) simplifies to

$$-\mathcal{B}p_{\text{att}} + H \cdot \mathcal{B}q_{\text{att}} - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f = 0.$$

Since $\mathcal{B}$ is a causal operator and p_{att} is a causal function, we have $\mathcal{B}p_{\text{att}} = H \cdot \mathcal{B}p_{\text{att}}$. This together with (43) implies that $-\mathcal{B}p_{\text{att}} + H \cdot \mathcal{B}q_{\text{att}} = H \cdot \mathcal{B}q_0$. Hence

$$H \cdot \mathcal{B}q_0 - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f = 0$$

and thus (42) holds. This proves the first direction of the theorem.

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⇐: To prove the opposite direction let

$$\tilde{p}_{\text{att}} := H \cdot q_{\text{att}} \qquad \text{such that} \qquad q_{\text{att}} = \tilde{p}_{\text{att}} + q_0 \,.$$

We prove that $p_{\text{att}} = \tilde{p}_{\text{att}}$ holds for $t > 0$. Similarly as in part a) of the proof it follows that

$$\nabla^2 \tilde{p}_{\text{att}} - \mathcal{B}\tilde{p}_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,\tilde{p}_{\text{att}} = H \cdot \left[\nabla^2 q_{\text{att}} - \mathcal{B}q_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,q_{\text{att}}\right]$$
$$+ H \cdot \mathcal{B}q_0 - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right)$$

holds. Since q_{att} solve problem (39), (40) and condition (42) is satisfied, the last identity simplifies to

$$\nabla^2 \tilde{p}_{\text{att}} - \mathcal{B}\tilde{p}_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,\tilde{p}_{\text{att}} = -\tilde{f}\,.$$

Hence we have shown that $\tilde{p}_{\text{att}}$ solves problem (32), (4) and since this problem has the unique solution p_{att}, it follows $\tilde{p}_{\text{att}} = p_{\text{att}}$. In summary we have shown that

$$p_{\text{att}} = \tilde{p}_{\text{att}} = q_{\text{att}} \qquad \text{for} \qquad t > 0\,,$$

which proves the assertion.

$\square$

Remark 5. In the absence of attenuation the operator $\mathcal{B}$ is the zero operator and condition (42) reduces to

$$f = \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right)\,.$$

In this case the solutions of

$$\boxed{\begin{array}{ll} \nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p = -f\,, & \nabla^2 q - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}q = 0\,, \\[2mm] p|_{t<0} = 0\,,\ \frac{\partial}{\partial t}p\big|_{t<0} = 0\,, & q|_{t=0} = \varphi\,,\ \frac{\partial}{\partial t}q\big|_{t=0} = \psi\,, \end{array}}$$

are identical for $t > 0$.

VI. THE THERMO-VISCOUS WAVE EQUATION

In this section we review that the *thermo-viscous* wave equation (see e.g. [7]) is not causal, which is a consequence of the fact that the attenuation coefficient α_* there does not satisfy the Kramers-Kronig relation (see Theorem VI.1). The formalism introduced here

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will enable us to derive a causal variant of the thermo-viscous equation which satisfies the same attenuation law.

The thermo-viscous wave equation models propagation of pressure waves in viscous media and reads as follows

$$\left(I + \tau_0 \frac{\partial}{\partial t}\right) \nabla^2 p_{\text{att}} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p_{\text{att}} = -F \ . \tag{47}$$

Here τ_0 and c_0 denotes the *relaxation time* and the *thermodynamic speed*, respectively and F models sources.

In the following we transform the thermo-viscous wave equation into the form (28), which enables us to deduce that the thermo-viscous equation is not causal. For these purpos we consider the attenuation coefficient

$$\beta_*(r, \omega) = \alpha_*(\omega) r \qquad \text{with} \qquad \alpha_*(\omega) = \frac{\mathbf{i}\omega}{c_0} - \frac{\mathbf{i}\omega}{c_0 \sqrt{1 - \tau_0 \mathbf{i}\omega}} \ . \tag{48}$$

and the time convolution operators $T^{1/2}$ and $L^{1/2}$ with kernels

$$K_{T^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{(1 - \mathbf{i}\tau_0\omega)^{-1/2}\right\} \qquad \text{and} \qquad K_{L^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{(1 - \tau_0 \mathbf{i}\omega)^{1/2}\right\} \ ,$$

respectively. Since K_* satisfies (26) it can be rewritten in the following form

$$K_* = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{\frac{\mathbf{i}\omega}{c_0} - \frac{\mathbf{i}\omega}{c_0 \sqrt{1 - \mathbf{i}\omega\tau_0}}\right\} = -\frac{1}{c_0} \frac{\partial}{\partial t} \delta_t + \frac{1}{c_0} \frac{\partial}{\partial t} K_{T^{1/2}} \ .$$

Therefore the according convolution operator D_* is given by

$$D_* = -\frac{1}{c_0} \frac{\partial}{\partial t} + \frac{1}{c_0} T^{1/2} \frac{\partial}{\partial t} \ . \tag{49}$$

In the following we summarize some properties of the operators $T^{1/2}$, $L^{1/2}$ and D_*, and the associated kernels.

Lemma VI.1. *The kernel functions $K_{T^{1/2}}$, $K_{L^{1/2}}$ and the operators $L^{1/2}$, $T^{1/2}$, D_* respectively, satisfy:*

1. *For $\tau_0 = 0$, $T^{1/2} = L^{1/2} = I$ and $K_{T^{1/2}} = K_{L^{1/2}} = \delta_t$.*

2.
$$K_{T^{1/2}}(t) = \frac{\sqrt{2\pi} H(t) \exp\left(-t/\tau_0\right)}{\Gamma(1/2) \tau_0^{1/2} t^{1/2}} \ . \tag{50}$$

3. *$L^{1/2}$ is the inverse of $T^{1/2}$,*

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4. Let $L := \left(L^{1/2}\right)^2$ and $T := \left(T^{1/2}\right)^2$, then

$$L = I + \tau_0 \frac{\partial}{\partial t} \qquad and \qquad T = L^{-1} . \tag{51}$$

5. $D'_* \equiv 0$ and for $\tau_0 = 0$ also $D_* \equiv 0$.

6.

$$\left[D_* + \frac{1}{c_0} \frac{\partial}{\partial t} \right]^2 = \frac{1}{c_0^2} T \frac{\partial^2}{\partial t^2}$$

Proof. 1. The first item is a trivial consequence of properties of the Fourier transform $\mathcal{F}^{-1}\{\cdot\}$.

2. With the substitution $s = -\mathrm{i}\omega\tau_0$ we derive the relation with the inverse Laplace transformation $\mathcal{L}^{-1}\{\cdot\}$ (for a definition and some basi properties see the appendix)

$$\begin{aligned}
\mathcal{F}^{-1}\left\{(1 - \mathrm{i}\omega\tau_0)^{-1/2}\right\}(t) &= \frac{1}{\mathrm{i}\tau_0\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp\left(st/\tau_0\right) \cdot (1+s)^{-1/2} ds \\
&= \frac{\sqrt{2\pi}}{\tau_0} \mathcal{L}^{-1}\left\{(1+s)^{-1/2}\right\}(t/\tau_0).
\end{aligned} \tag{52}$$

Using the properties (63) and (64) of the inverse Laplace transformation the assertion follows.

3. From

$$K_{T^{1/2}} *_t K_{L^{1/2}} = K_{L^{1/2}} *_t K_{T^{1/2}} = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\{1\} = \delta_t ,$$

it follows that for each function f

$$\begin{aligned}
T^{1/2} L^{1/2} f &= K_{T^{1/2}} *_t K_{L^{1/2}} *_t f = \delta_t *_t f = f \\
L^{1/2} T^{1/2} f &= K_{T^{1/2}} *_t K_{L^{1/2}} *_t f = \delta_t *_t f = f .
\end{aligned} \tag{53}$$

4. Since

$$K_{L^{1/2}} *_t K_{L^{1/2}} = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\{1 - \mathrm{i}\omega\tau_0\} = \delta_t - \tau_0 \frac{\partial}{\partial t} \delta_t,$$

it follows that

$$L f = K_{L^{1/2}} *_t K_{L^{1/2}} *_t f = \left(\delta_t - \tau_0 \frac{\partial}{\partial t} \delta_t \right) *_t f = \left(I + \tau_0 \frac{\partial}{\partial t} \right) f .$$

The assertion $T = L^{-1}$ is then a consequence of the previous item.

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5. Since K_* does not depend on $|\mathbf{x}|$ and K'_* is the kernel of D'_*, it follows that $K'_* = 0$, i.e. $D'_* \equiv 0$. The second statement is a direct consequence of Item 1 which states that $T^{1/2} = I$ for $\tau_0 = 0$.

6. Follows from (49).

$\square$

The thermo-viscous wave equation (47) can be put in formal relation to the wave equation (28) by identifying an appropriate operator D_* as in (49):

Utilizing Item 6 of Lemma VI.1 in equation (32) and taking into account that $D'_* \equiv 0$ (cf. Item 5 of Lemma VI.1) shows that the solution of the thermo-viscous wave equation (47) with $F := L f$ satisfies

$$\nabla^2 p_{\text{att}} - \left[D_* + \frac{1}{c_0} \frac{\partial}{\partial t} \right]^2 p_{\text{att}} = \nabla^2 p_{\text{att}} - \frac{1}{c_0^2} T \frac{\partial^2}{\partial t^2} p_{\text{att}} = -f \ .$$

Conversely, the solution of equation (32) with D_* defined as in (49) satisfies the thermo-viscous wave equation (47) with $F = L f$.

Theorem VI.1. *Let α_* be defined as in (48). Then $\Re(\alpha_*)$ and $\Im(\alpha_*)$ do not satisfy the Kramers-Kronig relation and consequently the solution operator $\mathcal{A}$ of the thermo-viscous wave equation does not have a causal domain of influence.*

Proof. Let $\alpha_*^c(\omega) := -\frac{i\omega}{c_0 \sqrt{1 - i\tau_0 \omega}}$. From [4, Theorem 7.4.3] it follows that the convolution operator with $K^c := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \{ \exp(-\alpha_*^c |\mathbf{x}|) \}$ is causal. As a consequence of Remark 3 $K_*^c := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \{ (-\alpha_*^c |\mathbf{x}|) \}$ is causal, and therefore α_*^c satisfies the Kramers-Kronig relation.

Moreover, the real part $\alpha := \Re(\alpha_*^c)$ is identical to the real part of α_*, as defined in (48), and therefore both models satisfy the same attenuation law. However, the according imaginary parts are different and therefore α_* from (48) does not satisfy the Kramers-Kronig relation and as a conseqeunce the according solution operator of the thermo–viscous wave equation is not causal. $\square$

Remark 6. From (48) it follows that the attenuation law $\alpha = \Re(\alpha_*)$ approximates for small frequencies the frequency power law with $\gamma = 2$.

VII. A CAUSAL THERMO-VISCOUS WAVE EQUATION

Below we discus a causal variant of the thermo-viscous wave equation.

Let $\alpha_1 \geq 0$. It follows from that proof of Theorem VI.1 that the attenuation operator with attenuation coefficient of standard form $\beta_*(r, \omega) = \alpha_*^c(\omega)r$ and

$$\alpha_*^c(\omega) = -\frac{\alpha_1 \mathbf{i}\omega}{c_0\sqrt{1 - \tau_0 \mathbf{i}\omega}} \tag{54}$$

has a causal domain of influence. The operator D_* and its kernel K_* read as follows

$$D_* := \frac{\alpha_1}{c_0} T^{1/2} \frac{\partial}{\partial t} \qquad \text{and} \qquad K_* = \frac{\alpha_1}{c_0}\frac{\partial}{\partial t} K_{T^{1/2}}. \tag{55}$$

Note that $D_*' \equiv 0$, since K_* does not depend on $|\mathbf{x}|$.

For $\alpha_1 = 0$, $D_* \equiv 0$ and thus equation (28) with operator D_* defined by (55) is the standard wave equation (without attenuation). Since

$$\left(D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right)^2 = \frac{1}{c_0^2}\left[I + \alpha_1 T^{1/2}\right]^2 \frac{\partial^2}{\partial t^2} \qquad \text{and} \qquad L = T^{-1},$$

(28) can be rewritten as

$$\left(I + \tau_0\frac{\partial}{\partial t}\right)\nabla^2 p_{\mathrm{att}} - \left[\alpha_1 I + L^{1/2}\right]^2 \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_{\mathrm{att}} = -\left(I + \tau_0\frac{\partial}{\partial t}\right)\delta_{\mathbf{x},t}. \tag{56}$$

The solution operator of this equation has a causal domain of influence, and satisfies the same attenuation law as the thermo-viscous wave equation, which is a small frequency approximation of the frequency power law with $\gamma = 2$.

Remark 7. In ultrasound imaging soft tissue is often modeled as a viscous fluid and therefore (56) is a potential model, on which thermoacoustic tomography can be based on. Moreover, the attenuation of tissue is frequently modeled as a power frequency law with $\gamma \in (1, 2)$.

VIII. CAUSAL WAVE EQUATIONS SATISFYING FREQUENCY POWER LAWS FOR SMALL FREQUENCIES WITH $\gamma \in (1, 2]$

In Example III.1 we have shown that the frequency power law does not yield to a causal wave equation when $\gamma \geq 1$. In this section we derive causal wave equations for attenuation laws which approximate frequency power laws *for small frequencies* with exponent $\gamma \in (1, 2]$, where for $\gamma = 2$ we get the causal variant of the thermo-viscous wave equation (56).

Here we follow the notation of the previous section and introduce the following families of operators: For constants $\gamma \in (1,2]$, $\tau_0 \geq 0$ and $\alpha_1 \geq 0$ let $T_\gamma^{1/2}$ and $L_\gamma^{1/2}$ denote time convolution operators with kernels:

$$K_{T_\gamma^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}\left\{ \left(1 + (-\mathbf{i}\omega\tau_0)^{\gamma-1}\right)^{-1/2} \right\},$$

$$K_{L_\gamma^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}\left\{ \left(1 + (-\mathbf{i}\omega\tau_0)^{\gamma-1}\right)^{1/2} \right\}.$$

We set $T_\gamma := \left(T_\gamma^{1/2}\right)^2$ and $L_\gamma := \left(L_\gamma^{1/2}\right)^2$. We emphasize that $T_2^{1/2} = T^{1/2}$, where $T^{1/2}$ is the operator in the thermo-viscous case. The operators $T_\gamma^{1/2}$ and $L_\gamma^{1/2}$ satisfy similar properties as the operators $T^{1/2}$ and $L^{1/2}$ in the thermo-viscous case:

The following lemma is proven analogously as Lemma VI.1.

Lemma VIII.1.   • *For $\tau = 0$ we have*

$$T_\gamma^{1/2} = L_\gamma^{1/2} = I \qquad and \qquad K_{T_\gamma^{1/2}} = K_{L_\gamma^{1/2}} = \delta_t \ .$$

- *$L_\gamma^{1/2}$ is the inverse of $T_\gamma^{1/2}$.*

- *Let $D_t^{\gamma-1}$ be the fractional derivative of order $\gamma - 1$, as defined as in (33), then*

$$L_\gamma = I + \tau_0^{\gamma-1} D_t^{\gamma-1} \ .$$

In analogy to Section VI we consider now the standard attenuation coefficient with

$$\alpha_*(\omega) = -\frac{\alpha_1 \mathbf{i}\omega}{c_0 \sqrt{1 + (-\mathbf{i}\tau_0\omega)^{\gamma-1}}} \ . \tag{57}$$

Here α_1, τ_0 and c_0 are positive constants that are medium specific. The operator D_* and its kernel K_* are given by

$$\boxed{D_* := \frac{\alpha_1}{c_0} T_\gamma^{1/2} \frac{\partial}{\partial t} \qquad and \qquad K_* = \frac{\alpha_1}{c_0} \frac{\partial}{\partial t} K_{T_\gamma^{1/2}} \ .} \tag{58}$$

Moroever, the kernel K, defined by (9), reads as follows

$$K(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left\{ \exp\left(\frac{\alpha_1 \mathbf{i}\omega |\mathbf{x}|}{c_0 \sqrt{1 + (-\mathbf{i}\tau_0\omega)^{\gamma-1}}} \right) \right\}(t) \tag{59}$$

For ω small we have

$$\alpha(\omega) \sim \alpha_0 |\tau_0\omega|^\gamma \qquad with \qquad \alpha_0 = \frac{\sin(\frac{\pi}{2}(\gamma-1))}{2\tau_0 c_0} \ .$$

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The wave equation (28) with D_* as in (58) reads as follows

$$\left(I + \tau_0^{\gamma-1} D_t^{\gamma-1}\right) \nabla^2 p_{\text{att}} - \left[\alpha_1 I + L_\gamma^{1/2}\right]^2 \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p_{\text{att}} = -F. \tag{60}$$

In particular, for $\gamma = 2$ we recover the causal variant of the thermo-viscous wave equation (56).

Theorem VIII.1. *The solution operator of equation (60) has a causal domain of influence.*

Proof. From [4, Theorem 7.4.3] it follows that K from (59) is causal and thus the solution operator of (60) has a causal domain of influence. □

IX. EXAMPLES

In this section we present some calculations, highlighting the effects of non-causality. In all examples β_* is of standard form (11) and the solution operator $\mathcal{A}$ determined by β_* has the Green function

$$G(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\exp\left(-\beta_*\right) \cdot \exp\left(-\mathbf{i} \cdot \left(t - \frac{|\mathbf{x}|}{c_0}\right)\right)}{4\pi |\mathbf{x}|} d\omega.$$

We recall that the operator $\mathcal{A}$ has a causal domain of influence if and only if $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$ is a causal function. In other words, non-causality can be observed if

$$\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}(t) \neq 0 \qquad \text{for some} \qquad t < 0.$$

All numerical simulations were performed in MATLAB with the fft-subroutine.

Frequency power law: Let $\alpha = \alpha_0 |\omega|^\gamma$ with some $\gamma > 0$. The extension α_* by the Kramers-Kroenig relation is given by (17). Fig. 1 shows simulations of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$, which illustrates that causality only holds for $\gamma \in [0, 1)$.

Szabos's model: Here $\alpha_*(\omega)$ is as in (38). In Fig. 2 we show simulations of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$. The numerical result confirm the mathematical considerations that causality only holds for $\gamma \in [0, 1)$.

Thermo-viscous wave equation: There α_* is as in (48). The left pictures in Fig. 3 shows a simulation of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$ for the thermo-viscous wave equation (47). Note that
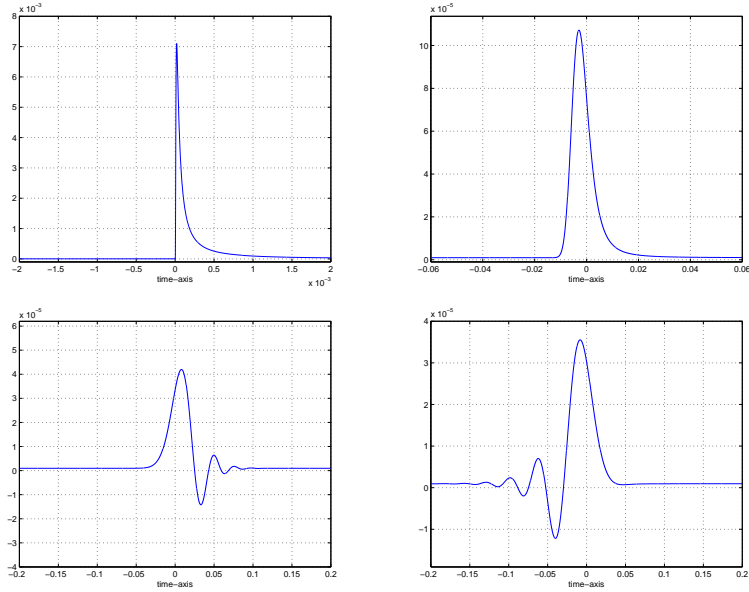
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FIG. 1: Simulation of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*(|\mathbf{x}|,\omega)\right)\right\}$ for the frequency power law with $(\gamma,\alpha_0) \in \{(0.5, 0.1581), (1.5, 0.0316), (2.7, 0.0071), (3.3, 0.0027)\}$, $c_0 = 1$ and $|\mathbf{x}| = \frac{1}{4}$. In the first example $\gamma < 1$ and thus the function is causal. For all other cases it is non causal as predicted by the theory.

according to (48) and (54) the attenuation laws of the thermo-viscous wave equation and the causal variant (56) differ just by a multiplicative constant α_1. A simulation of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}$ with $\alpha_1 = 1$ for the causal variant (56) of the thermo-viscous wave equation is shown in the right pictures of Fig. 3

X. APPENDIX: NOMENCLATURE AND ELEMENTARY FACTS

Real and Complex Numbers: $\mathbb{C}$ denotes the space of complex numbers, $\mathbb{R}$ the space of reals. For a complex number $c = a + \mathbf{i}b$ $a = \Re(c)$, $b = \Im(c)$ denotes the real and imaginary parts, respectively.

Differential Operators: ∇ denotes the gradient. $\nabla\cdot$ denotes divergence, and ∇^2 denotes the Laplacian.

Product: When we write $\cdot$ between two functions, then it means a pointwise product, it can be a scaler product or if the functions are vector valued an inner product. The
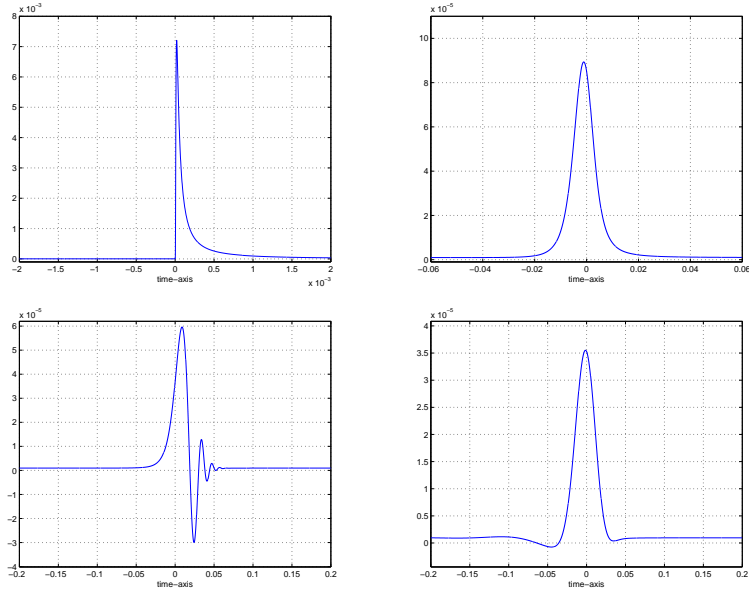
FIG. 2: Simulation of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*(|\mathbf{x}|,\omega)\right)\right\}$ for Szabo's frequency law with $(\gamma,\alpha_0) \in \{(0.5, 0.1581), (1.5, 0.0316), (2.7, 0.0071), (3.3, 0.0027)\}$, $c_0 = 1$ and $|\mathbf{x}| = \frac{1}{4}$.
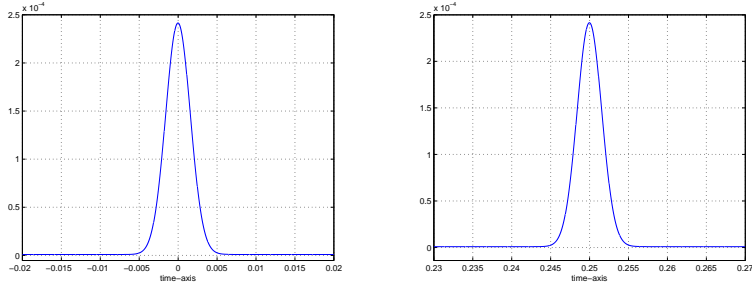

FIG. 3: *Left:* $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*(|\mathbf{x}|,\omega)\right)\right\}$ defined by the thermo-viscous wave equation (47) with $\tau_0 = 10^{-5}$, $c_0 = 1$ and fixed $|\mathbf{x}| = \frac{1}{4}$. *Right:* Causal variant (56) of the thermo-viscous wave equation with $\alpha_1 = 1$, $\tau_0 = 10^{-5}$, $c_0 = 1$ and fixed $|\mathbf{x}| = \frac{1}{4}$.

product between a function and a number is not explicitly stated.

Decomposition: The decomposition of operators $\mathcal{A}$ and $\mathcal{B}$ is written as $\mathcal{AB}$.

Special functions: The *signum* function is defined by

$$\mathrm{sgn} := \mathrm{sgn}(\mathbf{x}) := \frac{\mathbf{x}}{|\mathbf{x}|} \ .$$

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In $\mathbb{R}^3$ it satisfies

$$\nabla \cdot \mathrm{sgn} = \frac{2}{|\mathbf{x}|} \, . \tag{61}$$

The *Heaviside* function

$$H := H(t) := \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

satisfies

$$H := \frac{1}{2}(1 + \mathrm{sgn}) \, .$$

The Delta-distribution is the derivative of the Heaviside function at 0 and is denoted by $\delta_t := \delta_t(t)$. In our terminology δ_t denotes a *one*-dimensional distribution. The three dimensional Delta-distribution $\delta_{\mathbf{x}}$ is the product of three one-dimensional distribution δ_{x_i}, $i = 1, 2, 3$. Moreover,

$$\delta_{\mathbf{x},t} := \delta_{\mathbf{x},t}(\mathbf{x}, t) = \delta_{\mathbf{x}} \cdot \delta_t, \tag{62}$$

is a four dimensional distribution in space and time.

Properties related to functions: $\mathrm{supp}(g)$ denote the *support* of the function g, that is the closure of the set of points, where g does not vanish.

Derivative with respect to radial components: We use the notation

$$r := r(\mathbf{x}) = |\mathbf{x}| \, ,$$

and denote the derivative of a function f, which is only dependent on the radial component $|\mathbf{x}|$, with respect to r (i.e., with respect to $|\mathbf{x}|$) by $\cdot'$.

Let $\beta = \beta(r)$, then it is also identified with the function $\beta = \beta(|\mathbf{x}|)$ and therefore

$$\nabla \beta = \frac{\mathbf{x}}{|\mathbf{x}|} \beta' \, .$$

Convolutions: Three different types of convolutions are considered: $*_t$ and $*_\omega$ denote *convolutions* with respect to time and frequency, respectively. Let f, $\hat{f}$, g and $\hat{g}$ be functions defined on the real line with complex values. Then

$$f *_t g := \int_{\mathbb{R}} f(t - t') g(t') dt', \qquad \hat{f} *_\omega \hat{g} := \int_{\mathbb{R}} \hat{f}(\omega - \omega') \hat{g}(\omega') d\omega'.$$

$*_{\mathbf{x},t}$ denotes space–time convolution and is defined as follows: Let f, g be functions defined on the Euclidean space $\mathbb{R}^3$ with complex values, then

$$f *_{\mathbf{x},t} g := \int_{\mathbb{R}^3} \int_{\mathbb{R}} f(\mathbf{x} - \mathbf{x}', t - t')g(\mathbf{x}', t')d\mathbf{x}dt .$$

Fourier transform: For more background we refer to [4, 11, 12, 21]. All along this paper $\mathcal{F}\{\cdot\}$ is the Fourier Transformation with respect to t, and the inverse Fourier transform $\mathcal{F}^{-1}\{\cdot\}$ is with respect to ω. In this paper we use the following definition of the *Fourier transform* $\mathcal{F}\{\cdot\}$ and its inverse $\mathcal{F}^{-1}\{\cdot\}$

$$\mathcal{F}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(\mathbf{i}\omega t)f(t)dt, \quad \mathcal{F}^{-1}\{\hat{f}\}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\mathbf{i}\omega t)\hat{f}(\omega)d\omega .$$

The Fourier transform and its inverse have the following properties:

1.
$$\mathcal{F}\left\{\frac{\partial}{\partial t}f\right\}(\omega) = (-\mathbf{i}\omega)\mathcal{F}\{f\}(\omega) .$$

2.
$$\mathcal{F}\{f \cdot g\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f\} *_\omega \mathcal{F}\{g\} \text{ and}$$
$$\mathcal{F}\{f\} \cdot \mathcal{F}\{g\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f *_t g\} ,$$
$$\mathcal{F}^{-1}\left\{\hat{f} \cdot \hat{g}\right\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\hat{f}\right\} *_t \mathcal{F}^{-1}\{\hat{g}\} \text{ and}$$
$$\mathcal{F}^{-1}\left\{\hat{f}\right\} \cdot \mathcal{F}^{-1}\{\hat{g}\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\hat{f} *_\omega \hat{g}\right\} .$$

3. For $a \in \mathbb{R}$
$$\mathcal{F}\{f(t - a)\}(\omega) = \exp(-ia\omega) \cdot \mathcal{F}\{f(t)\}(\omega)$$

4. The Delta-distribution at $a \in \mathbb{R}$ satisfies
$$\delta_t(t - a) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(\mathbf{i}a\omega)\}(t) .$$

5. Let f be real and even, odd respectively, then $\mathcal{F}\{f\}$ is real and even, imaginary and odd, respectively.

The Hilbert transform is defined by

$$\mathcal{H}\{f\}(t) = \frac{1}{\pi}\fint_{\mathbb{R}} \frac{f(s)}{t - s}ds ,$$

where $\fint_{\mathbb{R}} f(s)ds$ denotes the Cauchy principal value of $\int_{\mathbb{R}} f(s)ds$. The Hilbert transform satisfies

- $\mathcal{H}\{\mathcal{F}\{f\}\}(\omega) = -i\mathcal{F}\{\mathrm{sgn}f\}(\omega)$,

- $\mathcal{H}\{\mathcal{H}\{f\}\} = -f$.

From the first of these properties the Kramers-Kronig relation can be formally derived as follows. Since $f(t)$ is a causal function if and only if $f = H \cdot f$ and $H = (1 + \mathrm{sgn})/2$, it follows that $\mathcal{F}\{f\} = [\mathcal{F}\{f\} + \mathbf{i}\mathcal{H}\{\mathcal{F}\{f\}\}]/2$, which is equivalent to $\mathcal{F}\{f\} = i\mathcal{H}\{\mathcal{F}\{f\}\}$, i.e.

$$\Re(\mathcal{F}\{f\}) = -\Im(\mathcal{H}\{\mathcal{F}\{f\}\}) \qquad \text{and} \qquad \Im(\mathcal{F}\{f\}) = \Re(\mathcal{H}\{\mathcal{F}\{f\}\}).$$

The inverse Laplace transform of f is defined by

$$\mathcal{L}^{-1}\{f\}(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \exp(st)f(s)ds, & \text{for} \quad t > 0, \end{cases}$$

where γ is appropriately chosen.

The inverse Laplace transform satisfies (see e.g. [3]) that

$$\mathcal{L}^{-1}\{h(s-a)\}(t) = \exp(at)\mathcal{L}^{-1}\{h(s)\}(t) \text{ for all } a, t \in \mathbb{R} \tag{63}$$

and

$$\mathcal{L}^{-1}\{s^{-r}\}(t) = \frac{H(t)t^{r-1}}{\Gamma(r)} \qquad (r > 0). \tag{64}$$

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[1] P. Burgholzer, H. Grün, M. Haltmeier, R. Nuster, and G. Paltauf. Compensation of acoustic attenuation for high-resolution photoacoustic imaging with line detectors. volume 6437, page 643724. SPIE, 2007.

[2] R. Courant, K. Friedrichs, and H. Lewy. On the partial difference equations of mathematical physics. *IBM J. Res. Develop.*, 11:215–234, 1967.

[3] H. Heuser. *Gewöhnliche Differentialgleichungen.* Teubner, Stuttgart, second edition, 1991.

[4] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer Verlag, New York, 2nd edition, 2003.

[5] Y. Hristova, P. Kuchment, and L. Nguyen. Reconstruction and time reversal in thermoacoustic tomography in acoustically homogeneous and inhomogeneous media. *Inverse Problems*, 24(5):055006 (25pp), 2008.

[6] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies.* Elsevier Science B.V., Amsterdam, 2006.

[7] L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders. *Fundamentals of Acoustics.* Wiley, New York, 2000.

[8] G. Ku, X. Wang, G. Stoica, and L. V. Wang. Multiple-bandwidth photoacoustic tomography. *Phys. Med. Biol.*, 49:1329–1338, 2004.

[9] P. Kuchment and L. A. Kunyansky. Mathematics of thermoacoustic and photoacoustic tomography. *European J. Appl. Math.*, 19:191–224, 2008.

[10] P. J. La Riviére, J. Zhang, and M. A. Anastasio. Image reconstruction in optoacoustic tomography for dispersive acoustic media. *Opt. Letters*, 31(6):781–783, 2006.

[11] M. J. Lighthill. *Introduction to Fourier Analysis and Generalized Functions. Student's Edition.* Cambridge University Press, London, 1964.

[12] A. Papoulis. *The Fourier Integral and its Applications.* McGraw-Hill, New York, 1962.

[13] S. K. Patch and O. Scherzer. Special section on photo- and thermoacoustic imaging. *Inverse Probl.*, 23:S1–S122, 2007.

[14] S.K. Patch and A. Greenleaf. Equations governing waves with attenuation according to power law. Technical report, Department of Physics, University of Wisconsin-Milwaukee, 2006.

[15] I. Podlubny. *Fractional differential equations*, volume 198 of *Mathematics in Science and Engineering.* Academic Press Inc., San Diego, CA, 1999.

[16] T.D.E. Rossing, editor. *Springer Handbook of Acoustics.* Springer Verlag, New York, 2007.

[17] N. V. Sushilov and R. S. C. Cobbold. Frequency-domain wave equation and its time-domain solution in attenuating media. *Journal of the Acoustical Society of America*, 115:1431–1436,

2005.

[18] T.L. Szabo. Time domain wave equations for lossy media obeying a frequency power law. *J. Acoust. Soc. Amer.*, 96:491–500, 1994.

[19] T.L. Szabo. Causal theories and data for acoustic attenuation obeying a frequency power law. *J. Acoust. Soc. Amer.*, 97:14–24, 1995.

[20] A. C. Tam. Applications of photoacoustic sensing techniques. *Rev. Modern Phys.*, 58(2):381–431, 1986.

[21] E. C. Titchmarch. *Theory of Fourier Integrals*. Clarendon Press, Oxford, 1948.

[22] L. V. Wang. Prospects of photoacoustic tomography. *Med. Phys.*, 35(12):5758–5767, 2008.

[23] K. R. Waters, M. S. Hughes, G. H. Brandenburger, and J. G. Miller. On a time-domain representation of the Kramers-Krönig dispersion relation. *J. Acoust. Soc. Amer.*, 108(5):2114–2119, 2000.

[24] S. Webb, editor. *The Physics of Medical Imaging*. Institute of Physics Publishing, Bristol, Philadelphia, 2000. reprint of the 1988 edition.

[25] M. Xu and L. V. Wang. Photoacoustic imaging in biomedicine. *Rev. Sci. Instruments*, 77(4):1–22, 2006. Article ID 041101.