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Abstract

In this paper we show that the standard causality condition for attenuated waves, i.e. the Kramers-Kronig relation that relates the attenuation law and the phase speed of the wave, is necessary but not sufficient for causality of a wave. By causality of a wave we understand the property that its wave front speed is bounded. Although this condition is not new, the consequences for wave attenuation have not been analysed sufficiently well. We derive the wave equation (for a homogeneous and isotropic medium) obeying attenuation and causality and with a generalization of the Paley-Wiener-Schwartz Theorem (cf. Theorem 7.4.3. in [5]), we perform a causality analysis of waves obeying the frequency power attenuation law. Afterwards the causality behaviour of Szabo's wave equation (cf. [11]) and the thermo-viscous wave equation are investigated. Finally, we present a generalization of the thermo-viscous wave equation that obeys causality and the frequency power law (for powers in $(1, 2]$ and) for small frequencies, which we propose for Thermoacoustic Tomography.

Key Words: Causal wave equations obeying attenuation, Kramers-Kronig relation, Szabo's wave equation, thermo-viscous wave equation

AMS: 45K05, 35Q72, 74J05, 42A38, 42A85

1 Introduction

In physics an attenuated wave is modeled by replacing the real frequency dependent wave number by a complex frequency dependent wave number. For an *attenuated spherical wave* with origin $(\mathbf{x}, t) = (\mathbf{0}, 0)$, this yields [3, 10, 7]

$$p_\alpha(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-\alpha(\omega)\,|\mathbf{x}|}\, e^{-i\,\omega\,\left(t - \frac{|x|}{c(\omega)}\right)}}{4\,\pi\,|\mathbf{x}|}\, \mathrm{d}\omega\,, \tag{1}$$

where $\alpha = \alpha(\omega) > 0$ is called the *attenuation law*, which is assumed to be a positive real-valued even function, and $c = c(\omega)$ is called the *phase speed*. The complex wave number is given by $k(\omega) = \frac{\omega}{c(\omega)} + i\,\alpha(\omega)$. With the notion

$$(2) \qquad \alpha_*(\omega) := \alpha(\omega) - i\,\left(\frac{\omega}{c(\omega)} - \frac{\omega}{c_0}\right) = -i\,k(\omega) + i\,\frac{\omega}{c_0} \qquad (c_0 > 0 \text{ const.})\,,$$

the standard causality requirement for waves of the form (1) read as follows (cf. [13] and Section 3.3 in [1])

$$(3) \qquad \mathrm{Re}(\alpha_*(\omega)) = -\mathrm{Im}(\mathcal{H}(\alpha_*(\omega))) \quad \text{and} \quad \mathrm{Im}(\alpha_*(\omega)) = \mathrm{Re}(\mathcal{H}(\alpha_*(\omega)))\,.$$

These relations are the *Kramers-Kronig relations* for α_* and are equivalent to the relation (cf. [13])

$$(4) \qquad \frac{\omega}{c(\omega)} - \frac{\omega}{c_0} = -\mathcal{H}(\alpha(\omega))\,.$$

Here $\mathcal{H}$ denotes the *Hilbert transform*.

In this paper we require that every wave has a finite front speed v_α. Of course, this is not a new requirement (cf. [2]), but the consequences for wave attenuation have not been analysed sufficiently well. If a wave p_α of the form (1) has a finite front speed v_α bounded from above by a constant $v_B > 0$, then the distribution

$$g_\alpha(\mathbf{x},t) := 4\,\pi\,|\mathbf{x}|\,p_\alpha\left(\mathbf{x}, t + \frac{|\mathbf{x}|}{v_B}\right) \qquad \mathbf{x} \in \mathbb{R}^3,\ t \in \mathbb{R}$$

is causal for any $\mathbf{x} \in \mathbb{R}^3$, i.e. its support lies in $[0,\infty)$ for any fixed $\mathbf{x} \in \mathbb{R}^3$. Conversely, if $g_\alpha(\mathbf{x},\cdot)$ is causal for any $\mathbf{x} \in \mathbb{R}^3$, then the front speed of the wave p_α is bounded from above by $v_B > 0$. Let $\hat{g}_\alpha(\mathbf{x},\omega)$ denote the *Fourier transform*[1] of $g_\alpha(\mathbf{x},t)$. Then g_α is causal if and only if the Kramers-Kronig relations for $\hat{g}_\alpha$ holds (cf. Section 3.3 in [1]), i.e.

$$(5) \qquad \mathrm{Re}(\hat{g}_\alpha) = -\mathrm{Im}(\mathcal{H}(\hat{g}_\alpha)) \quad \text{and} \quad \mathrm{Im}(\hat{g}_\alpha) = \mathrm{Re}(\mathcal{H}(\hat{g}_\alpha))\,,$$

If $g_\alpha(\mathbf{x},\cdot)$ is a Schwartz function then these conditions imply the Kramers-Kronig relations (3) for α_*. This can be seen as follows: Let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed and let $*_t$ denote the time-convolution. For a spherical wave of the form (1) with $c_0 := v_B$ we have

$$(6) \qquad g_\alpha(\mathbf{x},t) = \frac{1}{\sqrt{2\,\pi}}\,\mathcal{F}\left(e^{-\alpha_*(\omega)\,|\mathbf{x}|}\right)(t) \qquad (\mathbf{x} \in \mathbb{R}^3,\ t \in \mathbb{R})$$

with α_* defined as in (2). Since the causality of $g_\alpha(\mathbf{x},\cdot)$ implies the causality of

$$(7) \qquad \nabla g_\alpha(\mathbf{x},t) = -\frac{\mathbf{x}}{|\mathbf{x}|}\,\hat{\alpha}_*(t) *_t g_\alpha(\mathbf{x},t)\,,$$

$\hat{\alpha}_*(t)$ must be causal. But the causality of $\hat{\alpha}_*(t)$ means that the Kramers-Kronig relations (3) for $\alpha_*(\omega)$ hold. We note that in general, the causality of $\hat{\alpha}_*$ or ∇g_α does not imply the causality of g_α (cf. Section 4).

[1] If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f}(\omega) := \mathcal{F}(f)(\omega) := \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-i\,\omega\,t} f(t)\mathrm{d}t$ and $\check{f} := \mathcal{F}^{-1}(f)$. This convention implies $\widehat{f\,g} = \frac{1}{\sqrt{2\,\pi}}\,\hat{f} * \hat{g}$ and $\hat{f}\,\hat{g} = \frac{1}{\sqrt{2\,\pi}}\,\widehat{f * g}$.

The phase speeds induced by the frequency power laws $\alpha(\omega) = \alpha_0 \, |\omega|^\gamma$ with $\gamma \in (\mathbb{R}^+ \backslash \mathbb{N}) \cup \{1\}$ and the Kramers-Kronig relation (4), are derived in [13, 16, 15, 14, 12]. For example if $\gamma \in (0,\infty)$ and $\gamma \notin \mathbb{N}$, then the phase speed reads as follows

$$\frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} = \alpha_0 \, \tan\left(\frac{\pi}{2}\,\gamma\right) \, (|\omega|^{\gamma-1} - |\omega_0|^{\gamma-1}) \qquad (\omega \in \mathbb{R}) \, ,$$

which implies

$$\alpha_*(\omega) = \frac{\alpha_0 \, (-i\omega)^\gamma}{\cos(\frac{\pi}{2}\,\gamma)} + i\alpha_0 \, \tan\left(\frac{\pi}{2}\,\gamma\right) \, |\omega_0|^{\gamma-1} \, \omega \, . \tag{8}$$

In Section 4 it will be shown that the wave (1) with this α_* is causal only if $\omega_0 = 0$ and $\gamma \in (0,1)$. This shows that the Kramers-Kronig relation (4) for α_* is necessary but not sufficient. Moreover, this shows that the frequency power law for $\gamma \geq 1$ is not an *admissible* attenuation law.

A standard calculation shows that the wave (1) with α_* defined as in (8) satisfies the following wave equation:

$$\nabla^2 p_\alpha - \left[\frac{\alpha_0}{\cos(\frac{\pi}{2}\,\gamma)} \, D_t^\gamma + \frac{1}{c_0} \, \frac{\partial}{\partial t}\right]^2 \, p_\alpha = -\delta(\mathbf{x})\delta(t) \, , \tag{9}$$

where $\alpha_0 > 0$ and D_t^γ denotes the *Riemann-Liouville fractional derivative* defined by ([6, 9]))

$$\mathcal{F}^{-1}(D_t^\gamma(f))(\omega) := (-i\,\omega)^\gamma \, \check{f}(\omega) \, . \tag{10}$$

We note that the causality of the kernel of D_t^γ implies the Kramers-Kronig relation (4) and vice versa.

A different wave equation was derived by Szabo in [11]). A comparision of the dispersion relations of the $1d-$ thermo-viscous wave equation and the $1d-$ electromagnetic wave equation for conducting media lead Szabo to the following dispersion relation $k(\omega)^2 = \frac{\omega^2}{c_0^2} + i\,2\,\frac{\omega}{c_0}\,\alpha_0\,|\omega|^\gamma$ with $\gamma > 0$, which implies the wave equation:

$$\nabla^2 p_\alpha - \frac{1}{c_0^2}\frac{\partial^2 p_\alpha}{\partial t^2} + L_\gamma *_t p_\alpha = -\delta(\mathbf{x})\delta(t) \, . \tag{11}$$

For example, if $\gamma > 0$ and $\gamma \notin \mathbb{N}$, then

$$L_\gamma = -\frac{2\,\alpha_0}{\sqrt{2\,\pi}\,\cos(\frac{\pi}{2}\,\gamma)\,c_0} \, \mathcal{F}\left((-i\,\omega)^{\gamma+1}\right) \, . \tag{12}$$

Since L_γ is the kernel of the operator $\frac{2\,\alpha_0}{\cos(\frac{\pi}{2}\,\gamma)\,c_0}\,\frac{\partial}{\partial t}\,D_t^\gamma$, Szabo's equation can be obtained by neglecting the term $\frac{\alpha_0^2}{\cos^2(\frac{\pi}{2}\,\gamma)}\,D_t^{2\,\gamma}$ in equation (9). In Section 5 we show that Szabo's equation admits causality only if $\gamma \in (0,1)$.

This paper is organized as follows: The general properties of attenuated waves assumed in this paper are introduced and discussed in Section 2. With these assumptions we derive the general wave equation for a homogeneous and isotropic medium obeying attenuation and causality in Section 3. The first causality analysis of attenuated waves is performed in Section 4 for the case of the frequeny power law. Afterwards the causality analysis of Szabo' equation (Section 5) and the thermo-viscous wave equation (Section 6) are perfomed. Finally, a generalization of the thermo-viscous wave equation, is derived in Section 7, which admits causality and obeys the power frequency law for sufficiently small frequencies and $\gamma \in (1,2]$.

2 General properties of attenuated waves

In this section we postulate the basic properties of attenuated waves propagating in *homogeneous isotropic media* and infer the structure of attenuated waves. The main goal of this section is to clarify our assumptions on which we base our casuality analysis of attenuated waves.

Assumptions

Let p_0 denote the solution of the standard wave equation $\nabla^2 p_0 - \frac{1}{c_0^2}\frac{\partial^2 p_0}{\partial t^2} = -f$ with $p_0|_{t<0} = 0$ and $\frac{\partial p_0}{\partial t}|_{t<0} = 0$. Let $\mathcal{A}_\alpha$ denote the map that associates to each source term f the corresponding attenuated wave p_α. $\alpha = 0$ means no attenuation, i.e. $p_0 = \mathcal{A}_0(f)$. We assume a *homogeneous isotropic medium*, which implies that

$$\mathcal{A}_\alpha \text{ commutes with all translations in } \mathbf{x} \text{ and } t\,.$$

We also assume that for $\alpha \neq 0$:

$$\begin{array}{l}\mathcal{A}_\alpha \text{ is a linear continuous mapping}\\ \text{that maps } C_0^\infty(\mathbb{R}^4) \text{ into } \mathcal{S}(\mathbb{R}, C^\infty(\mathbb{R}^3))\,,\end{array}$$

and that

$$\mathcal{A}_\alpha(\delta(\mathbf{x})\,\delta(t)) \in \mathcal{S}(\mathbb{R}, C(\mathbb{R}^3))\,.$$

Here $\mathcal{S}$ denotes the space of Schwartz functions, $\delta(\mathbf{x})$ denotes the $3D-$delta distribution and $\delta(t)$ denotes the $1D-$delta distribution. The last two assumptions take into account that wave attenuation smoothes and decreases the wave.

Superposition law

Since the operator $\mathcal{A}_\alpha$ satisfies the assumptions of Theorem 4.2.1 in [5], $\mathcal{A}_\alpha$ is a space-time convolution operator with kernel $G_\alpha(\mathbf{x},t) := \mathcal{A}_\alpha(\delta(\mathbf{x})\,\delta(t))$. Since $\delta(\mathbf{x})\,\delta(t)$ is the source term of a spherical wave for $\alpha = 0$, we interprete G_α as an attenuated spherical wave and thus

$$\mathcal{A}_\alpha(f)(\mathbf{x},t) = G_\alpha(\mathbf{x},t) *_{\mathbf{x},t} f(\mathbf{x},t) \tag{13}$$

is nothing else but the superposition law of attenuated waves. In analogy to partial differential equations and linear system theory we call G_α the *Green function* of the wave model.

Causality condition

We require that any attenuated spherical wave G_α has a positive finite wave front speed v_α, which is equivalent to the requirement that

$$g_\alpha(\mathbf{x},t) := 4\,\pi\,|\mathbf{x}|\,G_\alpha\left(\mathbf{x},t+T(\mathbf{x})\right) \qquad (v_\alpha > 0) \tag{14}$$

is a causal distribution i.e. $\mathrm{supp}(g_\alpha(\mathbf{x},\cdot)) \subseteq [0,\infty)$. Here $T(\mathbf{x}) := \int_0^{|\mathbf{x}|} \frac{1}{v_\alpha(r)}\,\mathrm{d}r \geq 0$ denotes the *travel time* of G_α. Then any attenuated wave $\mathcal{A}_\alpha(f)$ with compactly supported source f has finite front speed if and only if the Green function $G_\alpha(\mathbf{x},\cdot)$ has finite front speed. We note that property (14) implies

$$\mathrm{supp}(G_\alpha(\mathbf{x},\cdot)) \subseteq [0,\infty) \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3\,.$$

Moreover, we assume that the front speed v_α is continuous, which implies $T(\mathbf{x}) \in C(\mathbb{R}^3)$ and $g_\alpha \in \mathcal{S}(\mathbb{R}, C(\mathbb{R}^3))$. For the standard case were the wave front speed is assumed to be constant, this assumption is satisfied.

General structure of attenuated waves

The property $g_\alpha \in \mathcal{S}(\mathbb{R}, C(\mathbb{R}^3))$ implies that

$$g_\alpha(\mathbf{x},t) = \frac{1}{\sqrt{2\,\pi}}\,\mathcal{F}\left(e^{-\beta_*(|\mathbf{x}|,\omega)}\right)\,, \tag{15}$$

with $\mathrm{Re}(\beta_*) := -\log\left(\sqrt{2\,\pi}|\check{g}_\alpha|\right)$ and $\mathrm{Im}(\beta_*) := -\arg\left(\check{g}_\alpha\right)$. Here $\check{g}_\alpha(\mathbf{x},\cdot)$ denotes the inverse Fourier transform of g_α. Since g_α is real-valued, it follows that $\mathrm{Re}(\beta_*)$ is even and $\mathrm{Im}(\beta_*)$ is odd with respect to ω. (We note that the causality of $g_\alpha(\mathbf{x},\cdot)$ implies that $z \in \mathbb{R}+i\,\mathbb{R}^+ \mapsto \check{g}_\alpha(\mathbf{x},z) \in \mathbb{C}$ is analytic (cf. Theorem 4.1) and thus $-\mathrm{Re}(\beta_*(z)) : \mathbb{R}+i\,\mathbb{R}^+ \mapsto [-\infty,\infty)$ is subharmonic (cf. Example 4.1.10 in [5]).)
If $\frac{\partial}{\partial r}\mathrm{Im}\left(\beta_*(r,\omega)\right) < \frac{\omega}{v_\alpha(r)}$, then β_* can be written as follows

$$\beta_*(|\mathbf{x}|,\omega) = a_S(\omega) + \int_0^{|\mathbf{x}|}\left[\alpha(r,\omega) - i\,\left(\frac{\omega}{c(r,\omega)} - \frac{\omega}{v_\alpha(r)}\right)\right]\,\mathrm{d}r\,, \tag{16}$$

where α and c are positive and even with respect to ω and $\lim_{\omega\to\infty}\mathrm{Re}(\beta_*) = \infty$. Moreover, a_S is a complex valued function such that

$$g_\alpha(\mathbf{0},t) \in \mathcal{S}(\mathbb{R})\,.$$

By S_α we denote the time-convolution operator with kernel $g_\alpha(\mathbf{0},t)$ that maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. If β_* depends only on ω, i.e. $\beta_* = a_S$, then the attenuated wave satisfies the standard wave equation with source term f replaced by $S_\alpha\,f$. We note that g_α is causal if and only if Theorem 4.1 (cf. Section 4) is satisfied.

The standard normal form (1) assumes that

$$S_\alpha = \mathrm{Id}, \quad v_\alpha = c_0 = \mathit{const}, \quad \alpha = \alpha(\omega) \quad \text{and} \quad 0 < \alpha(\omega) < \infty\,,$$

which together with the Kramers-Kronig relation (4) imply that $c = c(\omega)$. Moreover, $\beta_*(|\mathbf{x}|,\omega) = \alpha_*(\omega)\,|\mathbf{x}|$ with α_* defined as in (2). We note that g_α is causal if and only if α_* satisfies Lemma 4.2.

Since wave attenuation is an irreversible thermodynamic process, β_* can depend on time or equivalently on the distance to the origin. We will see that in general, wave attenuation depends on its history, even if α depends only on ω. In particular this is the case if the support of the kernel of S_α is not discrete. Moreover, it is not evident that the front speed of an attenuated spherical wave is constant. Therefore we see no reason to assume that β_* and the front speed v_α depend only on ω.

Remark 2.1. *If $v_\alpha(r) \leq v_B$ for all $r \geq 0$ ($v_B = const.$) and g_α is a causal distribution, then $\tilde{g}(\mathbf{x},t) := 4\pi\,|\mathbf{x}|\,G_\alpha\left(\mathbf{x}, t + \frac{|\mathbf{x}|}{v_B}\right)$ must also be a causal distribution. In this case $\tilde{\alpha}_*$ corresponding to $\tilde{g}(\mathbf{x},t)$ is given by (16) with v_α replaced by v_B and the Green function reads as follows $G_\alpha(\mathbf{x},t) = \tilde{g}_\alpha(\mathbf{x},t) *_t \frac{\delta(t-\frac{|\mathbf{x}|}{v_B})}{4\pi\,|\mathbf{x}|}$. This fact will be used to prove the non-causality of some wave models.*

3 Wave equation obeying attenuation and causality

Now we derive the wave equation satisfied by the attenuated waves described in Section 2 and discuss its Cauchy problem.

First we derive the wave equation for the Green function G_α. The most convenient derivation uses the representation of the Green function introduced in Remark 2.1:

$$G_\alpha = g_\alpha *_t G_B \qquad \text{with}$$

$$G_B(\mathbf{x},t) = \frac{\delta(t-\frac{|\mathbf{x}|}{v_B})}{4\pi\,|\mathbf{x}|} \quad \text{and} \quad g_\alpha(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}\left(e^{\beta_*(|\mathbf{x}|,\omega)}\right)(t)\,. \tag{17}$$

Here the constant v_B is an upper bound of the front speed of G_α and β_* is defined as in (16) with $v_\alpha(\mathbf{x})$ replaced by v_B. We recall that S_α is the time convolution operator such that $S_\alpha\,\delta(t) = g_\alpha(\mathbf{0},t) \in \mathcal{S}(\mathbb{R})$. To formulate the wave equation we need the time convolution operators $D_* : \mathcal{D}'_+(\mathbb{R}) \to \mathcal{D}'_+(\mathbb{R})$ and $D_{*,|x|} : \mathcal{D}'_+(\mathbb{R}) \to \mathcal{D}'_+(\mathbb{R})$ with causal kernels

$$K_*(r,t) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}\left(\frac{\partial \beta_*}{\partial r}(r,\omega)\right)(t) \quad \text{and} \quad K_{*,|x|}(r,t) = \frac{\partial K_*}{\partial r}(r,t) \tag{18}$$

for all r, $t > 0$, respectively. Here $\mathcal{D}'_+(\mathbb{R})$ denotes the space of causal distributions. From (17), it follows

$$\nabla g_\alpha = -\frac{\mathbf{x}}{|\mathbf{x}|}\,K_* *_t g_\alpha \qquad \text{and}$$

$$\nabla^2 g_\alpha = \left[-\frac{2}{|\mathbf{x}|}\,K_* - K_{*,|x|} + K_* *_t K_*\right] *_t g_\alpha\,.$$

This together with (17) imply

$$\nabla^2 G_\alpha - \frac{1}{v_B^2}\,\frac{\partial^2 G_\alpha}{\partial t^2} = \left[D_*^2 + \frac{2}{v_B}\,\frac{\partial}{\partial t}\,D_* - D_{*,|x|}\right] G_\alpha - S_\alpha\,\delta(t)\,\delta(\mathbf{x})\,.$$

Due to causality of g_α we have

$$G_\alpha|_{t<0} = 0\,, \qquad \left.\frac{\partial G_\alpha}{\partial t}\right|_{t<0} = 0\,. \tag{19}$$

From

$$g_\alpha,\ \frac{\partial g_\alpha}{\partial x_j},\ \nabla^2 g_\alpha \in \mathcal{S}(\mathbb{R}, C(\mathbb{R}^3)) \qquad \text{for } j = 1,\,2,\,3\,,$$

it follows that

$$K_{\Lambda_\alpha} := \left[D_*^2 + \frac{2}{v_B}\,\frac{\partial}{\partial t}\,D_* - D_{*,|x|}\right] G_\alpha \in \mathcal{S}(\mathbb{R}, C(\mathbb{R}^3))$$

and that the space-time-convolution operator Λ_α with kernel K_{Λ_α} maps $\mathcal{S}(\mathbb{R}, C^\infty(\mathbb{R}^3))$ into $\mathcal{S}(\mathbb{R}, C^\infty(\mathbb{R}^3))$. Since an arbitrary attenuated wave is of the form $p_\alpha = G_\alpha *_{\mathbf{x},t} f$, the general wave equation reads as follows:

$$\begin{aligned} \nabla^2 p_\alpha - \frac{1}{v_B^2}\,\frac{\partial^2 p_\alpha}{\partial t^2} &= (\Lambda_\alpha - S_\alpha)(f) \quad \text{with} \\ p_\alpha|_{t<0} = 0\,, \quad \left.\frac{\partial p_\alpha}{\partial t}\right|_{t<0} &= 0\,. \end{aligned} \tag{20}$$

This equation has for every $f \in \mathcal{S}(\mathbb{R}, C^\infty(\mathbb{R}^3))$ with compact support a unique solution

$$p_\alpha(\mathbf{x}, t) = \int_{\mathbb{R}^3} \frac{[\Lambda_\alpha(f) - S_\alpha(f)]\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{v_B}\right)}{4\,\pi\,|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{x}' \ \in\ \mathcal{S}(\mathbb{R}, C^\infty(\mathbb{R}^3))$$

with finite wave front speed.

If the attenuation law and the phase speed do not depend on the spatial position, then the operator D_* does not depend on the spatial position, too, and

$$D_*(G_\alpha) *_{\mathbf{x},t} f = D_*(p_\alpha) \tag{21}$$

holds. Moreover, then $D_{*,|x|}$ is the zero operator. In this case we can write the wave equation as follows

$$\nabla^2 p_\alpha - \left[D_* + \frac{1}{v_B}\,\frac{\partial}{\partial t}\right]^2 p_\alpha = -S_\alpha(f)\,, \tag{22}$$

with $p_\alpha|_{t<0} = 0$ and $\left.\frac{\partial p_\alpha}{\partial t}\right|_{t<0} = 0$. In general the supports of the kernels of D_* and S_α are subsets of $[0, \infty)$ with positive Lebesgue measure, which means that the attenuated wave depends on its history. Since the values of the wave in the past are required, the Cauchy problem of wave equation (22) is not reasonable. In the next theorem we formulate a generalization of the Cauchy problem and state its properties for the special case $\alpha = \alpha(\omega)$, $c = c(\omega)$ with $a_S = 0$.

Proposition 3.1. *Let D_* be the time-convolution operator with causal kernel K_* defined as in (18) and let $q \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}_0^-)$, $\varphi := q|_{t=0}$ and $\psi := \frac{\partial q}{\partial t}\Big|_{t=0}$. Moreover, let G_α denote the Green function of wave equation (22) with $S_\alpha = Id$ and wave front speed $\leq v_B$. Then the solution of the generalized Cauchy problem*

$$\nabla^2 p_\alpha - \left[D_* + \frac{1}{v_B}\frac{\partial}{\partial t}\right]^2 p_\alpha = 0 \qquad \text{on } \mathbb{R}^3 \times \mathbb{R}_0^+ ,$$

$$p_\alpha|_{t\leq 0} = q \quad \text{and} \quad \frac{\partial p_\alpha}{\partial t}\Big|_{t=0} = \psi ,$$

is given by

$$p_\alpha = G_\alpha *_{\mathbf{x},t} f \qquad \text{on } \mathbb{R}^3 \times \mathbb{R}_0^+ ,$$

with

$$f = -\frac{\psi}{v_B^2}\,\delta(t) - \frac{\varphi}{v_B^2}\,\delta'(t) - \left[D_*^2 + \frac{1}{v_B}\frac{\partial}{\partial t}\,D_*, M_{H(t)}\right] q\,.$$

Here $M_{H(t)}$ denotes the multiplication operator, $H = H(t)$ the Heaviside function and $[,]$ denotes the commutator, i.e. if A, B are operators then $[A, B] = A\,B - B\,A$.

Proof. Let p_α, G_α, f and $H(t)$ be defined as in the Proposition. For convenience let $A_* := D_*^2 + \frac{1}{v_B}\frac{\partial}{\partial t}\,D_*$. Then $\tilde{p}_\alpha := H(t)\,p_\alpha$ satisfies the following properties:

$$\nabla^2 \tilde{p}_\alpha = H(t)\,\nabla^2 p_\alpha ,$$

$$\frac{\partial^2 \tilde{p}_\alpha}{\partial t^2} = H(t)\,\frac{\partial^2 p_\alpha}{\partial t^2} + \psi\,\delta(t) + \varphi\,\delta'(t) ,$$

and

$$A_*\,\tilde{p}_\alpha = H\,A_*(p_\alpha) + [A_*, M_H]\,p_\alpha ,$$

since $\varphi = p_\alpha|_{t=0}$ and $\psi = \frac{\partial p_\alpha}{\partial t}\Big|_{t=0}$. From these properties we infer

$$\nabla^2 \tilde{p}_\alpha - \frac{1}{v_B^2}\frac{\partial^2 \tilde{p}_\alpha}{\partial t^2} - A_*\,\tilde{p}_\alpha = -\frac{\psi}{v_B^2}\,\delta(t) - \frac{\varphi}{v_B^2}\,\delta'(t) - [A_*, M_H]\,p_\alpha ,$$

$$\tilde{p}_\alpha|_{t<0} = 0 \quad \text{and} \quad \frac{\partial \tilde{p}_\alpha}{\partial t}\Big|_{t<0} = 0 ,$$

which has the solution $\tilde{p}_\alpha = G_\alpha *_{\mathbf{x},t} f$ and thus $p_\alpha = G_\alpha *_{\mathbf{x},t} f$ on $\mathbb{R}^3 \times \mathbb{R}_0^+$. This proves the Proposition. □

4 Causality analysis of attenuated spherical waves obeying the frequency power law

In this section we show that not every attenuated spherical wave of the form (1) has a finite front speed although its phase speed is properly related to the frequency power law via the Kramers-Kronig relation (4). Other attenuated spherical wave models are analysed in Section 5, 6 and 7

In the following we use the notions $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_0^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{R}_0^- = (-\infty, 0]$. The next Theorem is a reformulation of Theorem 7.4.3 in [5] for the case of causal tempered distributions.

Theorem 4.1. *A distribution $u \in \mathcal{S}'(\mathbb{R})$ is causal, i.e. supp$(u) \subseteq \mathbb{R}_0^+$, if and only if*

(A1) $\mathbb{R} + i\,\mathbb{R}^- \to \mathbb{C}$, $z \mapsto \hat{u}(z)$ *is analytic and*

(A2) $\exists \epsilon > 0\, \exists C > 0\, \exists N > 0\, \forall z \in \mathbb{R} + i\,(-\infty, -\epsilon) : \, |\hat{u}(z)| \leq C\,(1+|z|)^N\,.$

Applying Theorem 4.1 to attenuated spherical waves of the form (1) yields the following Lemma.

Lemma 4.2. *Let p_α be defined by (1) with real-valued functions $\alpha = \alpha(\omega) > 0$ and $c = c(\omega)$, and let $p_\alpha(\mathbf{x}, \cdot) \in \mathcal{S}'(\mathbb{R})$ for any $\mathbf{x} \in \mathbb{R}^3$. The wave defined by (1) has a finite wave front speed if and only if α_* defined by (2) satisfies the following conditions:*

(B1) $\alpha_* = \alpha_*(-z)$ *is analytic on* $\mathbb{R} + i\,\mathbb{R}^-$ *and*

(B2) $\exists \epsilon > 0\, \exists C > 0\, \exists N > 0\, \forall z \in \mathbb{R} + i\,(-\infty, -\epsilon) :$
$$-\,Re(\alpha_*(-z)) \leq C + N\,\log(1+|z|)\,.$$

Proof. According to Theorem 4.1 $g_\alpha(\mathbf{x}, \cdot)$ defined as in (14) is a causal distribution for any fixed $\mathbf{x} \in \mathbb{R}^3$, if properties (A1) and (A2) hold for $u := g_\alpha(\mathbf{x}, \cdot)$ for any $\mathbf{x} \in \mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed. According to (6) we have $g_\alpha(\mathbf{x}, \cdot) = \mathcal{F}\left(e^{-\alpha_*(\omega)\,|\mathbf{x}|}\right) = \mathcal{F}^{-1}\left(e^{-\alpha_*(-\omega)\,|\mathbf{x}|}\right)$ and thus

$$\hat{g}_\alpha(\mathbf{x}, \omega) = e^{-\alpha_*(-\omega)\,|\mathbf{x}|} \qquad (\omega \in \mathbb{R})\,. \tag{23}$$

Let $\beta_1(z) := -\mathrm{Re}(\alpha_*(-z))\,|\mathbf{x}|$ and $\beta_2(z) := -\mathrm{Im}(\alpha_*(-z))\,|\mathbf{x}|$. Since $e^{-\alpha_*(-z)\,|\mathbf{x}|}$ is analytic on $\mathbb{R} + i\,\mathbb{R}^-$, the Cauchy-Riemann equations are satisfied

$$\begin{aligned} \left[\frac{\partial \beta_1}{\partial x} - \frac{\partial \beta_2}{\partial y}\right] \cos\beta_2 &= \left[\frac{\partial \beta_1}{\partial y} + \frac{\partial \beta_2}{\partial x}\right] \sin\beta_2\,, \\ \left[\frac{\partial \beta_1}{\partial x} - \frac{\partial \beta_2}{\partial y}\right] \sin\beta_2 &= -\left[\frac{\partial \beta_1}{\partial y} + \frac{\partial \beta_2}{\partial x}\right] \cos\beta_2\,, \end{aligned} \tag{24}$$

which imply for all z with $\cos\beta_2(z) \neq 0$ and $\sin\beta_2(z) \neq 0$ the equations

$$\frac{\partial \beta_1}{\partial x} = \frac{\partial \beta_2}{\partial y} \qquad \text{and} \qquad \frac{\partial \beta_1}{\partial y} = -\frac{\partial \beta_2}{\partial x}\,.$$

This means that $\beta_1(z) + i\,\beta_2(z)$ is analytic for all $z \in \mathbb{R} + i\,\mathbb{R}^-$ satisfying $\cos\beta_2(z) \neq 0$ and $\sin\beta_2(z) \neq 0$. The same equations follow easily if $\sin\beta_2(z) \neq 0$ but $\cos\beta_2(z) = 0$, and for $\cos\beta_2(z) \neq 0$ but $\sin\beta_2(z) = 0$ and thus $\alpha_*(-z)$ is analytic on $\mathbb{R} + i\,\mathbb{R}^-$. This shows that condition (B1) is satisfied. Conversely if $\alpha_*(-z)$ is analytic on $\mathbb{R} + i\,\mathbb{R}^-$ then due to the chain rule $e^{-\alpha_*(-z)\,|\mathbf{x}|}$ must be analytic on $\mathbb{R} + i\,\mathbb{R}^-$, since the complex exponential function is analytic on $\mathbb{C}$.

From condition (A2) in Theorem 4.1 together with (23) we infer

$$e^{-\mathrm{Re}(\alpha_*(-z)\,|\mathbf{x}|)} \leq C\,(1+|z|)^N\,,$$

whereby we can assume without loss of generality $C > 1$. Let $\tilde{C} := \log C > 0$. Since the real (natural) logarithm function is monotonic increasing, we can apply it onto the previous inequality, which yields condition (B2) with C replaced by $\frac{\tilde{C}}{|\mathbf{x}|}$. Conversely, if (B2) holds then condition (A2) holds, too. This concludes the proof. □

For the analysis of the causality properties of the standard models of spherical attenuated waves we need the following Lemma.

Lemma 4.3. *Let* $\mathbf{x} \in \mathbb{R}^3$ *and* $\gamma > 0$ *be arbitrary but fixed. Moreover, let* $s(\gamma)$ *be the sign of* $\cos(\frac{\pi}{2}\gamma)$ *if* $\cos(\frac{\pi}{2}\gamma) \neq 0$ *and let* $s(\gamma) = 1$ *if* $\cos(\frac{\pi}{2}\gamma) = 0$. *The distribution* $\mathcal{F}((-i\,\omega)^\gamma)$ *is causal on* $\mathbb{R}$ *for any* $\gamma > 0$ *and the distribution* $\mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ *is causal if* $\gamma \in (0, 1]$ *and non-causal if* $\gamma > 1$.

Proof. i) First we prove the causality of $\mathcal{F}((-i\,\omega)^\gamma)$. Since $\omega \in \mathbb{R} \mapsto (-i\,\omega)^\gamma \in \mathbb{C}$ is a slowly increasing function, it is a tempered distribution and hence $t \in \mathbb{R} \mapsto \mathcal{F}((-i\,\omega)^\gamma)(t) \in \mathbb{R}$ is also a tempered distribution. Therefore Theorem 4.1 can be applied. Let $\gamma \in (0, \infty)$ be arbitrary but fixed and let U denote the complex plane without the non-negative real axis. Then $z^\gamma := e^{\gamma\,\log z}$ is analytic on U (cf.[8]). Therefore $(i\,z)^\gamma$ is analytic on $\mathbb{R} + i\,\mathbb{R}^-$ and condition (A1) in Theorem 4.1 is satisfied. Moreover, $z \in U \mapsto (i\,z)^\gamma \in \mathbb{C}$ satisfies condition (A2) in Theorem 4.1 with $C = 1$ and $N = \gamma$, which shows that $\mathcal{F}((-i\,\omega)^\gamma)(t)$ is causal on $\mathbb{R}$.

ii) For the rest of the proof let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed. Now we prove that $\sqrt{2\,\pi}\,g_\alpha(\mathbf{x}, t) := \mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ is causal for $\gamma \in (0, 1]$ and non-causal for $\gamma \in (1, \infty)$. Since $(-i\,\omega)^\gamma = |\omega|^\gamma\,[\cos(\frac{\pi}{2}\gamma) - i\,\mathrm{sgn}(\omega)\,\sin(\frac{\pi}{2}\gamma)]$, it follows that $\left|e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|}\right| = e^{-|\cos(\frac{\pi}{2}\gamma)|\,|\omega|^\gamma\,|\mathbf{x}|}$ is bounded for each $\gamma \in \mathbb{R}^+$ and thus $\mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ is a tempered distribution and Lemma 4.2 can be applied. Using the same notion as in (15), yields $\alpha_*(-z) = s(\gamma)\,(i\,z)^\gamma$.
Let $z = r\,e^{i\,\varphi}$ with $r > 0$ and $\varphi \in (-\pi, 0)$. Then $z \in \mathbb{R} + i\,\mathbb{R}^-$ and $\mathrm{Re}((i\,z)^\gamma) = \cos(\gamma(\varphi + \frac{\pi}{2}))\,|z|^\gamma$, and thus the inequality in (B2) reads as follows

$$(25) \qquad -s(\gamma)\,\cos(\gamma(\varphi + \frac{\pi}{2}))\,|z|^\gamma \leq C + N\,\log(1 + |z|) \qquad (\gamma > 0)\,.$$

This shows that condition (B2) is satisfied if and only if $s(\gamma)\,\cos(\gamma(\varphi + \frac{\pi}{2})) \geq 0$.
a) For $\gamma \in (0, 1]$ we get $s(\gamma) = 1$ and $\gamma(\varphi + \frac{\pi}{2}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for any $\varphi \in (-\pi, 0)$ and thus $s(\gamma)\,\cos(\gamma(\varphi + \frac{\pi}{2})) \geq 0$, i.e. condition (B2) is satisfied for any $\epsilon > 0$. Therefore $\mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ is causal for $\gamma \in (0, 1]$.
b) Now we prove the non-causality of $\mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ for $\gamma \in [3, 5] \cup [7, 9] \cup \cdots$. Since for these $\gamma-$values $s(\gamma) = 1$ we have to find a φ such that the sign of $\cos(\gamma(\varphi + \frac{\pi}{2}))$ is negative. For $\gamma > 1$ let $0 < \delta < \min\left(\frac{\pi}{4}, \frac{\gamma-1}{\gamma+1}\,\frac{\pi}{2}\right)$ and $\varphi_\delta := \left(\frac{1}{\gamma} - 1\right)\,\frac{\pi}{2} + \frac{\delta}{\gamma}$. Since $\varphi_\delta \in (-\frac{\pi}{2}, -\delta)$ and $\gamma(\varphi_\delta + \frac{\pi}{2}) = \frac{\pi}{2} + \delta \in (\frac{\pi}{2}, \frac{3\pi}{4})$, it follows that $z := r\,e^{i\,\varphi_\delta} \in \mathbb{R} + i\,\mathbb{R}^-$ and $\cos(\gamma(\varphi_\delta + \frac{\pi}{2})) < 0$ and hence condition (B2) cannot be satisfied for any $\epsilon > 0$.
c) Now we prove the non-causality of $\mathcal{F}(e^{-s(\gamma)\,(-i\,\omega)^\gamma\,|\mathbf{x}|})$ for $\gamma \in (1, 3) \cup (5, 7) \cup (9, 11) \cup \cdots$. Since for these $\gamma-$values $s(\gamma) = -1$ we have to find a φ such that the sign of $\cos(\gamma(\varphi + \frac{\pi}{2}))$ is positive. For $\varphi = -\frac{\pi}{2}$ it follows that $z = r\,e^{i\,\varphi} \in i\,\mathbb{R}^-$ $(r > 0)$ and $\cos(\gamma(\varphi + \frac{\pi}{2})) = 1$, which implies at once that condition (B2) cannot be satisfied for any $\epsilon > 0$. This proves the Corollary. □

The following two Corollaries clarify for which values of γ and ω_0 the frequency power law $\alpha(\omega) = \alpha_0\,|\omega|^\gamma$ together with the phase speed determined by the Kramers-Kronig relations (4), yield an attenuated spherical wave that satisfies the causality requirement. The derivation of the phase speed $c = c(\omega)$ corresponding to frequency power law with various $\gamma-$values can be found in [13, 16, 15, 14, 12].

Corollary 4.4. *Let α_0 and ω_0 be positive constants, $\gamma \in \mathbb{R}^+\backslash\mathbb{N}$ and let the attenuation law α and the phase speed c be defined as follows:*

$$\begin{aligned} \alpha(\omega) &= \alpha_0\,|\omega|^\gamma\,, \\ \frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} &= \alpha_0\,\tan\left(\frac{\pi}{2}\,\gamma\right)\,(|\omega|^{\gamma-1} - |\omega_0|^{\gamma-1})\,. \end{aligned} \tag{26}$$

Then α_* defined as in (2) reads as follows

$$\alpha_*(\omega) = \frac{\alpha_0\,(-i\omega)^\gamma}{\cos(\frac{\pi}{2}\,\gamma)} + i\alpha_0\,\tan\left(\frac{\pi}{2}\,\gamma\right)\,|\omega_0|^{\gamma-1}\,\omega \tag{27}$$

and the wave defined by (1) has finite front speed if $\gamma \in (0,1)$ and $\omega_0 = 0$. For these cases the wave front speed is equal to $c(0)$. If $\omega_0 = 0$ and $\gamma \in (1,\infty)\backslash\mathbb{N}$ or $\omega_0 > 0$ and $\gamma \in \mathbb{R}^+\backslash\mathbb{N}$, then the wave defined by (1) cannot have a finite front speed.

Proof. Let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed. From (26) and (2) we get at once identity (27).
i) Let $\omega_0 = 0$, then according to Lemma 4.3 $\mathcal{F}(e^{-\alpha_*\,|\mathbf{x}|})(t)$ with α_* defined as in (27) is causal if $\gamma \in (0,1)$ and non-causal if $\gamma \in (1,\infty)\backslash\mathbb{N}$.
ii) Now let $\omega_0 > 0$. Then it follows that

$$-\mathrm{Re}\,(\alpha_*(-z)) = -a_1(\gamma)\,|z|^\gamma + a_2(\gamma)\,|\mathrm{Im}(z)| \qquad \text{for all } z \in \mathbb{R} + i\,\mathbb{R}^-\,,$$

where $a_1(\gamma) := \alpha_0\,\frac{\cos((\varphi+\frac{\pi}{2})\,\gamma)}{\cos(\frac{\pi}{2}\,\gamma)}$, $\varphi \in (-\pi, 0)$ is the argument of z and $a_2(\gamma) := \alpha_0\,\tan\left(\frac{\pi}{2}\,\gamma\right)\,|\omega_0|^{\gamma-1}$.
a) If $\gamma \in (0,1)$ then a_1, $a_2 > 0$ and

$$-\mathrm{Re}\,(\alpha_*(-i\,z_2)) = -a_1(\gamma)\,|z_2|^\gamma + a_2(\gamma)\,|z_2| \qquad \text{for all } z_2 < 0\,,$$

i.e. $-\mathrm{Re}\,(\alpha_*(-i\,z_2))$ growths like $|z_2|$ for sufficiently large $-z_2$. Since this term cannot be bounded by $\log(1+|z_2|)$, condition (B2) in Lemma 4.2 cannot be satisfied for $\omega_0 > 0$ and $\gamma \in (0,1)$.
b) Now let $\gamma > 1$ and $\gamma \notin \mathbb{N}$. We note that $a_1(\gamma)$ has the same sign as $s(\gamma)\,\cos((\varphi + \frac{\pi}{2})\,\gamma)$, where $s(\gamma)$ is defined as in Lemma 4.3. As in the proof of Lemma 4.3 one shows that for an appropriate choice of $\varphi \in (-\pi, 0)$ the constant a_1 is negative, which shows that $-\mathrm{Re}\,(\alpha_*(-z(r)))$ growth like $|z|^\gamma$ for sufficiently large $|z|$. Therefore condition (B2) in Lemma 4.2 cannot be satisfied for $\omega_0 > 0$ and $\gamma \in (0,1)$. In summary we have shown that the front speed of the wave defined by (1) and (27) cannot be finite if $\omega_0 > 0$ and $\gamma \in \mathbb{R}^+\backslash\mathbb{N}$.
iii) Now we show that the front speed for the case $\omega_0 = 0$ with $\gamma \in (0,1)$ is equal to $c(0)$. We recall that if $g_\alpha(\mathbf{x},t)$ is causal, then the front wave speed satisfies the condition $v_\alpha \leq c(\omega_0)$. If the front wave speed v_α at $\mathbf{x}$ is smaller than $c(0)$ ($\omega_0 = 0$), then $g_\alpha(\mathbf{x}, t + \delta\,|\mathbf{x}|)$ is causal for some $\delta > 0$. This means that there exist constants $\epsilon > 0$, $C > 0$ and $N > 0$ such that for all $z \in \mathbb{R} + i\,(-\infty, -\epsilon)$:

$$-\mathrm{Re}(\alpha_*(-z)) - \mathrm{Re}(i\,(-z)\,\delta) \leq C + N\,\log(1+|z|)\,. \tag{28}$$

Since $\omega_0 = 0$, we have $a_2 = 0$. As above we get for all $z = i\,z_2 \in \mathbb{R}^-$

$$-\mathrm{Re}\,(\alpha_*(-z)) = -a_1\,|z_2|^\gamma + \delta\,|z_2| \qquad (a_1 > 0)\,,$$

which grows like $|z_2|$ if $\delta > 0$ and thus condition (28) cannot be satisfied. This contradicts the fact that $g_\alpha(\mathbf{x},t)$ is causal and hence we conclude that $\delta = 0$. This proves that the wave front speed is equal to $c(0)$ and concludes the proof. $\square$

Now we come to the special case $\gamma = 1$.

Corollary 4.5. *Let α_0, $\omega_0 > 0$ be constants and*

$$\alpha_*(\omega) := \lim_{\gamma\to 1}\left[\frac{\alpha_0\,(-i\omega)^\gamma}{\cos(\frac{\pi}{2}\,\gamma)} + i\alpha_0\,\tan\left(\frac{\pi}{2}\,\gamma\right)\,|\omega_0|^{\gamma-1}\,\omega\right], \tag{29}$$

then

$$\alpha(\omega) = \alpha_0\,|\omega|\,, \qquad \frac{1}{c(\omega)} - \frac{1}{c(\omega_0)} = -\alpha_0\,\frac{2}{\pi}\,\log\left|\frac{\omega}{\omega_0}\right| \tag{30}$$

and the wave defined by (1) cannot have a finite front speed. Moroever, $\mathcal{F}(\alpha_*(\omega))$ is not causal.

Proof. Definition (29) implies

$$\alpha_*(\omega) = \alpha_0\,|\omega| - i\alpha_0\,\omega\,\lim_{\epsilon\to 0+}\frac{|\omega|^\epsilon - |\omega_0|^\epsilon}{\cot(\frac{\pi}{2}\,(1+\epsilon))}\,.$$

Since both the numerator and the denominator of the last expression converge to zero, we can apply the *rule of de l'Hospital*, which yields

$$\begin{aligned}\alpha_*(\omega) - \alpha_0\,|\omega| &= -i\alpha_0\,\omega\,\lim_{\epsilon\to 0+}\frac{|\omega|^\epsilon\,\log|\omega| - |\omega_0|^\epsilon\,\log|\omega_0|}{-\frac{\pi}{2}\,\sin^2(\frac{\pi}{2}\,(1+\epsilon))}\\ &= i\alpha_0\,\frac{2}{\pi}\,\omega\,(\log|\omega| - \log|\omega_0|)\,.\end{aligned}$$

The result for the limit $\epsilon \to 0-$ follows analogously. Comparing α_* with (2) yields (30). For $(x,y)\in\mathbb{R}^2$ let

$$u(x,y) := \alpha_0\,\sqrt{x^2+y^2} - \alpha_0\,\frac{2}{\pi}\,y\,(\log\sqrt{x^2+y^2} - \log|\omega_0|)$$

and

$$v(x,y) := \alpha_0\,\frac{2}{\pi}\,x\,(\log\sqrt{x^2+y^2} - \log|\omega_0|)\,.$$

Then $\alpha_*(z) = u(x,y) + i\,v(x,y)$ for $z\in\mathbb{C}$ with $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, and $\frac{\partial u(x,y)}{\partial x} \neq \frac{\partial v(x,y)}{\partial y}$ for every $(x,y)\in\mathbb{R}^2\backslash(0,0)$. Since $u(x,y)$ and $v(x,y)$ do not satisfy the Cauchy-Riemann equations for every $(x,y)\in\mathbb{R}^2\backslash(0,0)$, $\alpha_*(-z)$ is not analytic on $\mathbb{R}+i\,\mathbb{R}^-$, i.e. condition (B1) in Lemma 4.2 is not satisfied. This shows that for fixed $\mathbf{x}\in\mathbb{R}^3$ $\mathcal{F}(\alpha_*(\omega))(t)$ and $\mathcal{F}(e^{-\alpha_*(\omega)\,|\mathbf{x}|})(t)$ cannot be causal and concludes the proof. $\square$

5 Causality analysis of Szabo's wave equation

The Green function of Szabo's equation (11) for $\gamma > 0$, $\gamma \notin \mathbb{N}$ is given by (1) with the following attenuation law and phase speed:

$$\alpha(\omega) = \mathrm{Re}(\alpha_*(\omega)) \qquad \text{and} \qquad \frac{1}{c(\omega)} - \frac{1}{c_0} = -\frac{\mathrm{Im}(\alpha_*(\omega))}{\omega}\,,$$

where

$$\alpha_*(\omega) = \frac{1}{c_0}\sqrt{(-i\,\omega)^2 + 2\,\alpha_0\,c_0\,\frac{(-i\,\omega)^{\gamma+1}}{\cos\left(\frac{\pi}{2}\,\gamma\right)}} + i\,\frac{\omega}{c_0} \tag{31}$$

(cf. dispersion relation above equation (11).) Here the square root is understood as the primitive square root, since $\alpha(\omega)$ has to be positive. Since for $\gamma \in (0,1)$

$$\alpha_*(\omega) \approx \frac{(-i\,\omega)^\gamma}{\cos\left(\frac{\pi}{2}\,\gamma\right)} \qquad \text{for } |\omega| >> 1$$

and for $\gamma > 1$, $\gamma \notin \mathbb{N}$

$$\alpha_*(\omega) \approx \frac{(-i\,\omega)^\gamma}{\cos\left(\frac{\pi}{2}\,\gamma\right)} \qquad \text{for } |\omega| << 1\,,$$

Szabo's model is a high frequency approximation of equation (9) if $\gamma \in (0,1)$ and as a small frequency approximation of equation (9) if $\gamma > 1$, $\gamma \notin \mathbb{N}$.

The next Proposition investigates the causality behaviour of Szabo's equation.

Proposition 5.1. *Let $\gamma \in \mathbb{R}^+\backslash\mathbb{N}$. The Green function G_α of Szabo's equation (11) has finite front speed only if $\gamma \in (0,1)$. For $\gamma \in (0,1)$, the front speed of G_α is c_0.*

Proof. For the proof let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed. The Green function $G(\mathbf{x},t)$ of wave equation (11) has a front speed $\leq c_0 < \infty$, if $G(\mathbf{x}, t + \frac{|\mathbf{x}|}{c_0})$ is causal, i.e. if $\alpha_*(\omega)$ defined by (31) satisfies Lemma 4.2. For convenience we set $\tilde{\alpha}_0 := \frac{\alpha_0}{\cos\left(\frac{\pi}{2}\,\gamma\right)}$.

i) First we prove the Proposition for $\gamma \in (0,1)$. Since $\sqrt{(i\,z)^2 + 2\,\tilde{\alpha}_0\,c_0\,(i\,z)^{\gamma+1}}$ (primitive square root) maps $\mathbb{R} + i\,\mathbb{R}^-$ analytically into $\mathbb{R} + i\,\mathbb{R}^-$, $\alpha_*(-z)$ maps $\mathbb{R} + i\,\mathbb{R}^-$ analytically into $\mathbb{R} + i\,\mathbb{R}^-$. This proves condition (B1).

Let $z = z_1 + i\,z_2$ with $z_1 \in \mathbb{R}$, $z_2 \in \mathbb{R}^-$ and

$$B(z) := \sqrt{1 + 2\,\tilde{\alpha}_0\,c_0\,\left(\frac{1}{i\,z}\right)^{1-\gamma}} - 1\,,$$

then $\mathrm{Re}\,(\alpha_*(-z)) = \mathrm{Re}\,(i\,z\,B(z)) = -z_1\,\mathrm{Im}\,(B(z)) - z_2\,\mathrm{Re}\,(B(z))$. Moreover, for Proposition 5.1 property (B2) in Lemma 4.2 reads as follows: there exist constants $\epsilon > 0$, $C > 0$ and $N > 0$ such that for all $z \in \mathbb{R} + i\,(-\infty, -\epsilon)$:

$$\frac{1}{c_0}\,\left(z_1\,\mathrm{Im}\,(B(z)) + z_2\,\mathrm{Re}\,(B(z))\right) \leq C + N\,\log(1 + |z|)\,. \tag{32}$$

We prove this inequality by showing that its left hand side is always negative or zero.
a) First we prove that $\mathrm{Re}\,(B(z)) \geq 0$ if $z_2 \in \mathbb{R}^-$, which implies $z_2\,\mathrm{Re}\,(B(z)) \leq 0$ if $z_2 \in \mathbb{R}^-$. From $\gamma \in (0,1)$ and $z_2 < 0$, it follows that $(i\ z)^{1-\gamma}$ has positive real part. Since the inversion of a complex number with positive real part yields a complex number with positive real part, $1 + 2\,\tilde{\alpha}_0\,c_0\,\left(\frac{1}{i\,z}\right)^{1-\gamma}$ has a real part ≥ 1. This implies that $B(z)$ has positive real part.
b) Now we show that $z_1\,\mathrm{Im}\,(B(z)) \leq 0$ if $z_2 \in \mathbb{R}^-$ and $z_1 \in \mathbb{R}$. Let $z_1 > 0$, then the imaginary part of $(i\,z)^{1-\gamma}$ is positive and since the inversion of a complex number with positive imaginary part yields a complex number with negative imaginary part, $1 + 2\,\tilde{\alpha}_0\,c_0\,\left(\frac{1}{i\,z}\right)^{1-\gamma}$ has negative imaginary part. This implies that $B(z)$ has negative imaginary part and therefore $z_1\,\mathrm{Im}\,(B(z)) \leq 0$ if $z_2 \in \mathbb{R}^-$ and $z_1 > 0$. Now let $z_1 < 0$. Then the imaginary part of $(i\,z)^{1-\gamma}$ is negative and since the inversion of a complex number with negative imaginary part yields a complex number with positive imaginary part, $1 + 2\,\tilde{\alpha}_0\,c_0\,\left(\frac{1}{i\,z}\right)^{1-\gamma}$ has positive imaginary part. This implies that $B(z)$ has positive imaginary part and therefore $z_1\,\mathrm{Im}\,(B(z)) \leq 0$ if $z_2 \in \mathbb{R}^-$ and $z_1 < 0$. Clearly, if $z_1 = 0$ then $z_1\,\mathrm{Im}\,(B(z)) = 0$. In summary we have proven that the left hand side of (32) is smaller or equal to zero and thus the inequality holds.
ii) The second part of the theorem is first proven for $\gamma \in (1,3)\backslash\{2\}$ and then for $\gamma > 3$ with $\gamma \notin \mathbb{N}$.
a) Let $\gamma \in (1,3)\backslash\{2\}$. (Indeed the following arguments hold for any $\gamma \in (4\,n + 1, 4\,n+3)\backslash\mathbb{N}$ with $n \in \mathbb{N}_0$.) Then $\tilde{\alpha}_0 < 0$ and for $z = i\,z_2$ with $z_2 < 0$, condition (32) simplifies to

$$-\frac{z_2}{c_0} - \frac{1}{c_0}\,\mathrm{Re}\left(\sqrt{z_2^2 + 2\,\tilde{\alpha}_0\,c_0\,(-z_2)^{\gamma+1}}\right) \leq C + N\,\log(1+|z_2|)\,.$$

Because $\gamma > 1$ and $\tilde{\alpha}_0 < 0$, the term under the root is negative for sufficently large $-z_2$ and thus the real part of the root vanishes, which leads to the contradiction $\frac{|z_2|}{c_0} \leq C + N\,\log(1+|z_2|)$. Therefore condition (B2) cannot be valid for any $\gamma \in (1,3)\backslash\{2\}$.
b) Now let $\gamma > 3$ and $z = z_1 + i\,z_2$ with $z_1 \in \mathbb{R}$ and $z_2 < 0$. We recall that $B(z) = \sqrt{1 + 2\,\tilde{\alpha}_0\,c_0\,(i\,z)^{\gamma-1}} - 1$. Let $\tilde{z}(r) = r\,e^{i\,\varphi}$ with $r > 0$ and $\varphi := \frac{\pi}{\gamma-1} - \frac{\pi}{2}$. Since $\gamma > 3$, we have $\varphi \in (-\frac{\pi}{2}, 0)$ and thus $\tilde{z}_1 > 0$ and $\tilde{z}_2 < 0$. Since $(\gamma-1)\,(\varphi + \frac{\pi}{2}) = \pi$ we have $\cos((\gamma-1)\,(\varphi+\frac{\pi}{2})) = -1$ and $\sin((\gamma-1)\,(\varphi+\frac{\pi}{2})) = 0$. This shows that

$$1 + 2\,\tilde{\alpha}_0\,c_0\,(i\,\tilde{z}(r))^{\gamma-1} < 0 \qquad \text{for sufficiently large } r. \tag{33}$$

Therefore $\mathrm{Re}(B(\tilde{z})) = -1 < 0$, which together with $\tilde{z}_2 < 0$ implies $\tilde{z}_2\,\mathrm{Re}(B(\tilde{z})) = -\tilde{z}_2 > 0$. This shows that the first left hand side term of (32) is positive. Moreover, (33) implies that $\mathrm{Im}(B(\tilde{z})) = \mathrm{Im}(B(\tilde{z})+1) > 0$ for sufficiently large $-\tilde{z}_2$. From this together with $\tilde{z}_1 > 0$ we obtain $\tilde{z}_1\,\mathrm{Im}(B(\tilde{z})) > 0$, which shows that the second term on the left hand side of (32) is positive, too. If r is sufficiently large then $\mathrm{Im}(B(\tilde{z}(r)))$ is of the order of $|r|^{\frac{\gamma-1}{2}}$ with $\gamma > 3$ which cannot be bounded by $C + N\,\log(1+r)$. Hence condition (B2) cannot be true for $\gamma > 3$. In summary we have shown that the Green function of Szabo's equation cannot be causal for $\gamma \in (1,\infty)\backslash\mathbb{N}$.
iii) Now we show that the front speed of G_α is c_0, if $\gamma \in (0,1)$. Since $G(\mathbf{x}, t + \frac{|\mathbf{x}|}{c_0})$

is causal, the front wave speed v_α satisfies $v_\alpha \leq c_0$. If v_α at $\mathbf{x}$ is smaller than c_0, then $G(\mathbf{x}, t + \frac{|\mathbf{x}|}{c_0} + \delta\,|\mathbf{x}|)$ is causal for some $\delta > 0$. This means that there exist constants $\epsilon > 0$, $C > 0$ and $N > 0$ such that for all $z \in \mathbb{R} + i\,(-\infty, -\epsilon)$ (cf. (28) in Corollary 4.4):

$$-\mathrm{Re}(iz\,(B(z) - \delta)) \leq C + N\,\log(1 + |z|)\,,$$

since

$$\mathrm{Re}(\alpha_*(-z)) + \mathrm{Re}(i\,(-z)\,\delta) = -\mathrm{Re}(iz\,(B(z) - \delta))\,.$$

For the setting $z = i\,z_2$ with $z_2 < 0$, we obtain

$$\mathrm{Re}\left(|z_2|\,\left[\delta + 1 - \sqrt{1 + 2\,\tilde{\alpha}_0\,c_0\,\left|\frac{1}{z_2}\right|^{1-\gamma}}\right]\right) \leq C + N\,\log(1 + |z_2|)\,,$$

which cannot be true if $|z_2|$ is sufficiently large. Hence we conclude that $\delta = 0$. This proves that the wave front speed is c_0 and concludes the proof. □

6 Causality analysis of the thermo-viscous wave equation

The operator of the *thermo-viscous wave equation* (cf. e.g. [7])

$$P(D) := \left(\mathrm{Id} + \tau_0\,\frac{\partial}{\partial t}\right)\,\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}$$

has order 3 and *principal part* $P_3(\mathbf{X}) = \tau_0\,t\,\sum_{j=1}^3 x_j$, where $\mathbf{X} := (t, x_1, x_2, x_3)^T \in \mathbb{R}^4$. Since $P_3(\mathbf{N}) = 0$ for $\mathbf{N} := (1, 0, 0, 0)^T$, the plane $\{\mathbf{X} \in \mathbb{R}^4\,|\,\langle \mathbf{X}, \mathbf{N}\rangle = 0\}$ in $\mathbb{R}^4$ is *characteristic* with respect to P(D). According to Theorem 8.6.7 in [5] the thermo-viscous wave equation with vanishing source term has a solution $p_{tv} \in C^\infty(\mathbb{R}^4)$ such that $\mathrm{supp}(p_{tv}) = \mathbb{R}^3 \times \mathbb{R}_0^-$. This shows that the Green function of the thermo-viscous wave equation is not uniquely determined. (The existence is guaranteed, since $P(\mathbf{X}) := (1 + \tau_0\,t)\,\sum_{j=1}^3 x_j^2 - \frac{t^2}{c_0^2}$ is not the zero polynomial.) Theorem 6.2 below shows that the Green function of the thermo-viscous wave equation cannot have a finite front speed and that a solution of the thermo-viscous wave equation depends on its history. This explains why its Cauchy problem has no unique solution. For this theorem and Theorem 7.1 we need the following lemma.

Lemma 6.1. *Let $\mathcal{S}_+ = \{f \in \mathcal{S}(\mathbb{R})\,|\,supp(f) \subseteq \mathbb{R}_0^+\}$ and $\gamma \in (1, 2]$. The time-convolution operator $T_\gamma^{\frac{1}{2}}$ defined by the kernel*

$$k_{T_\gamma^{\frac{1}{2}}}(t) := \frac{1}{\sqrt{2\,\pi}}\,\mathcal{F}\left(\frac{1}{\sqrt{1 + (-i\,\tau_0\,\omega)^{\gamma-1}}}\right)(t)$$

is an isomorphism of $\mathcal{S}_+$. Here the square root is understood as the primitive square root. The inverse of $T_\gamma^{\frac{1}{2}}$ is the time-convolution operator $L_\gamma^{\frac{1}{2}}$ with the kernel

$$k_{L_\gamma^{\frac{1}{2}}}(t) := \frac{1}{\sqrt{2\,\pi}}\,\mathcal{F}\left(\sqrt{1 + (-i\,\tau_0\,\omega)^{\gamma-1}}\right)(t)\,.$$

Again the square root is understood as the primitive square root.

Proof. First we show that the kernel of $T_\gamma^{\frac{1}{2}}$ and $L_\gamma^{\frac{1}{2}}$ have supports in $[0,\infty)$. Let $B(z) := \sqrt{1+(i\,\tau_0\,z)^{\gamma-1}}$ and $z \in \mathbb{R} + i\,\mathbb{R}^-$. Then

$$\mathrm{supp}\left(k_{T_\gamma^{\frac{1}{2}}}\right) \subseteq [0,\infty) \quad \text{and} \quad \mathrm{supp}\left(k_{L_\gamma^{\frac{1}{2}}}\right) \subseteq [0,\infty)\,,$$

if and only if $\hat{u}_1(z) := B(z)$ and $\hat{u}_2(z) := \frac{1}{B(z)}$ satisfy Theorem 4.1. Since $\sqrt{1+(i\,\tau_0\,z)^{\gamma-1}}$ maps $\mathbb{R}+i\,\mathbb{R}^-$ analytically into $[1,\infty)+i\,\mathbb{R}$ and $1+(i\,\tau_0\,z)^{\gamma-1}$ cannot vanish on $\mathbb{R}+i\,\mathbb{R}^-$, $\hat{u}_1(z)$ and $\hat{u}_2(z)$ map $\mathbb{R}+i\,\mathbb{R}^-$ analytically into $[1,\infty)+i\,\mathbb{R}$. Therefore property (A1) in Lemma 4.2 is satisfied. We have for all $z \in \mathbb{R}+i\,\mathbb{R}^-$ with $|z| >> 1$:

$$|B(z)| \leq C_1\,|z|^{(\gamma-1)/2} \quad \text{and} \quad \left|\frac{1}{B(z)}\right| \leq C_2\,\left(\frac{1}{|z|}\right)^{(\gamma-1)/2} \qquad (\gamma \in (1,2])$$

for some constants $C_1,\, C_2 > 0$. Therefore property (A2) in Theorem 4.1 is satisfied for $\hat{u}_1(z)$ and $\hat{u}_2(z)$, which proves that the kernels of $T_\gamma^{\frac{1}{2}}$ and $L_\gamma^{\frac{1}{2}}$ have support in $[0,\infty)$.

Since

$$T_\gamma^{\frac{1}{2}}\,f \in \mathcal{D}'_+ \qquad \text{for every } f \in \mathcal{S}_+\,,$$

and

$$\mathcal{F}^{-1}\left(T_\gamma^{\frac{1}{2}}\,f\right)(\omega) = \frac{\check{f}(\omega)}{\sqrt{1+(-i\,\tau_0\,\omega)^{\gamma-1}}} \in \mathcal{S} \qquad \text{for every } f \in \mathcal{S}_+\,,$$

the convolution operator $T_\gamma^{\frac{1}{2}}$ is well-defined on $\mathcal{S}_+$. Since $\mathcal{S}$ is invariant under multiplication by a polynomial, it follows analogously that $L_\gamma^{\frac{1}{2}}$ maps $\mathcal{S}_+$ into $\mathcal{S}_+$ and is well-defined. Since the Fourier transform is an isomorphism on $\mathcal{S}$ and

$$\mathcal{F}^{-1}\left(L_\gamma^{\frac{1}{2}}\,T_\gamma^{\frac{1}{2}}\,f\right) = \check{f} = \mathcal{F}^{-1}\left(T_\gamma^{\frac{1}{2}}\,L_\gamma^{\frac{1}{2}}\,f\right) \qquad \text{for every } \check{f} \in \mathcal{S}\,,$$

it follows that $L_\gamma^{\frac{1}{2}} : \mathcal{S}_+ \to \mathcal{S}_+$ is the inverse of $T_\gamma^{\frac{1}{2}} : \mathcal{X}_+ \to \mathcal{S}_+$ and $T_\gamma^{\frac{1}{2}}$ is an isomorphism of $\mathcal{S}_+$. □

With the help of the Laplace transform table in [4] (cf. Appendix 2), the kernel of $T_2^{\frac{1}{2}}$ can be calculated as follows:

$$\begin{aligned} k_{T_2^{\frac{1}{2}}}(t) &= \frac{1}{\sqrt{2\,\pi}}\,\mathcal{F}\left(\frac{1}{\sqrt{1-i\,\tau_0\,\omega}}\right)(t) = \sqrt{2\,\pi}\,\mathcal{L}^{-1}\left(\frac{1}{\sqrt{1+\tau_0\,p}}\right)(t) \\ &= 2\,\sqrt{\frac{\pi}{\tau_0\,t}}\,e^{-\frac{t}{\tau_0}}\,H(t)\,, \end{aligned}$$

where $H(t)$ denotes the Heaviside function. This shows that for $\gamma = 2$ the kernel of $T_2^{\frac{1}{2}}$ decreases exponentially. In the following we denote $T_\gamma^{\frac{1}{2}}\,T_\gamma^{\frac{1}{2}}$ by T_γ.

Proposition 6.2. *Let $\tau_0,\, c_0 > 0$ be constants and $A(\omega) := 1 + \sqrt{1+(\tau_0\,\omega)^2}$ for any $\omega \in \mathbb{R}$. The Green function of the thermo-viscous wave equation*

$$(34) \qquad \left(I + \tau_0\,\frac{\partial}{\partial t}\right)\,\nabla^2 G(\mathbf{x},t) - \frac{1}{c_0^2}\,\frac{\partial^2 G(\mathbf{x},t)}{\partial t^2} = -\delta(\mathbf{x})\,\delta(t) \qquad \mathbb{R}^3 \times \mathbb{R}$$

cannot have a finite front speed and is given by $G = T_2\, p_\alpha$, where p_α is defined as in (1) with attenuation law

$$\alpha(\omega) = \frac{\tau_0}{\sqrt{2\,A(\omega)}\,(A(\omega)-1)}\,\frac{\omega^2}{c_0} \tag{35}$$

and phase speed

$$c(\omega) = \frac{\sqrt{2}\,(A(\omega)-1)}{\sqrt{A(\omega)}}\,c_0\,. \tag{36}$$

Proof. Applying the inverse Fourier transform to the thermo-viscous wave equation yields

$$\begin{aligned} &\nabla^2 \check{G}(\mathbf{x},\omega) + k^2(\omega)\,\check{G}(\mathbf{x},\omega) = -\frac{\delta(\mathbf{x})}{\sqrt{2\,\pi}\,(1-i\,\tau_0\,\omega)} \qquad \text{with} \\ &k(\omega) := \frac{\pm\,\omega}{c_0\,\sqrt{1-i\,\tau_0\,\omega}}\,. \end{aligned} \tag{37}$$

This problem has the solution $\check{G}(\mathbf{x},\omega) = \frac{1}{\sqrt{2\,\pi}\,(1-i\,\tau_0\,\omega)}\,\frac{e^{i\,k(\omega)\,|\mathbf{x}|}}{4\,\pi\,|\mathbf{x}|}$, where the square root of $1-i\,\tau_0\,\omega$ is understood as the root with positive real part. We assume that G satisfies the causality requirement (14) which in particular implies that $\mathrm{supp}(G(\mathbf{x},\cdot)) \in [0,\infty)$. Then the Green function can be written as follows

$$G(\mathbf{x},t) = T_2\left(\frac{1}{\sqrt{2\,\pi}}\int\limits_{\mathbb{R}} \frac{e^{-i\,(\omega\,t-k(\omega)\,|\mathbf{x}|)}}{4\,\pi\,|\mathbf{x}|}\,\mathrm{d}\omega\right) =: T_2\,p_\alpha(\mathbf{x},t)\,, \tag{38}$$

where T_2 denotes the time-convolution operator in Lemma 6.1 for $\gamma = 2$. The last identity is equivalent to

$$\left(\mathrm{Id} + \tau_0\,\frac{\partial}{\partial t}\right) G(\mathbf{x},t) = p_\alpha(\mathbf{x},t)\,,$$

which shows that p_α has finite front speed if and only if G has finite front speed. Let $c_1 > 0$ be arbitrary but fixed. We prove a contradiction by showing that $p_\alpha(\mathbf{x}, t + \frac{|\mathbf{x}|}{c_1})$ cannot be a causal distribution for any $\mathbf{x} \in \mathbb{R}^3$. Comparing (38) and (1) shows that $\alpha(\omega) = \mathrm{Im}(k(\omega))$ and $\frac{1}{c(\omega)} = \frac{\mathrm{Re}(k(\omega)}{\omega}$. Since $\alpha > 0$ is required, the imaginary part of $k(\omega)$ must be positive and therefore we choose the positive sign for k, i.e.

$$k(\omega) = \frac{\omega}{c_0}\,\frac{1}{A(\omega)-1}\left(\sqrt{\frac{A(\omega)}{2}} + i\,\frac{\tau_0\,\omega}{\sqrt{2\,A(\omega)}}\right) \tag{39}$$

with $A(\omega) := 1 + \sqrt{1+(\tau_0\,\omega)^2}$. From this we get the attenuation law (35) and the phase speed (36). If $p_\alpha(\mathbf{x}, t+\frac{|\mathbf{x}|}{c_1})$ is causal, then property (B2) in Lemma 4.2 must be satisfied for $-i\,k(\omega) + i\,\frac{\omega}{c_1}$, i.e. for each $\mathbf{x} \in \mathbb{R}^3$ there exist constants $\epsilon > 0$, $C > 0$ and $N > 0$ such that for all $z \in \mathbb{R} + i\,(-\infty, -\epsilon)$:

$$\mathrm{Re}(i\,k(-z)) + \mathrm{Re}\left(i\,\frac{z}{c_1}\right) \le C + N\,\log(1+|z|)\,.$$

For $z = i\,z_2$ with $z_2 < 0$ we get

$$\frac{1}{\sqrt{1-\tau_0\,z_2}}\,\frac{z_2}{c_0} - \frac{z_2}{c_1} \le C + N\,\log(1+|z_2|)\,,$$

which cannot be true for sufficiently large $-z_2$ and finite $c_1 > 0$. Hence property (B2) cannot be satisfied for α_* of the wave p_α if c_1 is finite. This contradiction proves that the front speed of G cannot be finite and concludes the proof. □

7 Causal wave equations obeying attenuation power laws with $\gamma \in (1,2]$ for small frequencies

As we have seen in Section 4, the frequency power law for $\gamma \ge 1$ cannot hold. Now we present admissible attenuation laws that permit approximate frequency power laws with $\gamma \in (1,2]$ for sufficiently small frequencies.

For given constants $\gamma \in (1,2]$, $0 < c_1 < \infty$ and $\tau_0 \ge 0$ we define

$$\alpha_*(\omega) := \frac{-i\,\omega}{c_1\,\sqrt{1+(-i\,\tau_0\,\omega)^{\gamma-1}}}\,, \tag{40}$$

where the square root is understood as the primitive square root. This implies for the attenuation law:

$$\alpha(\omega) \approx \alpha_0\,|\tau_0\,\omega|^\gamma \quad \text{with} \quad \alpha_0 = \frac{\sin(\frac{\pi}{2}\,(\gamma-1))}{\tau_0\,c_1}$$

for sufficiently small frequencies. Moreover, let $T_\gamma^{\frac{1}{2}}$ and $L_\gamma^{\frac{1}{2}}$ be defined as in Lemma 6.1 and let the operators D_* and $D_{*,|\mathbf{x}|}$ be defined as in Section 3. Then

$$D_* = \frac{1}{c_1}\,T_\gamma^{\frac{1}{2}}\,\frac{\partial}{\partial t} \qquad \text{and} \qquad D_{*,|\mathbf{x}|} = 0$$

and wave equation (22) (with v_B replaced by c_0) reads as follows

$$\nabla^2 p_\alpha - \left[\mathrm{Id} + \frac{1}{c_1}\,T_\gamma^{\frac{1}{2}}\right]^2\,\frac{1}{c_0^2}\,\frac{\partial^2 p_\alpha}{\partial t^2} = -f\,. \tag{41}$$

For $\gamma = 1$ we obtain the classical wave equation without damping and for $\gamma = 2$, we obtain a modified thermo-viscous wave equation. Since $\mathrm{Re}(\alpha_*)$ is equal to (35) for $\gamma = 2$, the modified thermo-viscous wave equation obeys for $\gamma = 2$ the same attenuation law as the thermo-viscous wave equation (if the source term f is replaced by $L_2\,f$).

Proposition 7.1. *The Green function of wave equation (41) has finite and constant wave front speed c_0.*

Proof. For the proof let $\mathbf{x} \in \mathbb{R}^3$ be arbitrary but fixed. The Green function $G(\mathbf{x},t)$ of wave equation (41) has a front speed $\le c_0 < \infty$, if $G_{tv}(\mathbf{x}, t + \frac{|\mathbf{x}|}{c_0})$ is causal, i.e. if $\alpha_*(\omega)$ defined by (40) satisfies Lemma 4.2. We recall that the square root in the definition of α_* is understood as the primitive square root.

According to the proof of Lemma 6.1 $\frac{\alpha_*(-z)}{i\,z}$ satisfies property (B1) and thus $\alpha_*(-z)$ satisfies property (B1), too.

Property (B2) in Lemma 4.2 reads as follows: there exist constants $\epsilon > 0$, $C > 0$ and $N > 0$ such that for all $z \in \mathbb{R} + i\,(-\infty, -\epsilon)$:

$$-\mathrm{Re}\,(\alpha_*(-z)) = -\mathrm{Re}\left(\frac{i\,z}{c_1\,\sqrt{1+(i\,\tau_0\,z)^{\gamma-1}}} \right) \leq C + N\,\log(1+|z|)\,. \tag{42}$$

Therefore (B2) is satisfied if $-\mathrm{Re}\,(\alpha_*(-z)) \leq 0$ for each $z \in \mathbb{R} + i\,\mathbb{R}^-$. Let $B(z) := c_1\,\sqrt{1+(i\,\tau_0\,z)^{\gamma-1}}$ and $z = z_1 + i\,z_2 \in \mathbb{R} + i\,\mathbb{R}^-$ then

$$-\mathrm{Re}\,(\alpha_*(-z)) = z_1\,\mathrm{Im}\left(B^{-1}(z)\right) + z_2\,\mathrm{Re}\left(B^{-1}(z)\right)\,.$$

Since $\gamma \in (1,2]$ and $z_2 < 0$, it follows that $(i\,\tau_0\,z)^{\gamma-1}$ has positive real part and thus $B(z)$ has also positive real part. The inversion of a complex number with positive real part yields a complex number with positive real part and thus $\mathrm{Re}\left(B^{-1}(z)\right) > 0$ for any $z_2 < 0$. This proves that

$$z_2\,\mathrm{Re}\left(B^{-1}(z)\right) < 0 \qquad \text{for any } z \in \mathbb{R} + i\,\mathbb{R}^-\,. \tag{43}$$

For $z_1 > 0$ the imaginary part of $(i\,\tau_0\,z)^{\gamma-1}$ is positive and thus $\mathrm{Im}\,(B(z)) > 0$. The inversion of a complex number with positive imaginary part yields a complex number with negative imaginary part and thus $\mathrm{Im}\left(B^{-1}(z)\right) < 0$. Therefore we infer that $z_1\,\mathrm{Im}\left(B^{-1}(z)\right) < 0$ for any $z \in \mathbb{R}^+ + i\,\mathbb{R}^-$ with $z_1 > 0$. For $z_1 < 0$ the imaginary part of $(i\,\tau_0\,z)^{\gamma-1}$ is negative and thus $\mathrm{Im}\,(B(z)) < 0$. Since the inversion of a complex number with negative imaginary part yields a complex number with positive imaginary part we conclude that $\mathrm{Im}\left(B^{-1}(z)\right) > 0$. Hence $z_1\,\mathrm{Im}\left(B^{-1}(z)\right) < 0$ for any $z \in \mathbb{R}^- + i\,\mathbb{R}^-$ with $z_1 < 0$. For $z_1 = 0$ we get $z_1\,\mathrm{Im}\left(B^{-1}(z)\right) = 0$. In summary we have proven that

$$z_1\,\mathrm{Im}\left(B^{-1}(z)\right) \leq 0 \qquad \text{for any } z \in \mathbb{R} + i\,\mathbb{R}^-\,. \tag{44}$$

(43) and (44) imply that the left hand side of (42) is always non-positive and therefore (42) is true. This shows that $G(\mathbf{x},t)$ has a front speed $\leq c_0 < \infty$.

Now we show that the front speed of G is equal to c_0. If the front speed is $v_\alpha(\mathbf{x}) < c_0$ for any $\mathbf{x} \in \mathbb{R}^3$, then (42) must hold for

$$\alpha_*(-z) := \alpha_*(-z) + i\,\epsilon\,(-z)\,.$$

For $z := i\,z_2$ with sufficiently large $-z_2$ we obtain

$$-\mathrm{Re}\,(\alpha_*(-z)) = \frac{z_2}{c_1\,\sqrt{1+(-\tau_0\,z_2)^{\gamma-1}}} + \epsilon\,(-z_2)\,,$$

which is positive and of the order $|z_2|$. This shows that condition (B2) can only be true if $\epsilon = 0$. This concludes the proof. □

Remark 7.2. *For $\gamma = 2$ let G_{c_0} denote the solution of wave equation (41) and let G_{tv} denote the solution of the thermo-viscous wave equation (34) with c_0 replaced by c_1. Then one can show that*

$$\lim_{c_0\to\infty} L\,G_{c_0}(\mathbf{x},t) = G_{tv}(\mathbf{x},t) \qquad \textit{for each } \mathbf{x} \in \mathbb{R}^3 \textit{ and } t \in \mathbb{R}^+\,,$$

which shows again that the front speed of G_{tv} is infinite.

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