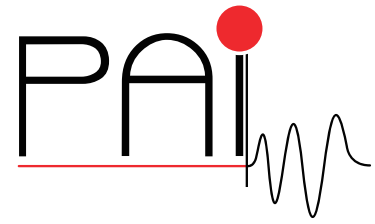


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## **Spatial Resolution in Photoacoustic Tomography: Effects of Detector Size and Detector Bandwidth**

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# Spatial Resolution in Photoacoustic Tomography: Effects of Detector Size and Detector Bandwidth

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High spatial resolution is one of the major aims in tomographic imaging. Two main factors limiting the resolution of photo- and thermoacoustic tomography are the detector size and the finite bandwidth of the ultrasound detection system. In this paper we present a quantitative analysis of those effects for “approximate point detectors” and for “approximate line detectors”.

**Keywords.** Photoacoustic tomography; thermoacoustic; tomography; image reconstruction; wave equation; resolution; bandwidth.

**AMS classifications.** 44A12, 65R32, 35L05, 92C55.

## 1 Introduction and main results

Photoacoustic tomography (PAT) is new imaging modality for visualizing the optical absorption coefficient of a probe. It has proven great promise for a variety of biomedical applications, such as imaging of animals, early cancer diagnostics, and imaging of vasculature (see [9, 21, 27, 30, 39, 40, 41, 44]). PAT is based on the excitation of acoustic pressure waves by illuminating a probe with short light pulses. The acoustic pressure is recorded outside of the object and used to reconstruct an image of the probe. We refer the reader to [37, Section 1.5] for a detailed mathematical description.

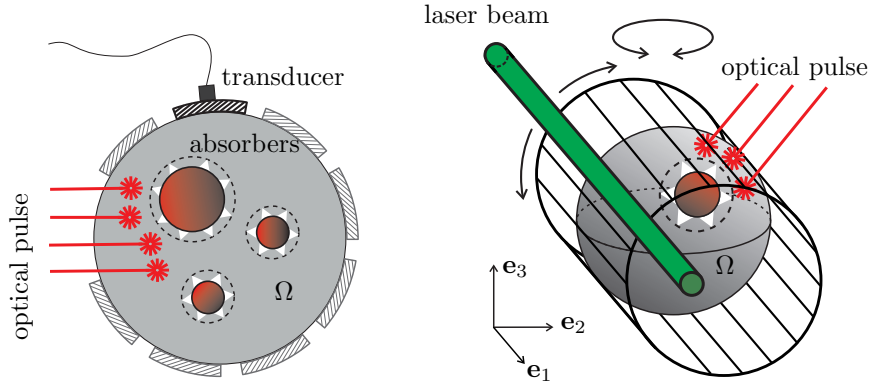


Figure 1: Photoacoustic tomography with piezoelectric transducer as an “approximate point detector” (left) and a laser beam as “approximate line detector” (right).

If we assume that the probe is acoustically homogeneous, the induced acoustic pressure  $p : \mathbb{R}^3 \times (0, \infty) \rightarrow \mathbb{R}$  satisfies

$$(\partial_t^2 - \Delta) p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^3 \times (0, \infty), \quad (1a)$$

$$p(\mathbf{x}, 0) = f(\mathbf{x}), \quad \partial_t p(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad (1b)$$

where  $\Delta$  denotes the Laplacian with respect to the spatial variable  $\mathbf{x}$  and  $\partial_t$  is the derivative with respect to the temporal variable  $t$ . The goal of PAT is to reconstruct the initial pressure  $f$  (representing the probe) from measurements of  $p(\mathbf{x}, t)$  taken outside of  $\Omega$ . We assume throughout that initial pressure  $f$  is an element of the space  $C_c^\infty(\Omega)$  of all smooth functions with compact support in  $\Omega$ .

We denote by  $\mathcal{W}^{3D} : C_c^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3 \times (0, \infty))$  the operator that maps the compactly supported  $f \in C_c^\infty(\mathbb{R}^3)$  to the solution of (1a), (1b).

## 1.1 Approximate point detectors

The classical approach in PAT is to assume that point-wise measured data

$$(\mathbf{P} f)(\mathbf{x}, t) := g(\mathbf{x}, t) := (\mathcal{W}^{3D} f)(\mathbf{x}, t), \quad \text{with } (\mathbf{x}, t) \in \partial\Omega \times (0, \infty),$$

are given (see [22, 23, 44]). The operator  $\mathbf{P} : C_c^\infty(\Omega) \rightarrow C^\infty(\partial\Omega \times (0, \infty))$  that maps the initial data of the wave equation to the data measured by an ideal point detector can be inverted uniquely (see [2, 3]) and stably (see [26, 32, 46]). Analytic expressions for its inverse

$$\mathbf{P}^{-1} : \text{ran}(\mathbf{P}) \subset C^\infty(\partial\Omega \times (0, \infty)) \rightarrow C_c^\infty(\Omega)$$

via eigenfunction expansions are derived in [1, 25]. (Here  $\text{ran}(\mathbf{P}) := \{\mathbf{P} f : f \in C_c^\infty(\Omega)\}$  denotes the range of  $\mathbf{P}$ .) Efficient reconstruction methods for arbitrary  $\Omega$  are based on time reversal (see [7, 8, 11, 18, 19, 38]). In the case that  $\Omega$  is a ball, filtered back-projection type inversion formulas are derived in [11, 24, 29, 43].

In practical applications, the detection system has a finite bandwidth. Moreover, standard ultrasound transducers, which integrate the pressure over its surface, are used to approximate point data (see [42]). We assume for simplicity that the transducer surface is part of the measurement surface. Then, the data measured by approximate point detectors are given by (see [42])

$$(\mathbf{P}_{\phi,w} f)(\mathbf{x}, t) = \left[ \phi *_t \int_{\partial\Omega} w(\mathbf{x}, \mathbf{x}') (\mathcal{W}^{3D} f)(\mathbf{x}', \cdot) dS(\mathbf{x}') \right] (t),$$

for  $(\mathbf{x}, t) \in \partial\Omega \times (0, \infty)$ . (2)

Here  $w(\mathbf{x}, \cdot)$  represents the sensitivity of the detector located at position  $\mathbf{x} \in \partial\Omega$ ,  $\phi(t)$  denotes the impulse response function of the ultrasound detection system, and  $*_t$  is the convolution with respect to the temporal variable  $t$ .

Inexact knowledge of  $w$  and  $\phi$ , as well as the ill-posedness of deblurring problems make it impossible to stably reconstruct the function  $f$  from the data in (2). It is therefore common to apply the exact inverse of  $\mathbf{P}$  to the data  $\mathbf{P}_{\phi,w} f$ . This results in a blurred reconstruction, where the blurring depends on the detector size and the detector bandwidth.

For the following result we assume that  $\Omega = B_R$  is a ball of radius  $R$  and that the detector surface is rotationally symmetric (see Figure 1), that is, we have  $w(\mathbf{x}, \mathbf{x}') = w_{\text{point}}(|\mathbf{x} - \mathbf{x}'|)$  for some one dimensional function  $w_{\text{point}}$ .

**Theorem 1.1.** *Let  $f \in C_c^\infty(B_R)$ , and let  $\phi, w_{\text{point}} : \mathbb{R} \rightarrow \mathbb{R}$  be even functions such that  $\mathbf{x} \in \mathbb{R}^3 \mapsto \phi'(|\mathbf{x}|)/|\mathbf{x}|$  and  $\mathbf{x} \in \partial B_R \mapsto w_{\text{point}}(|\mathbf{x} - \mathbf{x}_0|)$  are absolutely integrable, for some  $\mathbf{x}_0 \in \partial B_R$ . Moreover, assume that  $\text{supp}(\phi) \subset [-\tau, \tau]$ , where  $\tau := \text{dist}(\text{supp}(f), \partial B_R)$  and  $\text{supp}(\phi) := \{t : \phi(t) \neq 0\}$ .*

Then  $\mathbf{P}_{\phi,w} f \in \text{ran}(\mathbf{P})$  and

$$(\mathbf{P}^{-1} \mathbf{P}_{\phi,w} f)(\mathbf{x}) = \left[ \Phi_{\text{band}} * \int_{\mathbb{R}^3} W_{\text{point}}(\cdot, \mathbf{x}') f(\mathbf{x}') d\mathbf{x}' \right] (\mathbf{x}), \quad \mathbf{x} \in B_R,$$

with the blurring kernels

$$\Phi_{\text{band}}(\mathbf{x}) := -\pi \phi'(|\mathbf{x}|)/(2|\mathbf{x}|), \quad \mathbf{x} \in \mathbb{R}^3, \quad (3)$$

$$W_{\text{point}}(\mathbf{x}, \mathbf{x}') := \frac{R^2}{|\mathbf{x}|^2} \delta(|\mathbf{x}| - |\mathbf{x}'|) w_{\text{point}}(|\mathbf{x} - \mathbf{x}'| R/|\mathbf{x}|), \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^3. \quad (4)$$

Here  $\delta$  denotes the one-dimensional delta distribution.

*Proof.* See Section 2. □

As a consequence of Theorem 1.1, the detector aperture causes blurring in the lateral direction, which becomes more severe near to the recording surface. The finite bandwidth, on the other hand, causes spatially invariant blurring.

**Remark 1.2.** In the case of an arbitrary domain, one obtains the same spatially invariant blurring kernel (4) due to the detector bandwidth (see Proposition 2.3). The blurring kernel (3) due to the detector size, however, depends to the shape of the detection surface. In Subsection 2.4 we also derive analytic expressions for the blurring kernel for a planar and cylindrical detection surface (Theorems 2.4, 2.5).

## 1.2 Approximate line detectors

In order to partly overcome the size and shape limitations of point detectors, in [6, 35] we proposed PAT with integrating line detectors, where certain integrals of  $\mathcal{W}^{3D} f$  over lines are used to reconstruct the function  $f$ . There it is assumed that

$$\Omega := \{\Phi_\theta(\mathbf{y}) = (y_1\theta_1, y_1\theta_2, y_2) : \theta = (\theta_1, \theta_2) \in S^1 \text{ and } \mathbf{y} = (y_1, y_2) \in D\} \quad (5)$$

is a volume of revolution generated by rotating a domain  $\{0\} \times D$ , with  $D \subset \mathbb{R}^2$ , around the  $(0, 0, 1)$  axis. The data measured by ideal line detectors are given by

$$\begin{aligned} (\mathbf{L}f)(\theta, \mathbf{y}, t) &:= \int_{\mathbb{R}} (\mathcal{W}^{3D} f)(y_1\theta + s\theta^\perp, y_2, t) ds \\ &= \int_{\ell_{\theta, \mathbf{y}}} (\mathcal{W}^{3D} f)(\mathbf{z}, t) d\ell(\mathbf{z}), \quad (\theta, \mathbf{y}, t) \in S^1 \times \partial D \times (0, \infty), \end{aligned}$$

where  $\ell_{\theta, \mathbf{y}} = \{\Phi_\theta(\mathbf{y}) + s(\theta^\perp, 0) : s \in \mathbb{R}\}$  is the unique line passing through  $\Phi_\theta(\mathbf{x})$  and pointing in direction  $(\theta^\perp, 0) := (-\theta_2, \theta_1, 0)$ . The operator  $\mathbf{L} : C_c^\infty(\Omega) \rightarrow C^\infty(S^1 \times \partial D \times (0, \infty))$  can be inverted uniquely and stably by means of a two step procedure described in [5, 13, 34]. In the case that  $D$  is a disc, exact inversion formulas are derived in [5, 10, 14, 24, 25, 29]. Efficient reconstruction methods for arbitrary domain  $D$  are again based on time reversal (see [7, 18, 19, 38]).

In practical applications line detector are usually implemented by optical line detection, where the pressure is integrated over the volume of a cylindrical laser beam. Thus, the actually available data are given by (see [33, 36])

$$\begin{aligned} (\mathbf{L}_{\phi, w} f)(\theta, \mathbf{y}, t) &= \left[ \phi *_t \int_{\mathbb{R}^3} w(\text{dist}(\ell_{\theta, \mathbf{y}}, \mathbf{x})) (\mathcal{W}^{3D} f)(\mathbf{x}, t) d\mathbf{x} \right] \\ &\quad \text{with } (\theta, \mathbf{y}, t) \in S^1 \times \partial D \times (0, \infty). \end{aligned}$$

Here  $w(r)$  represents the radial intensity profile of laser beam,  $\ell_{\theta, \mathbf{y}}$  is its central axis,  $\phi(t)$  is the impulse response function of the detection system, and  $\text{dist}(\ell_{\theta, \mathbf{y}}, \mathbf{x})$  denotes the distance between the line  $\ell_{\theta, \mathbf{y}}$  and the point  $\mathbf{x} \in \mathbb{R}^3$ . Again, application of  $\mathbf{L}^{-1}$  to the data  $\mathbf{L}_{\phi, w} f$  leads to a blurred reconstruction. The laser beam, however, can be made very thin, suggesting that the one dimension approximation with approximate line detectors gives less blurred images than the zero dimension approximation with approximate point detectors.<sup>1</sup>

<sup>1</sup>Note that the finite length of an integrating detector does *not* influence the resultion, see [13, 36, 37].

**Theorem 1.3.** *Let  $D \subset \mathbb{R}^2$ , let  $\Phi_\theta, \Omega$  be as in (5), let  $f \in C_c^\infty(\Omega)$ , and let  $\phi, w : \mathbb{R} \rightarrow \mathbb{R}$  be even functions such that  $\Phi_{\text{band}}(\mathbf{x}) = -\phi'(|\mathbf{x}|)/(2|\mathbf{x}|)$  and*

$$W_{\text{line}}(\mathbf{x}) := -\frac{1}{\pi} \int_{|\mathbf{x}|}^{\infty} \frac{\partial_\xi w(\xi)}{\sqrt{\xi^2 - |\mathbf{x}|^2}} d\xi, \quad \mathbf{x} \in \mathbb{R}^3, \quad (6)$$

*are absolutely integrable. Moreover, assume that the support of  $\phi * w$  satisfies  $\text{supp}(\phi * w) \subset [-\tau, \tau]$ , with  $\tau := \text{dist}(\text{supp}(f), \partial\Omega)$ .*

*Then we have  $\mathbf{L}_{\phi, w} f \in \text{ran}(\mathbf{L})$ , and*

$$(\mathbf{L}^{-1} \mathbf{L}_{\phi, w} f)(\mathbf{x}) = (\Phi_{\text{band}} * W_{\text{line}} * f)(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

*Here both blurring effects are spatially invariant.*

*Proof.* See Section 3. □

### 1.3 Prior work and innovations

Analytic expressions for the blurring kernels with approximate line detectors are presented for the first time. Exact blurring kernels for approximate point detectors have also been derived in [42] (see [45] for an updated version) and due to the finite detector bandwidth in [4, Section 5]. However, we present a completely different and probably simpler analysis, which is based on geometric arguments and the rotational and translational invariance of the wave equation. Moreover, in [4, 42] the authors did not show that  $\mathbf{P}_{\phi, w} f \in \text{ran}(\mathbf{P})$ . Instead, they applied a particular inversion formula (i.e. an extension of  $\mathbf{P}^{-1}$  outside the range of  $\mathbf{P}$ ) to the blurred data. Hence their results depend on the used extension, whereas our results are independent of any particular inversion formula.

A preliminary version of this paper has been presented at the MATHMOD 2009 conference in Vienna (see [15]) where the case  $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$  has been treated.

## 2 Approximate point detectors

The main goal in this section is the derivation of the analytic expressions for the blurring kernels due to the detector size and bandwidth. The proof of Theorem 1.1 will follow from Propositions 2.1 and 2.3 and will be given in Subsection 2.3.

### 2.1 Effects of detector size

**Proposition 2.1.** *Let  $f \in C_c^\infty(B_R)$ , let  $w_{\text{point}} : \mathbb{R} \rightarrow \mathbb{R}$  be an even function such that  $\mathbf{x} \in \partial B_R \mapsto w_{\text{point}}(|\mathbf{x}_0, \mathbf{x}|) = w_{\text{point}}(|\mathbf{x}_0 - \mathbf{x}|)$  is absolutely integrable for some  $\mathbf{x}_0 \in \partial B_R$ , and define, for  $\mathbf{x} \in \mathbb{R}^3$ ,*

$$f_w(\mathbf{x}) := \int_{\mathbb{R}^3} \left[ \frac{R^2}{|\mathbf{x}|^2} \delta(|\mathbf{x}| - |\mathbf{x}'|) w_{\text{point}}(|\mathbf{x} - \mathbf{x}'| R / |\mathbf{x}|) \right] f(\mathbf{x}') d\mathbf{x}'. \quad (7)$$

Then, for every  $(\mathbf{x}, t) \in \partial B_R \times (0, \infty)$ , we have

$$(\mathbf{P} f_w)(\mathbf{x}, t) = \int_{\partial B_R} w_{\text{point}}(|\mathbf{x} - \mathbf{x}'|)(\mathbf{P} f)(\mathbf{x}', t) dS(\mathbf{x}'). \quad (8)$$

*Proof.* We first calculate an expression for  $f_w$  without the Dirac  $\delta$  distribution. To that end, denote  $\mathbf{x} = \rho\sigma$ ,  $\mathbf{x}' = \rho'\sigma'$  with  $\rho, \rho' \in (0, \infty)$  and  $\sigma, \sigma' \in S^2$ . Then, we have

$$\begin{aligned} f_w(\mathbf{x}) &= R^2 \int_{S^2} \left( \int_0^\infty \delta(\rho - \rho') w_{\text{point}}(|\rho\sigma - \rho'\sigma'| R/\rho) f(\rho'\sigma') d\rho' \right) d\sigma' \\ &= R^2 \int_{S^2} w_{\text{point}}(|\sigma - \sigma'| R) f(\rho\sigma') d\sigma', \end{aligned}$$

which is the desired expression.

Now define, for  $\mathbf{x} = \rho\sigma \in \mathbb{R}^3$ ,

$$p_w(\rho\sigma, t) := R^2 \int_{S^2} w_{\text{point}}(|\sigma - \sigma'| R) (\mathcal{W}^{3D} f)(\rho\sigma', t) d\sigma'.$$

We will show that  $p_w = \mathcal{W}^{3D} f_w$ , which then implies (8), since

$$p_w(R\sigma, t) = \int_{\partial B_R} w_{\text{point}}(|R\sigma - \mathbf{x}'|) (\mathcal{W}^{3D} f)(\mathbf{x}', t) dS(\mathbf{x}').$$

For every  $(\vartheta, \phi) \in (0, \pi) \times (0, 2\pi)$ , let  $Q[\vartheta, \phi] : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  denote any rotation with

$$Q[\vartheta, \phi] \mathbf{e}_3 = (\sin \vartheta \cos \phi, \sin \vartheta \sin \phi, \cos \vartheta).$$

Writing  $\sigma'$  in spherical coordinates around  $\sigma$  as north pole then shows that

$$\begin{aligned} p_w(\rho\sigma, t) &= R^2 \int_0^\pi \int_0^{2\pi} w_{\text{point}}(|\sigma - Q[\vartheta, \phi]\sigma| R) (\mathcal{W}^{3D} f)(\rho Q[\vartheta, \phi]\sigma, t) \sin \vartheta d\vartheta d\phi \\ &= R^2 \int_0^\pi w_{\text{point}}(2R(1 - \cos \vartheta)) \sin \vartheta \int_0^{2\pi} (\mathcal{W}^{3D} f)(Q[\vartheta, \phi](\rho\sigma), t) d\vartheta d\phi. \end{aligned}$$

The rotational invariance of the wave equation implies that

$$(\rho\sigma, t) \mapsto (\mathcal{W}^{3D} f)(Q[\vartheta, \phi](\rho\sigma), t)$$

is a solution of the wave equation and its linearity implies that also  $p_w(\rho\sigma, t)$  is a solution of the wave equation. Since the initial conditions  $p_w(\mathbf{x}, 0) = f_w(\mathbf{x})$  and  $\partial_t p_w(\mathbf{x}, 0) = 0$  immediately follow from the definition of  $p_w$  this implies  $p_w = \mathcal{W}^{3D} f_w$  and concludes the proof.  $\square$

**Remark 2.2.** The main ingredient of the proof of Proposition 2.1 is the rotational invariance of the wave equation. In an analogous manner (see Subsection 2.4) one derives analytic expressions of the blurring kernel for a planar (using translational invariance) and for a cylindrical detection surface (using translational and rotational invariance).

## 2.2 Finite bandwidth

The following proposition deals with the blurring to the finite bandwidth of the detection system. It states that it is equivalent to either convolute the solution  $\mathcal{W}^{3D} f$  in time with the function  $\phi$ , or to convolute the initial data in space with  $-\pi\phi'(|\mathbf{x}|)/(2|\mathbf{x}|)$ . This fundamental and simple looking property seems to be a new result.

**Proposition 2.3.** *Let  $f \in C_c^\infty(\mathbb{R}^3)$  and let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be an even compactly supported function such that  $\Phi_{\text{band}}(\mathbf{x}) = -\pi\phi'(|\mathbf{x}|)/(2|\mathbf{x}|)$  is absolutely integrable in  $\mathbb{R}^3$ . Then, the identity*

$$\mathcal{W}^{3D}(\Phi_{\text{band}} * f)(\mathbf{x}, t) = (\phi *_t \mathcal{W}^{3D} f)(\mathbf{x}, t)$$

holds for all  $(\mathbf{x}, t) \in \mathbb{R}^3 \times (0, \infty)$  with  $\mathbf{x}$  outside the support of  $\Phi_{\text{band}} * f$ .

*Proof.* Let  $\Psi(\mathbf{x}) = \psi(|\mathbf{x}|)$  be a radially symmetric absolutely integrable function of compact support. Our aim is to find an analytic expression for  $\mathcal{W}^{3D}(\Psi * f)$  in terms of the solution  $\mathcal{W}^{3D} f$  of (1a), (1b) and the to adjust  $\Psi$  such that  $\mathcal{W}^{3D}(\Psi * f) = \phi *_t \mathcal{W}^{3D} f$  outside the support of  $\Psi * f$ .

The solution formula for the three dimensional wave equation (see [20]) applied to the initial data  $\Psi * f$  reads

$$\mathcal{W}^{3D}(\Psi * f)(\mathbf{x}, t) = \frac{1}{4\pi} \partial_t t \int_{S^2} \left( \int_{\mathbb{R}^3} f(\mathbf{x}') \psi(|\mathbf{x} + t\omega - \mathbf{x}'|) d\mathbf{x}' \right) dS(\omega).$$

Substituting  $\mathbf{x}' = \mathbf{x} + \rho\sigma$ , with  $\rho > 0$  and  $\sigma \in S^2$ , in the inner integral, and applying Fubini's Theorem leads to

$$\begin{aligned} \mathcal{W}^{3D}(\Psi * f)(\mathbf{x}, t) &= \frac{1}{4\pi} \partial_t t \int_{S^2} \left( \int_{S^2} \int_0^\infty f(\mathbf{x} + \rho\sigma) \psi(|\rho\sigma - t\omega|) \rho^2 d\rho dS(\sigma) \right) dS(\omega) \\ &= \partial_t t \int_0^\infty \int_{S^2} f(\mathbf{x} + \rho\sigma) \left( \frac{1}{4\pi} \int_{S^2} \psi(|\rho\sigma - t\omega|) dS(\omega) \right) dS(\sigma) \rho^2 d\rho \end{aligned}$$

The inner integral in the last expression is the (spherical) mean of  $\Psi(\mathbf{x}) = \psi(|\mathbf{x}|)$  over a sphere with radius  $t$  centered at  $\rho\sigma$ . Denoting by  $I_\psi$  a primitive of  $s \mapsto \psi(\sqrt{s})$ , then [16, Lemma 5.1] assures that

$$\frac{1}{4\pi} \int_{S^2} \psi(|\rho\sigma - t\omega|) dS(\omega) = \frac{I_\psi((\rho+t)^2) - I_\psi((\rho-t)^2)}{4t\rho},$$

for  $\rho, t > 0$ . Consequently

$$\mathcal{W}^{3D}(\Psi * f)(\mathbf{x}, t) = \frac{1}{4} \partial_t \int_0^\infty \left( I_\psi((\rho+t)^2) - I_\psi((\rho-t)^2) \right) \left( \rho \int_{S^2} f(\mathbf{x} + \rho\sigma) dS(\sigma) \right) d\rho.$$

Differentiating under the integral leads

$$\mathcal{W}^{3D}(\Psi * f)(\mathbf{x}, t) = \frac{1}{2} \int_0^\infty (\rho - t) \psi(|\rho - t|) \left( \rho \int_{S^2} f(\mathbf{x} + \rho\sigma) dS(\sigma) \right) d\rho \quad (9)$$



for  $\mathbf{x} \notin \text{supp}(\Psi * f)$ .

On the other hand, we have

$$(\phi *_t \mathcal{W}^{3D} f)(\mathbf{x}, t) = \frac{1}{4\pi} \int_0^\infty \phi'(t - \rho) \left( \rho \int_{S^2} f(\mathbf{x} + \rho\sigma) dS(\sigma) \right) d\rho. \quad (10)$$

The right hand sides in equations (9) and (10) coincide if  $s\psi(|s|) = -\frac{\pi}{2}\phi'(s)$  for all  $s \in \mathbb{R}$ . Since  $\phi$  is assumed to be an even function, this is the case if

$$\psi(s) = -\frac{\pi}{2} \frac{\phi'(s)}{s}, \quad s \geq 0.$$

By taking  $\Psi = \Phi_{\text{line}}$ , this concludes the proof.  $\square$

Now we are ready to prove Theorem 1.1.

### 2.3 Proof of Theorem 1.1

Propositions 2.1 and 2.3 imply that

$$\begin{aligned} (\mathbf{P}_{\phi,w} f)(\mathbf{x}, t) &= \left( \phi *_t \int_{\partial B_R} w_{\text{point}}(|\mathbf{x} - \mathbf{x}'|) (\mathcal{W}^{3D} f)(\mathbf{z}', \cdot) dS(\mathbf{x}') \right) (t) \\ &= \left( \phi *_t (\mathcal{W}^{3D} f_w)(\mathbf{x}, \cdot) \right) (t) = \left( \mathcal{W}^{3D} (\Phi_{\text{band}} * f_w) \right) (\mathbf{x}, t) \end{aligned}$$

if  $\mathbf{x} \notin \text{supp}(\Phi_{\text{band}} * f)$ . Here  $f_{w_{\text{point}}}$  is defined as in (7). Together with support hypothesis on  $\phi$  this shows that  $\mathbf{P}_{\phi,w} f = \mathbf{P}(\Phi_{\text{band}} * f_{w_{\text{point}}})$ . Therefore  $\mathbf{P}_{\phi,w} f \in \text{ran}(\mathbf{P})$  and  $\mathbf{P}^{-1} \mathbf{P}_{\phi,w} f = \Phi_{\text{band}} * f_{w_{\text{point}}}$  which concludes the proof.

### 2.4 Planar and cylindrical detection surface

For the following theorems recall the definition of the data (2) of approximate point detectors.

**Theorem 2.4** (Blurring kernels for planar detection surface). *Let  $\Omega := (0, \infty) \times \mathbb{R}^2$  denote a half space in  $\mathbb{R}^3$  and assume that  $w(\mathbf{x}, \mathbf{x}') = w_{\text{point}}(\mathbf{x} - \mathbf{x}')$ , where  $w_{\text{point}}$  is absolutely integrable on  $\partial\Omega = \{0\} \times \mathbb{R}^2$ . Moreover, let  $f \in C_c^\infty(\Omega)$ , let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be an even function such that  $\Phi_{\text{band}}(\mathbf{x}) = \phi'(|\mathbf{x}|)/|\mathbf{x}|$  is integrable, and assume that  $\text{supp}(\phi) \subset [-\tau, \tau]$ , where  $\tau := \text{dist}(\text{supp}(f), \partial B_R)$ .*

Then  $\mathbf{P}_{\phi,w} f \in \text{ran}(\mathbf{P})$  and

$$(\mathbf{P}^{-1} \mathbf{P}_{\phi,w} f)(\mathbf{x}) = (\Phi_{\text{band}} * W_{\text{point}} * f)(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

with the blurring kernel

$$W_{\text{point}}(x, \mathbf{y}) := \delta(x) w_{\text{point}}(0, \mathbf{y}), \quad \text{for } \mathbf{x} = (x, \mathbf{y}) \in \mathbb{R}^3.$$

*Proof.* Theorem 2.4 is proven along the lines of Theorem 1.1. For showing an analogue of Proposition 2.1, one uses the translational invariance of the wave equation.  $\square$

In the case of cylindrical surface the blurring kernel is a combination of the planar and the spherical case and follows from the translational and the rotational invariance of the wave equation.

**Theorem 2.5** (Blurring kernels for cylindrical detection surface). *Let  $\Omega := \mathbb{R} \times D_R$  denote a cylinder of radius  $R$ , and assume that*

$$w(\mathbf{x}, \mathbf{x}') = w_{\text{point}}(x - x', |\mathbf{y} - \mathbf{y}'|), \quad \text{for } \mathbf{x} = (x, \mathbf{y}) \text{ and } \mathbf{x}' = (x', \mathbf{y}') \in \mathbb{R} \times \partial D_R.$$

Here  $w_{\text{point}}(x, |\mathbf{y} - \mathbf{y}_0|)$  is assumed to be absolutely integrable on  $\mathbb{R} \times \partial D_R$  for some  $\mathbf{y}_0 \in \partial D_R$ . Moreover, let  $f \in C_c^\infty(\Omega)$ , let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be an even function such that  $\Phi_{\text{band}}(\mathbf{x}) = \phi'(|\mathbf{x}|)/|\mathbf{x}|$  is integrable, and assume that  $\text{supp}(\phi) \subset [-\tau, \tau]$ , where  $\tau := \text{dist}(\text{supp}(f), \partial\Omega)$ .

Then  $\mathbf{P}_{\phi, w} f \in \text{ran}(\mathbf{P})$ , and

$$(\mathbf{P}^{-1} \mathbf{P}_{\phi, w} f)(\mathbf{x}) = \left[ \Phi_{\text{band}} * \int_{\mathbb{R}^3} W_{\text{point}}(\cdot, \mathbf{x}') f(\mathbf{x}') d\mathbf{x}' \right](\mathbf{x}), \quad \text{for } \mathbf{x} = (x, \mathbf{y}) \in \mathbb{R}^3$$

where

$$W_{\text{point}}((x, \mathbf{y}), (x', \mathbf{y}')) := \frac{R}{|\mathbf{x}|} \delta(|\mathbf{y}| - |\mathbf{y}'|) w_{\text{point}}(x - x', |\mathbf{y} - \mathbf{y}'|),$$

for  $\mathbf{x} = (x, \mathbf{y})$  and  $\mathbf{x}' = (x', \mathbf{y}') \in \mathbb{R} \times D_R$ .

*Proof.* Theorem 2.5 is again proven along the lines of Theorem 1.1. In order to show an analogue of Proposition 2.1, one uses the translational and the rotational invariance of the wave equation.  $\square$

### 3 Approximate line detectors

In this section we derive the analytic expressions for the blurring kernels due to the detector size and bandwidth given in Theorem 1.3. In the following let  $\mathcal{W}^{2D} : C_c^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2 \times (0, \infty))$  denote the operator that maps a compactly supported initial data to the solution of the two dimensional wave equation

$$(\partial_t^2 - \Delta) p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times (0, \infty), \quad (11a)$$

$$p(\mathbf{x}, 0) = f(\mathbf{x}), \partial_t p(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^2. \quad (11b)$$

The symmetry of the Greens function of the two dimensional wave equation implies the following well know result:

**Lemma 3.1** (Acoustic reciprocal principle in two dimensions). *Let  $f \in C_c^\infty(\mathbb{R}^2)$  and let  $\Psi$  be a compactly supported absolutely integrable function. Then  $\Psi * (\mathcal{W}^{2D} f) = \mathcal{W}^{2D}(\Psi * f)$ .*

### 3.1 Finite beamwidth

First we calculate the blurring kernel due to the detector size.

**Proposition 3.2.** *Let  $f \in C_c^\infty(\mathbb{R}^3)$ , let  $w : \mathbb{R} \rightarrow \mathbb{R}$  be a compactly supported even function such that  $W_{\text{line}}(\mathbf{x}) = -1/\pi \int_{|\mathbf{x}|}^\infty \partial_\xi w(\xi) / \sqrt{\xi^2 - |\mathbf{x}|^2} d\xi$  is absolutely integrable. Then, for any line  $\ell \subset \mathbb{R}^3$ , we have*

$$\int_{\mathbb{R}^3} w(\text{dist}(\ell, \mathbf{x})) (\mathcal{W}^{3D} f)(\mathbf{x}, t) d\mathbf{x} = \int_\ell \mathcal{W}^{3D} (W_{\text{line}} * f)(\mathbf{x}, t) d\ell(\mathbf{x}),$$

with  $W_{\text{line}}$  as defined in (6).

*Proof.* Without loss of restriction we shall assume that  $\ell$  is of the form  $\ell = \mathbb{R}(1, 0, 0) + (0, y_1, y_2)$ . Moreover we write  $\mathbf{x} = (x, \mathbf{y})$  with  $x \in \mathbb{R}$  and  $\mathbf{y} = (y_1, y_2)$  in  $\mathbb{R}^2$ , and denote by  $\mathbf{X}$  the X-ray transform restricted to lines pointing in  $(1, 0, 0)$  direction,

$$(\mathbf{X} h)(\mathbf{y}) := \int_{\mathbb{R}} h(x, \mathbf{y}) dx_1, \quad h \in C_c^\infty(\mathbb{R}^3).$$

The commutation relation of the Laplacian with the X-ray transform (see [17]) implies that  $\mathcal{W}^{2D} \mathbf{X} = \mathbf{X} \mathcal{W}^{3D}$ . After writing  $\mathbf{x}' = (x', \mathbf{y}')$  with  $x' \in \mathbb{R}$  and  $\mathbf{y}' = (y'_1, y'_2)$  in  $\mathbb{R}^2$ , we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} w(\text{dist}(\ell, \mathbf{x})) (\mathcal{W}^{3D} f)(\mathbf{x}', t) d\mathbf{x}' \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} w(|\mathbf{y} - \mathbf{y}'|) (\mathcal{W}^{3D} f)((x', \mathbf{y}'), t) dx' d\mathbf{y}' \\ &= \int_{\mathbb{R}^2} w(|\mathbf{y} - \mathbf{y}'|) (\mathbf{X} \mathcal{W}^{3D} f)(\mathbf{y}', t) d\mathbf{y}' \\ &= (w(|\cdot|) * (\mathbf{X} \mathcal{W}^{3D} f))(\mathbf{y}, t) = (w(|\cdot|) * (\mathcal{W}^{2D} \mathbf{X} f))(\mathbf{y}, t). \end{aligned} \quad (12)$$

Now let  $U(\mathbf{x}) = u(|\mathbf{x}|)$  be any radially symmetric integrable function. Then one easily verifies that  $(\mathbf{X} U)(\mathbf{y}) = I_u(|\mathbf{y}|)$  with  $I_u(\xi) := \int_{\mathbb{R}} u((s^2 + \xi^2)^{1/2}) ds$ . Moreover, the relation  $\mathbf{X} \mathcal{W}^{3D} = \mathcal{W}^{2D} \mathbf{X}$ , the convolution theorem for the X-ray transform, and the acoustic reciprocal principle Lemma 3.1, imply

$$\begin{aligned} \int_\ell \mathcal{W}^{3D} (U * f)(\mathbf{x}, t) d\ell(\mathbf{x}') &= ((\mathbf{X} \mathcal{W}^{3D} (U * f))(\mathbf{y}, t) \\ &= (\mathcal{W}^{2D} \mathbf{X} (U * f))(\mathbf{y}, t) \\ &= (\mathcal{W}^{2D} (\mathbf{X} U * \mathbf{X} f))(\mathbf{y}, t) \\ &= (\mathcal{W}^{2D} (I_u(|\cdot|) * \mathbf{X} f))(\mathbf{y}, t) \\ &= (I_u(|\cdot|) * (\mathcal{W}^{2D} \mathbf{X} f))(\mathbf{y}, t). \end{aligned} \quad (13)$$

Consequently the left hand sides of (12) and (13) coincide, if

$$w(\xi) = I_u(\xi) = \int_{\mathbb{R}} u((s^2 + \xi^2)^{1/2}) ds = 2 \int_{\xi}^{\infty} u(\eta) \frac{\eta d\eta}{\sqrt{\eta^2 - \xi^2}}.$$

This is an Abel integral equation for the function  $u$ . Its solution is (see [12, 28])  $u(\eta) = -\frac{1}{\pi} \int_{\eta}^{\infty} \frac{\partial_{\xi} w(\xi)}{\sqrt{\xi^2 - \eta^2}} d\xi$ . This concludes the proof by taking  $W_{\text{line}} = U$ .  $\square$

### 3.2 Finite bandwidth

Next we calculate the point spread function due to the finite bandwidth.

**Proposition 3.3.** *Let  $f \in C_c^{\infty}(\mathbb{R}^3)$  and let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be an even compactly supported function such that  $\Phi_{\text{band}}(\mathbf{x}) = -\pi\phi'(|\mathbf{x}|)/(2|\mathbf{x}|)$  is absolutely integrable. Then*

$$\phi *_{\mathbf{t}} \int_{\ell} (\mathcal{W}^{3D} f)(\mathbf{x}', t) d\ell(\mathbf{x}') = \int_{\ell} (\mathcal{W}^{3D} (\Phi_{\text{band}} * f))(\mathbf{x}', t) d\ell(\mathbf{x}') \quad (14)$$

for any line  $\ell$  with  $\ell \cap \text{supp}(\Phi_{\text{band}} * f) = \emptyset$ .

*Proof.* Proposition 2.3 states that  $\phi *_{\mathbf{t}} (\mathcal{W}^{3D} f) = \mathcal{W}^{3D} (\Phi_{\text{band}} * f)$  outside the support of  $\Phi_{\text{band}} * f$ . Integrating this identity over  $\ell$  proves (14).  $\square$

### 3.3 Proof of Theorem 1.3

According to Propositions 3.2 and 3.3 we have

$$\begin{aligned} \phi *_{\mathbf{t}} \int_{\mathbb{R}^3} w(\text{dist}(\ell, \mathbf{x}')) (\mathcal{W}^{3D} f)(\mathbf{x}, t) d\mathbf{x}' &= \phi *_{\mathbf{t}} \int_{\ell} \mathcal{W}^{3D} (W_{\text{line}} * f)(\mathbf{x}', t) d\ell(\mathbf{x}') \\ &= \int_{\ell} \mathcal{W}^{3D} (\Phi_{\text{band}} * W_{\text{line}} * f)(\mathbf{x}', t) d\ell(\mathbf{x}'). \end{aligned}$$

By taking  $\ell = \ell_{\theta, \mathbf{y}}$ ,  $\mathbf{y} \in D$ , and using the support hypothesis on  $\phi * w$ , this shows  $\mathbf{L}_{\phi, w} f = \mathbf{L}(\Phi_{\text{band}} * W_{\text{line}} * f)$ . Therefore  $\mathbf{L}_{\phi, w} f \in \text{ran}(\mathbf{L})$  and  $\mathbf{L}^{-1} \mathbf{L}_{\phi, w} f = \Phi_{\text{band}} * W_{\text{line}} * f$ , which concludes the proof.

## 4 Discussion

In this paper we derived analytic expression for the point spread functions in PAT due to the finite detector size and the finite bandwidth of the ultrasound detection system for approximate point and approximate line detectors. We showed, that the point spread functions due to the finite bandwidth is spatial invariant in both cases. The point spread function due to the detector size is only spatial invariant in the case of line detectors. The intensity profile of the detecting laser beam is usually well approximated by  $w(r) = I_0 \exp(-\frac{r^2}{2b})$ , see [33, 36]. In this case, up to some constant, the radial profile of the point spread function  $w_{\text{line}}$  coincides with the beam profile  $w$ .

The full width half maximum of the point spread function is a typical parameter to measure spatial resolution. Ignoring effects of finite bandwidth, Theorems 1.1 and 1.3 show that the lateral resolution of PAT with “approximate point detectors” is  $a_{\text{trans}} |\mathbf{x}|/R$ , where  $a_{\text{trans}}$  is the diameter of the ultrasound transducer, and the (uniform) resolution of PAT with “approximate line detectors” is given by the width  $a_{\text{laser}}$  of the detecting laser beam. Typical values  $a_{\text{trans}} = 1$  cm and  $a_{\text{laser}} = 0.1$  cm (see [33, 42]) point out the high spatial resolution of PAT with integrating line detectors.

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