Causality Analysis of Frequency Dependent Wave Attenuation

January 13, 2011

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Abstract

The work is inspired by thermo-and photoacoustic imaging, where recent efforts are devoted to take into account attenuation and varying wave speed parameters. In this paper we derive and analyze causal equations describing propagation of attenuated pressure waves. We also review standard models, like frequency power laws, and the thermo-viscous equation and show that they lack causality in the parameter range relevant for biological photoacoustic imaging. To discuss causality in mathematical rigor we use the results and concepts of linear system theory. We present some numerical experiments, which show the physically unmeaningful behavior of standard attenuation models, and the realistic behavior of the novel models.

Keywords: Attenuation, Causality, Mathematical Analysis, Power laws, Thermo Viscous Equation, Photoacoustic Modelling

1 Introduction

The work is inspired by thermo-and photoacoustic imaging (see e.g. [24, 11, 29, 17, 26, 1] for some articles related to the subject), where the problem is the reconstruction of the *absorption density* from measurements of the pressure outside of the object. This is the *Inverse Problem* according to the forward problem, which maps the absorption density onto the pressure by solving the *standard* wave equation. Various reconstruction methods have been suggested in the literature for photoacoustic imaging, which can for instance be found in the excellent survey [12]. Recent efforts have been made to take into account attenuation [18, 13, 4] and varying wave speed [8]. The standard model of attenuation (which is reviewed in Section 3) is formulated in the frequency range and models the physical reality that high frequency components of waves are attenuated more rapidly over time.

In this paper we review standard attenuation models, like *power laws* [20, 22, 23, 21, 27] and the *thermo-viscous* wave equation [10]. In this context, we discuss causality, which is the desired feature of attenuation models. The lack of causality of standard models in the parameter range[1] relevant for photoacoustic imaging requires to investigate novel equations, which are derived in Section 3 and the following.

We base the derivation of causal attenuation models on the mathematical concept of *linear system theory*, which can for instance be found in the book of Hörmander [7]. In Section 4 the abstract formulations are translated in equations which are formally similar to the wave equation. However, in general, the novel equations are integro-differential equations. An important issue is that the equations are formulated as inhomogeneous equations with homogeneous initial conditions, which is not standard for attenuated wave equations, where typically the equations are considered homogeneous with inhomogeneous initial conditions. For the standard wave equation these two concepts are equivalent, but only the one considered here, is mathematically sensible for the attenuation model.

The approach leads to some novel causal attenuation models, in particular power law models (valid for a bounded frequency range), which are documented in the literature to be relevant for biological specimen (in the terminology used later on this means that $\gamma \in (1, 2]$ – see [28, Chapter 7]) and also for instance also for castor oil, which satisfies a power law with index $\gamma = 1.66$ [22]. These models are presented in Section 8. The rotationally symmetric examples, presented in Section 9, illustrate the unphysical behavior of some existing attenuation models. Aside from unmeaningly physical effects, the stable and convergent numerical implementation of attenuated, non-causal wave equations is an unconsidered problem since these equations lack the Courant-Friedrich-Levy (CFL) condition [5]. The attenuation models considered here have a finite front wave speed and therefore can be implemented in a stable manner. Thus aside from physical considerations also from a point of view of stable numerical solution of wave equations questions of causality are most relevant. We note that in [15] an equation for acoustic propagation in inhomogeneous media with relaxation losses is derived which satisfies causality and a frequency power law with exponent 2 for small frequencies.

Concerning the presentation of the paper, the basic notation and mathematical results are summarized in the appendix.

[1] According to experiments, the attenuation law of some biological tissue satisfies a power law for small frequencies with exponent lying in the range $(1, 2]$. See [3].

2 Linear System Theory

This section surveys *linear system theory* (see e.g. [16, 7]), which provides the link between linear systems and convolution operators. This analysis is essential for the analysis. For notational convenience, when we speak about functions they are understood in the most wide meaning of the word, and can for instance be distributions.

In the following, we give a characterization of *causal* functions and operators.

Definition 2.1. 1. A function $f := f(\mathbf{x}, t)$ defined on the Euclidean space over time (i.e. in $\mathbb{R}^4$) is said to be *causal* if it satisfies

$$f(\mathbf{x}, t) = 0 \qquad \text{for} \qquad t < 0 \, .$$

2. In this paper $\mathcal{A}$ (with and without subscripts) denotes a real (that is, it is a mapping between sets of real functions on $\mathbb{R}^4$) and bounded operator.

   - $\mathcal{A}$ is *translation invariant* if for every function f and every linear transformation $L := L(\mathbf{x}, t) := (\mathbf{x} - \mathbf{x}_0, t - t_0)$, with $\mathbf{x}_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}$, it holds that

     $$\mathcal{A}(f \circ L) = (\mathcal{A} f) L \, .$$

     Here $\circ$ denotes the decomposition, i.e., $(f \circ L)(\mathbf{x}, t) = f(L(\mathbf{x}, t))$
   - $\mathcal{A}$ is called *causal*, if it maps causal functions to causal functions.
   - The operator $\mathcal{A}$ has a *causal domain of influence* if the function

     $$T(\mathbf{x}) := \sup\{t : \mathcal{A}\delta_{\mathbf{x},t}(\mathbf{x}, \tau) = 0 \text{ for all } \tau \leq t\} \qquad \text{for all} \qquad \mathbf{x} \in \mathbb{R}^3$$

     is rotationally symmetric and the derivative with respect to the radial component T' satisfies

     $$0 < \left(T'(r)\right)^{-1} \leq c_0 < \infty \, . \tag{1}$$

     For convenience of notation we identify $T(|\mathbf{x}|) = T(\mathbf{x})$.
     The function T, presumably it exists, corresponds to the travel time of a wavefront initiated in $\mathbf{0}$ at $t = 0$. (1) guarantees that the wavefront speed is finite.

Remark 1. If the operator $\mathcal{A}$ models a physical process in a *homogeneous and isotropic medium*, then $\mathcal{A}$ is shift invariant and $\mathcal{A}\delta_{\mathbf{x},t}$ is rotationally symmetric.

If T exists, and in addition satisfies (1), then the property of a causal domain of influence guarantees that a wavefront can propagate with a speed of at most c_0.

Now, we recall a fundamental mathematical theorem (see [7, Theorem 4.2.1]) of systems theory, which relates invariant operators with *space–time* convolutions.

Theorem 2.1. *[7, Theorem 4.2.1] Every linear (causal and) translation invariant operator $\mathcal{A}$ can be written as a space–time convolution operator with (causal) kernel G. That is, for arbitrary f from a suitable class of functions we have*

$$\mathcal{A} f = G *_{\mathbf{x},t} f \, . \tag{2}$$

In analogy to linear system theory we call the kernel G the *Green function* of $\mathcal{A}$. According to Definition 2.1 the considered operators are real and therefore the according Green functions are real valued too. From the definition of the Green function it follows that

$$G = \mathcal{A}\delta_{\mathbf{x},t} \ .$$

In the following example we review the Green function and the convolution operator according to the wave equation.

Example 2.1. We consider the standard wave equation in an isotropic medium with phase speed $c_0 \in (0,\infty)$:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} p = -f, \tag{3}$$

together with *initial conditions*

$$p|_{t<0} = 0 \qquad \text{and} \qquad \left.\frac{\partial}{\partial t} p\right|_{t<0} = 0 \ . \tag{4}$$

With source term $f = \delta_{\mathbf{x},t}$, the according solution G_0 of (3) and (4) is the *Green function*

$$G_0(\mathbf{x},t) = \frac{\delta_t\left(t - \frac{|\mathbf{x}|}{c_0}\right)}{4\pi\,|\mathbf{x}|} \ . \tag{5}$$

Because of (5) G_0 is commonly denoted *spherical wave* originating from $\mathbf{x} = \mathbf{0}$ at time $t = 0$.

In the space–frequency domain the Green function can be expressed by

$$\mathcal{F}\{G_0\} := \mathcal{F}\{G_0\}(\mathbf{x},\omega) = \frac{1}{\sqrt{2\pi}}\frac{\exp\left(\mathbf{i}\omega\frac{|\mathbf{x}|}{c_0}\right)}{4\pi\,|\mathbf{x}|} \ .$$

It satisfies

$$\nabla\mathcal{F}\{G_0\} = \left[\frac{\mathbf{i}\omega}{c_0} - \frac{1}{|\mathbf{x}|}\right]\cdot\mathcal{F}\{G_0\}\cdot \mathrm{sgn}, \tag{6}$$

and is the solution of the *Helmholtz equation*

$$\nabla^2\mathcal{F}\{G_0\} + \frac{\omega^2}{c_0^2}\mathcal{F}\{G_0\} = -\frac{1}{\sqrt{2\pi}}\delta_{\mathbf{x}} \ . \tag{7}$$

The operator

$$\mathcal{A}_0 f := G_0 *_{\mathbf{x},t} f$$

is causal and maps a causal function f onto the solution of (3) and (4).

3 Attenuation

In the chapter we investigated causality of attenuation models in a homogeneous, isotropic medium. In mathematical terms, it is common to describe attenuation by a *multiplicative law* in the frequency range:

Definition 3.1. A real, bounded, linear, translation invariant operator $\mathcal{A}$ with causal domain of influence is called *attenuation operator* if there exists a complex function $\beta_* := \beta_*(r,\omega)$ such that the associated Green function $G := \mathcal{A}\delta_{\mathbf{x},t}$ satisfies

$$\boxed{\mathcal{F}\{G\}(\mathbf{x},\omega) = \exp\left(-\beta_*(|\mathbf{x}|,\omega)\right) \cdot \mathcal{F}\{G_0\}(\mathbf{x},\omega) \qquad \text{for all} \qquad \mathbf{x} \in \mathbb{R}^3,\, \omega \in \mathbb{R}\,.} \tag{8}$$

Here, $\mathcal{F}$ is the Fourier transform (see Appendix).

We rewrite (8) in the space–time domain by using

$$K := K(\mathbf{x},t) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{\exp\left(-\beta_*\right)\right\}(|\mathbf{x}|,t)\,. \tag{9}$$

Therefore

$$\boxed{G(\mathbf{x},t) = [K *_t G_0](\mathbf{x},t) = \frac{1}{4\pi\,|\mathbf{x}|} K\left(\mathbf{x}, t - \frac{|\mathbf{x}|}{c_0}\right)\,.} \tag{10}$$

Since in the context of this paper the operator $\mathcal{A}$ is real, the associated Green function is real-valued, and consequently $\beta_*(r,\omega)$ has to be even with respect to ω (cf. Property 5 in Appendix).

Remark 2. In physical terms attenuation is a result of frequency dependent energy dissipation and therefore the ratio of the attenuated and un-attenuated wave amplitude must be smaller or equal to 1. That is

$$\exp\left(-\Re(\beta_*)\right) = \left|\frac{\mathcal{F}\{G\}}{\mathcal{F}\{G_0\}}\right| \leq 1\,.$$

This implies that the *attenuation coefficient* β_* satisfies $\Re(\beta_*) \geq 0$.

In the literature a special form of the attenuation coefficient is assumed:

Definition 3.2 (Standard Form)**.** The *standard form* of β_* considered in the literature is (see e.g. [20])

$$\beta_*(r,\omega) = \alpha_*(\omega) r \qquad \text{for} \qquad r > 0,\, \omega \in \mathbb{R}. \tag{11}$$

The function $\alpha := \Re(\alpha_*)$ is called *attenuation law.*

For the standard form β_* several properties for the attenuation operator are at hand. For instance the following results concerning travel time and causality.

Theorem 3.1. *Let $\mathcal{A}$ be an attenuation operator with β_* of standard form. Then the travel time satisfies $T(|\mathbf{x}|) = |\mathbf{x}|/c$ for some constant $0 < c \leq c_0$.*

Proof. The definition of the travel time T in Definition 2.1 states that $T(|\mathbf{x}|)$ is the largest positive number such that for the Green function $G = \mathcal{A}\delta_{\mathbf{x},t}$

$$G(\mathbf{x},t) = 0 \qquad \text{for} \qquad t < T(|\mathbf{x}|)\,.$$

This condition is equivalent to the condition that the function

$$(\mathbf{x},t) \to G(\mathbf{x}, t + T(|\mathbf{x}|)) \qquad \text{is causal.} \tag{12}$$

The operator $\mathcal{A}$ is causal and has causal domain of influence, which implies that $T(0) = 0$ and $(T'(r))^{-1} \leq c_0$. Consequently

$$T(|\mathbf{x}|) = \int_0^{|\mathbf{x}|} T'(r)dr \geq \frac{|\mathbf{x}|}{c_0} \,. \tag{13}$$

$\tau(r) := T(r) - r/c_0$ denotes the largest number such that $K(\mathbf{x}, \cdot + \tau(|\mathbf{x}|)) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{\exp(-\beta_*)\}$ is causal. From (8) it follows that

$$K(\mathbf{x}, t) = K(\mathbf{x}/2, t) *_t K(\mathbf{x}/2, t) \,.$$

This and the Theorem of Supports (cf. [7]) imply that $\tau(r) = 2\tau(r/2)$, and consequently τ is linear in r and after all T is linear as well. $\square$

In particular, from (10) and Theorem 3.1 it follows that $\mathcal{A}$ has a causal domain of influence if and only if K is a causal function.

Remark 3. In the literature (for instance in [27]) causality is aimed to be enforced by demanding that

$$\mathcal{F}^{-1}\{\alpha_*\} \qquad \text{is causal.} \tag{14}$$

This is equivalent to that the *Kramers-Kronig* relations for the m-th derivative $\alpha_*^{(m)}$ of some function α_* are satisfied, i.e., there exists a non-negative integer m, such that

$$\Im(\alpha_*^{(m)}) = \mathcal{H}\left\{\Re(\alpha_*^{(m)})\right\} = \mathcal{H}\left\{\alpha^{(m)}\right\} \quad \text{and} \quad \alpha^{(m)} := \Re(\alpha_*^{(m)}) = -\mathcal{H}\left\{\Im(\alpha_*^{(m)})\right\}, \tag{15}$$

where $\mathcal{H}\{\cdot\}$ is the Hilbert Transform (see Appendix).

(14) follows already from the causality of K: From the definition of K it follows that

$$|\nabla K| = \frac{1}{\sqrt{2\pi}} \left|\mathcal{F}^{-1}\{\alpha_* \cdot \exp(-\alpha_* |\mathbf{x}|)\}\right| \,.$$

Using some sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n \neq \mathbf{0}$ and $\mathbf{x}_n \to \mathbf{0}$ shows that

$$\lim |\nabla K| (\mathbf{x}_n, t) = \frac{1}{\sqrt{2\pi}} \left|\mathcal{F}^{-1}\{\alpha_*\}\right| (t) \,.$$

Due to the causality of K the left hand side is zero for $t < 0$, and thus $\mathcal{F}^{-1}\{\alpha_*\}$ is causal.

However, as we show in Example 3.1 below, causality of $\mathcal{F}^{-1}\{\alpha_*\}$ does not imply causality of K. In other words, in general, from the causality of $\mathcal{F}^{-1}\{\alpha_*\}$ it cannot be deduced that $\mathcal{A}$ has a causal domain of influence. As a consequence several attenuation models considered in the literature lack causality.

Example 3.1 (Frequency power laws [20, 23])**.** Consider the *frequency power law*,

$$\boxed{\alpha(\omega) = \alpha_0 |\omega|^\gamma \,,} \tag{16}$$

where $\gamma, \alpha_0 \geq 0$ and $\gamma \notin \mathbb{N}$. The Kramers-Kronig relation with differentiation index $m = 1$ is satisfied for the one-parametric family of complex extensions (as considered [27, 23])

$$\boxed{\alpha_*(\omega) = \frac{\alpha_0}{\cos\left(\frac{\pi}{2}\gamma\right)}(-\mathbf{i}\omega)^\gamma \,.} \tag{17}$$

Basically, [7, Theorem 7.4.3] implies that for every polynomial p in $-\mathbf{i}\omega$ with nonegative real exponents, $\mathcal{F}^{-1}\{p\}$ is causal. Hence if $\gamma > 1$, then

$$\alpha_*^{II}(\omega) := \alpha_*(\omega) + a_0\,(-\mathbf{i}\omega) \quad a_0 \in \mathbb{R},$$

has the same real part as α_* and $\mathcal{F}^{-1}\left\{\alpha_*^{II}\right\}$ is causal. As a consequence the attenuation law α together with the causality condition (14) does not uniquely determine the attenuation operator $\mathcal{A}$ (cf. Definition 3.1).

Let α_* be defined as in (17), then according to [7, Theorem 7.4.3] K, as defined in (9), is causal if and only if $\gamma \in [0,1)$. Consequently, for frequency power laws with $\gamma > 1$ the according operator $\mathcal{A}$, defined in Definition 3.1 does **not** have a causal domain of influence.

4 Equations for Attenuated Pressure Waves

In this section we formulate a causal wave equation which takes into account attenuation and review the literature (cf. [22, 23, 21, 13, 18]).

Let $\mathcal{A}$ denote a translation invariant operator with causal domain of influence with travel time function T and c_0 as in Definition 2.1. The Green function G satisfies (10) and (9) and therefore the according attenuation coefficient is given by

$$\beta_*(\mathbf{x},\omega) = -\log\left\{2\sqrt{(2\pi)^3}\,|\mathbf{x}|\,\mathcal{F}\left\{G\left(\mathbf{x},\cdot+\frac{|\mathbf{x}|}{c_0}\right)\right\}(\omega)\right\}. \tag{18}$$

In the following we rewrite the term $\nabla^2\mathcal{F}\{G\}$ from which we derive the Helmholtz equation for $\mathcal{F}\{G\}$. Using (10), which states that $G = K *_t G_0 = \mathcal{A}\delta_{\mathbf{x},t}$, and the product rule yields

$$\frac{1}{\sqrt{2\,\pi}}\,\nabla^2\mathcal{F}\{G\} = \nabla^2\mathcal{F}\{K\}\cdot\mathcal{F}\{G_0\} + 2\nabla\mathcal{F}\{K\}\cdot\nabla\mathcal{F}\{G_0\} + \mathcal{F}\{K\}\cdot\nabla^2\mathcal{F}\{G_0\}. \tag{19}$$

To evaluate this expression, we calculate $\nabla\mathcal{F}\{K\}$ and $\nabla^2\mathcal{F}\{K\}$. From (9) it follows that

$$\nabla\mathcal{F}\{K\} = -\beta_*'\cdot\mathcal{F}\{K\}\cdot\mathrm{sgn}\,, \tag{20}$$

where β_*' denotes the derivative of $\beta_*(r,\omega)$ with respect to r. This together with the formula (62) in the Appendix implies that

$$\begin{aligned}
\nabla^2\mathcal{F}\{K\} &= -\,\nabla\cdot(\beta_*'\cdot\mathcal{F}\{K\}\cdot\mathrm{sgn})\\
&= -\,(\nabla\cdot\mathrm{sgn})\cdot\beta_*'\cdot\mathcal{F}\{K\} - (\mathrm{sgn}\cdot\nabla\beta_*')\cdot\mathcal{F}\{K\} - (\mathrm{sgn}\cdot\nabla\mathcal{F}\{K\})\cdot\beta_*'\\
&= \left[-\frac{2}{|\mathbf{x}|}\cdot\beta_*' - \beta_*'' + (\beta_*')^2\right]\cdot\mathcal{F}\{K\}\,.
\end{aligned} \tag{21}$$

Inserting (20) and (21) into (19) and using the identity $G = K *_t G_0$ (cf. (10)), shows that

$$\begin{aligned}
\frac{1}{\sqrt{2\,\pi}}\,\nabla^2\mathcal{F}\{G\} =& \frac{1}{\sqrt{2\,\pi}}\left[-\frac{2}{|\mathbf{x}|}\cdot\beta_*' - \beta_*'' + (\beta_*')^2\right]\cdot\mathcal{F}\{G\}\\
&- 2\beta_*'\cdot\mathcal{F}\{K\}\cdot(\mathrm{sgn}\cdot\nabla\mathcal{F}\{G_0\}) + \mathcal{F}\{K\}\cdot\nabla^2\mathcal{F}\{G_0\}\,.
\end{aligned} \tag{22}$$

Together with (6) and (7), the last identity simplifies to

$$\nabla^2 \mathcal{F}\{G\} = \left[-\frac{2}{|\mathbf{x}|}\cdot\beta_*' - \beta_*'' + (\beta_*')^2\right]\cdot\mathcal{F}\{G\} - 2\left[\frac{\mathbf{i}\omega}{c_0} - \frac{1}{|\mathbf{x}|}\right]\cdot\beta_*'\cdot\mathcal{F}\{G\} - \frac{\omega^2}{c_0^2}\cdot\mathcal{F}\{G\} - \mathcal{F}\{K\}\cdot\delta_{\mathbf{x}}\,. \tag{23}$$

Since $\mathcal{F}\{K\}(\mathbf{x},\omega)\cdot\delta_{\mathbf{x}} = \mathcal{F}\{K\}(\mathbf{0},\omega)\cdot\delta_{\mathbf{x}}$, we obtain from (23) the Helmholtz equation

$$\boxed{\begin{aligned} &\nabla^2\mathcal{F}\{G\} - \left[\beta_*' + \frac{(-\mathbf{i}\omega)}{c_0}\right]^2\cdot\mathcal{F}\{G\} \\ = &-\beta_*''\cdot\mathcal{F}\{G\} - \mathcal{F}\{K\}(\mathbf{0},\omega)\cdot\delta_{\mathbf{x}} \\ = &-\beta_*''\cdot\mathcal{F}\{G\} - \frac{1}{\sqrt{2\pi}}\exp\left(-\beta_*(\mathbf{0},\omega)\right)\cdot\delta_{\mathbf{x}}\,. \end{aligned}} \tag{24}$$

To reformulate (24) in space–time coordinates, we introduce two convolution operators:

$$D_* f := K_* *_t f \qquad \text{and} \qquad D_*' f := K_*' *_t f, \tag{25}$$

where the kernels K_* and K_*' are given by

$$K_* := K_*(\mathbf{x},t) := K_*(|\mathbf{x}|,t) \qquad \text{and} \qquad K_*(r,t) := \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta_*'\}(r,t) \tag{26}$$

and

$$K_*' := K_*'(\mathbf{x},t) := K_*'(|\mathbf{x}|,t) \qquad \text{and} \qquad K_*'(r,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta_*''\}(r,t). \tag{27}$$

Using these operators and applying the inverse Fourier transform to (24) gives

$$\boxed{\nabla^2 G - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 G = -D_*' G - K(\mathbf{0},t)\delta_{\mathbf{x}}.} \tag{28}$$

For a general source term f, $p_{\text{att}} := \mathcal{A}f = G *_{\mathbf{x},t} f$ solves the equation

$$\boxed{\nabla^2 p_{\text{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p_{\text{att}} = -\mathcal{A}_s f\ ,} \tag{29}$$

where $\mathcal{A}_s$ denotes the space–time convolution operator with kernel

$$K_s := K_s(\mathbf{x},t) := -\mathcal{B}G + D_*' G + K(\mathbf{0},t)\cdot\delta_{\mathbf{x}} \tag{30}$$

where

$$\mathcal{B} := D_*^2 + \frac{2}{c_0}D_*\frac{\partial}{\partial t}\ . \tag{31}$$

Equation (29) is the pressure wave equation that obeys attenuation with attenuation coefficient (18).

Remark 4. In this remark we consider again the standard model, as in Definition 3.2. For this case the wave equation (29) can be casted in a form that resembles the standard attenuation wave equation (cf. Example 4.1). Since K is causal, it follows that K_* is causal too (the argumentation is analogous to Remark 3) and therefore the operator D_* is well-defined for all causal functions. Moreover, since $K_*' = 0$, it follows that $D_*' \equiv 0$. Using that K_* depends only on t it follows that

$$(D_* G) *_{\mathbf{x},t} f = [K_* *_t G] *_{\mathbf{x},t} f = K_* *_t [G *_{\mathbf{x},t} f] = D_*(G *_{\mathbf{x},t} f).$$

Convolving each term in (28) with a function f, using the previous identity and that $D_*' \equiv 0$, it follows that

$$\nabla^2 p_{\mathrm{att}} - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 p_{\mathrm{att}} = -f\,. \tag{32}$$

Since D_* does not depend on $\mathbf{x}$, a solution of (32) with source term

$$f(\mathbf{x},t) = \delta_x(x - x_0)\, f_0(t) \qquad \mathbf{x} = (x,y,z)^T \in \mathbb{R}^3$$

depends only on x and t by symmetry, i.e. p_{att} is a plane wave satisfying

$$\frac{\partial^2 p_{\mathrm{att}}}{\partial x^2}(x,t) - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 p_{\mathrm{att}}(x,t) = -\delta_x(x - x_0)\, f_0(t)\,.$$

Because D_* depends on the same $\alpha_*(\omega)$ as in the $3D$ case, the standard attenuation form does not depend on the space dimension. However, we emphasize that

$$f_0(t) = \delta_t(t) \qquad \text{and} \qquad f_0(t) = \frac{\partial \delta_t}{\partial t}(t)$$

produce qualitatively different plane waves. Which type of plane wave is usually used to model $1D$ attenuation is not clear, since the source term is not specified in the literature (cf. [27, 23]). Anyway, we see that our approach starting from a $3D$ spherical wave G, together with the superposition principle, also contains the $1D$ attenuation models.

In the following we review some wave equations obeying attenuation, which are frequently considered in the literature:

Example 4.1.

- For $\gamma > 0$ and $\gamma \notin \mathbb{N}$, denote by D_t^γ be the *Riemann-Liouville fractional derivative* (see [9, 19])) with respect to time. It is defined in the Fourier domain by

  $$\mathcal{F}\{D_t^\gamma f\} = (-\mathbf{i}\omega)^\gamma \mathcal{F}\{f\} \tag{33}$$

  and satisfies

  $$D_t^{2\gamma} f = D_t^\gamma D_t^\gamma f \qquad \text{and} \qquad \frac{\partial}{\partial t} D_t^\gamma f = D_t^\gamma \frac{\partial}{\partial t} f = D_t^{\gamma+1} f. \tag{34}$$

  Now, we consider the attenuation coefficient

  $$\beta_*(r,\omega) := \tilde{\alpha}_0 (-\mathbf{i}\omega)^\gamma r \qquad \text{with} \qquad \tilde{\alpha}_0 := \alpha_0 / \cos(\gamma\pi/2)\,, \tag{35}$$

which satisfies the attenuation law

$$\Re(\beta_*)(r,\omega) = \alpha(\omega) r \qquad \text{and} \qquad \alpha(\omega) = \alpha_0 |\omega|^\gamma$$

(cf. Example 3.1 and [27, 23]). Let D_* denote the time-convolution operator with kernel K_* defined by (26) and (35). Then form (33) and $K_* = \mathcal{F}^{-1}\{\tilde{\alpha}_0(-\mathbf{i}\omega)^\gamma\}/\sqrt{2\pi}$ it follows that $D_* = \tilde{\alpha}_0 D_t^\gamma$. In [21, 13] (see also [22, 23]) the following equation for the pressure function p_{att} of attenuated waves is investigated:

$$\boxed{\nabla^2 p_{\mathrm{att}} - \left[\tilde{\alpha}_0 D_t^\gamma + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 p_{\mathrm{att}} = -f\ ,} \tag{36}$$

which is equivalent to equation (32) with operator $D_* = \tilde{\alpha}_0 D_t^\gamma$. Let $\mathcal{A}$ denote the solution operator of (36), then from [7, Theorem 7.4.3] it follows that $\mathcal{A}$ has a causal domain of influence only for $\gamma \in [0,1)$.

- Let $\gamma > 0, \gamma \notin \mathbb{N}$. Neglecting in (36) the operator $\tilde{\alpha}_0^2 D_t^{2\gamma}$ (which one finds after expanding the decomposition operator) one finds Szabo's equation [22]

$$\nabla^2 p_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} p_{\mathrm{att}} - \frac{2\tilde{\alpha}_0}{c_0} D_t^{\gamma+1}(p_{\mathrm{att}}) = -f\ . \tag{37}$$

This equation is equivalent to equation (32) if we define the kernel of D_* by (26) with

$$\beta_*(r,\omega) := \tilde{\alpha}_0(-\mathbf{i}\omega)^\gamma r \quad \text{and} \quad \alpha_*(\omega) = \frac{\mathbf{i}\omega}{c_0} + \frac{1}{c_0}\sqrt{(-\mathbf{i}\omega)^2 + 2\tilde{\alpha}_0 c_0(-\mathbf{i}\omega)^{\gamma+1}}\ . \tag{38}$$

Remark 5. We note that the attenuation laws (real part of $\alpha_*(\omega)$) in Example 4.1 are very similar for small frequencies. Indeed, for the experimental measurable frequencies both laws are very similar

Again, if $\mathcal{A}$ denotes the solution operator of (37), then [7, Theorem 7.4.3] implies that $\mathcal{A}$ has a causal domain of influence only for $\gamma \in [0,1)$.

In the literature, the standard attenuation models (36) and (37) are considered as *homogeneous Cauchy problems* with *inhomogeneous* initial conditions. In contrast, in our setting, we consider *inhomogeneous Cauchy problems* with *homogeneous* initial conditions. In the following section we show that the two concepts can be equivalent. However, in general, only the concept suggested here leads to a rigorous framework, in which we can define solution operators for attenuated wave equations.

For the readers convenience, we summarize some important notation and facts in the following table. Note the difference between K, K_* and K'_*, respectively, with respect to the involved exponential function.

Kernel	General	Standard Form	Convolution Operator
K (9)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(-\beta_*)\}$	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(-\alpha_* \lvert\mathbf{x}\rvert)\}$	
K_* (26)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta'_*\}$	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\alpha_*\}$	D_* (25)
K'_* (26)	$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\beta''_*\}$	0	D'_* (25)

5 The homogeneous Cauchy problem with memory

We consider the standard attenuation model $\beta_*(r,\omega) = \alpha_*(\omega) r$. Let $\mathcal{A}$ denote a translation invariant operator with causal domain of influence and and let the operator D_* be as defined as in (25).

In this section we investigate under which conditions the inhomogeneous wave equation (32) with homogeneous initial conditions (4) (where p is replaced by p_{att}) and the homogeneous equation

$$\nabla^2 q_{\mathrm{att}} - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 q_{\mathrm{att}} = 0 \tag{39}$$

with the inhomogeneous initial conditions

$$q_{\mathrm{att}} = q_0 \qquad \text{for} \qquad t \leq 0 \qquad \text{and} \qquad \left.\frac{\partial}{\partial t} q_{\mathrm{att}}\right|_{t=0+} = \left.\frac{\partial}{\partial t} q_0\right|_{t=0-} \tag{40}$$

are equivalent. That is, both equations have the same solution for $t > 0$.

Theorem 5.1. *Assume that (39), (40) and (32), (4) have unique solutions, respectively. Then $q_{att} = p_{att}$ for $t > 0$ if and only if q_0 and p_{att} are related by the following conditions*

$$\varphi := \lim_{t\to 0-} q_0 = \lim_{t\to 0+} p_{att}\,, \qquad \psi := \lim_{t\to 0-} \frac{\partial}{\partial t} q_0 = \lim_{t\to 0+} \frac{\partial}{\partial t} p_{att} \tag{41}$$

and

$$\boxed{H \cdot \mathcal{B} q_0 = -f + \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right)\,,} \tag{42}$$

with $\mathcal{B}$ is as in (31) and H is the Heaviside function.

Proof. $\Rightarrow$: Assume that $q_{\mathrm{att}} = p_{\mathrm{att}}$ for $t > 0$. Then, using that $p_{\mathrm{att}} = 0$ for $t < 0$, implies that

$$p_{\mathrm{att}} = H \cdot q_{\mathrm{att}} \qquad \text{and} \qquad q_{\mathrm{att}} = p_{\mathrm{att}} + q_0\,. \tag{43}$$

In particular, property (41) holds. Moreover, (43) implies

$$\nabla^2 p_{\mathrm{att}} = H \cdot \nabla^2 q_{\mathrm{att}} \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2} p_{\mathrm{att}} = H \cdot \frac{\partial^2}{\partial t^2} q_{\mathrm{att}} + \psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t. \tag{44}$$

Since

$$\mathcal{B} + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} = \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 \tag{45}$$

it follows from (44), (45) and (43) that

$$\begin{aligned}\nabla^2 p_{\mathrm{att}} - \mathcal{B} p_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\, p_{\mathrm{att}} + f &= H \cdot \left[\nabla^2 q_{\mathrm{att}} - \mathcal{B} q_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\, q_{\mathrm{att}}\right] \\ &\quad - \mathcal{B} p_{\mathrm{att}} + H \cdot \mathcal{B} q_{\mathrm{att}} - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f\,.\end{aligned} \tag{46}$$

Using the definitions of q_{att} and p_{att}, (46) simplifies to

$$-\mathcal{B}p_{\mathrm{att}} + H \cdot \mathcal{B}q_{\mathrm{att}} - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f = 0\,.$$

Since $\mathcal{B}$ is a causal operator and p_{att} is a causal function, we have $\mathcal{B}p_{\mathrm{att}} = H \cdot \mathcal{B}p_{\mathrm{att}}$. This together with (43) implies that $-\mathcal{B}p_{\mathrm{att}} + H \cdot \mathcal{B}q_{\mathrm{att}} = H \cdot \mathcal{B}q_0$. Hence

$$H \cdot \mathcal{B}q_0 - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right) + f = 0$$

and thus (42) holds. This proves the first direction of the theorem.

$\Leftarrow$: To prove the opposite direction let

$$\tilde{p}_{\mathrm{att}} := H \cdot q_{\mathrm{att}} \qquad \text{such that} \qquad q_{\mathrm{att}} = \tilde{p}_{\mathrm{att}} + q_0\,.$$

We prove that $p_{\mathrm{att}} = \tilde{p}_{\mathrm{att}}$ holds for $t > 0$. Similarly as in part a) of the proof it follows that

$$\begin{aligned}\nabla^2\tilde{p}_{\mathrm{att}} - \mathcal{B}\tilde{p}_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,\tilde{p}_{\mathrm{att}} &= H \cdot \left[\nabla^2 q_{\mathrm{att}} - \mathcal{B}q_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,q_{\mathrm{att}}\right] \\ &\quad + H \cdot \mathcal{B}q_0 - \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right)\end{aligned}$$

holds. Since q_{att} solve problem (39), (40) and condition (42) is satisfied, the last identity simplifies to

$$\nabla^2\tilde{p}_{\mathrm{att}} - \mathcal{B}\tilde{p}_{\mathrm{att}} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\,\tilde{p}_{\mathrm{att}} = -\tilde{f}\,.$$

Hence we have shown that $\tilde{p}_{\mathrm{att}}$ solves problem (32), (4) and since this problem has the unique solution p_{att}, it follows $\tilde{p}_{\mathrm{att}} = p_{\mathrm{att}}$. In summary we have shown that

$$p_{\mathrm{att}} = \tilde{p}_{\mathrm{att}} = q_{\mathrm{att}} \qquad \text{for} \qquad t > 0\,,$$

which proves the assertion.

□

Remark 6. In the absence of attenuation the operator $\mathcal{B}$ is the zero operator and condition (42) reduces to

$$f = \frac{1}{c_0^2}\left(\psi \cdot \delta_t + \varphi \cdot \frac{\partial}{\partial t}\delta_t\right)\,.$$

In this case the solutions of

$$\boxed{\begin{array}{ll} \nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}p = -f\,, & \nabla^2 q - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}q = 0\,, \\ p\big|_{t<0} = 0\,,\ \frac{\partial}{\partial t}p\big|_{t<0} = 0\,, & q\big|_{t=0} = \varphi\,,\ \frac{\partial}{\partial t}q\big|_{t=0} = \psi, \end{array}}$$

are identical for $t > 0$.

6 The thermo-viscous wave equation

In this section we show that the *thermo-viscous* wave equation (see e.g. [10]) is not causal (see Theorem 6.1 below). The formalism introduced here will enable us to derive a causal variant of the thermo-viscous equation which satisfies the same attenuation law.

The thermo-viscous wave equation models propagation of pressure waves in viscous media and reads as follows

$$\boxed{\left(I + \tau_0 \frac{\partial}{\partial t}\right) \nabla^2 p_{\mathrm{att}} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p_{\mathrm{att}} = -F\ .} \tag{47}$$

Here τ_0 and c_0 denote the *relaxation time* and the *thermodynamic speed*, respectively and F models sources.

In the following we transform the thermo-viscous wave equation into the form (28), which enables us to deduce that the thermo-viscous equation is not causal. For these purpose we consider the attenuation coefficient

$$\beta_*(r,\omega) = \alpha_*(\omega) r \qquad \text{with} \qquad \alpha_*(\omega) = \frac{\mathbf{i}\omega}{c_0} - \frac{\mathbf{i}\omega}{c_0\sqrt{1-\tau_0 \mathbf{i}\omega}}\ . \tag{48}$$

and the time convolution operators $T^{1/2}$ and $L^{1/2}$ with kernels

$$K_{T^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{(1-\mathbf{i}\tau_0\omega)^{-1/2}\right\} \qquad \text{and} \qquad K_{L^{1/2}} := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{(1-\tau_0\mathbf{i}\omega)^{1/2}\right\}\ ,$$

respectively. Since K_* satisfies (26) it can be rewritten in the following form

$$K_* = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{\frac{\mathbf{i}\omega}{c_0} - \frac{\mathbf{i}\omega}{c_0\sqrt{1-\mathbf{i}\omega\tau_0}}\right\} = -\frac{1}{c_0}\frac{\partial}{\partial t}\delta_t + \frac{1}{c_0}\frac{\partial}{\partial t} K_{T^{1/2}}\ . \tag{49}$$

Therefore the according convolution operator D_* is given by

$$D_* = -\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{1}{c_0} T^{1/2} \frac{\partial}{\partial t}\ . \tag{50}$$

In the following we summarize some properties of the operators $T^{1/2}$, $L^{1/2}$ and D_*, and the associated kernels.

Lemma 6.1. *The kernel functions $K_{T^{1/2}}$, $K_{L^{1/2}}$ and the operators $L^{1/2}$, $T^{1/2}$, D_* respectively, satisfy:*

1. *For $\tau_0 = 0$, $T^{1/2} = L^{1/2} = I$ and $K_{T^{1/2}} = K_{L^{1/2}} = \delta_t$.*
2. $$K_{T^{1/2}}(t) = \frac{\sqrt{2\pi} H(t) \exp(-t/\tau_0)}{\Gamma(1/2)\tau_0^{1/2} t^{1/2}}\ . \tag{51}$$
3. *$L^{1/2}$ is the inverse of $T^{1/2}$,*
4. *Let $L := \left(L^{1/2}\right)^2$ and $T := \left(T^{1/2}\right)^2$, then*

$$L = I + \tau_0 \frac{\partial}{\partial t} \qquad \textit{and} \qquad T = L^{-1}\ . \tag{52}$$

5. $D'_* \equiv 0$ *and for* $\tau_0 = 0$ *also* $D_* \equiv 0$.

6.

$$\left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 = \frac{1}{c_0^2} T \frac{\partial^2}{\partial t^2}$$

Proof. 1. The first item is a trivial consequence of properties of the Fourier transform $\mathcal{F}^{-1}\{\cdot\}$.

2. With the substitution $s = -\mathbf{i}\omega\tau_0$ we derive the relation with the inverse Laplace transformation $\mathcal{L}^{-1}\{\cdot\}$ (for a definition and some basic properties see the appendix of this paper)

$$\begin{aligned} \mathcal{F}^{-1}\left\{(1-\mathbf{i}\omega\tau_0)^{-1/2}\right\}(t) &= \frac{1}{\mathbf{i}\tau_0\sqrt{2\pi}} \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \exp\left(st/\tau_0\right) \cdot (1+s)^{-1/2} ds \\ &= \frac{\sqrt{2\pi}}{\tau_0} \mathcal{L}^{-1}\left\{(1+s)^{-1/2}\right\}(t/\tau_0). \end{aligned} \tag{53}$$

Using the properties (64) and (65) of the inverse Laplace transformation the assertion follows.

3. From

$$K_{T^{1/2}} *_t K_{L^{1/2}} = K_{L^{1/2}} *_t K_{T^{1/2}} = \frac{1}{\sqrt{2}\,\pi} \mathcal{F}^{-1}\{1\} = \delta_t\,,$$

it follows that for each function f

$$\begin{aligned} T^{1/2} L^{1/2} f &= K_{T^{1/2}} *_t K_{L^{1/2}} *_t f = \delta_t *_t f = f \\ L^{1/2} T^{1/2} f &= K_{T^{1/2}} *_t K_{L^{1/2}} *_t f = \delta_t *_t f = f\,. \end{aligned} \tag{54}$$

4. Since

$$K_{L^{1/2}} *_t K_{L^{1/2}} = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\{1 - \mathbf{i}\omega\tau_0\} = \delta_t - \tau_0 \frac{\partial}{\partial t}\delta_t,$$

it follows that

$$L\,f = K_{L^{1/2}} *_t K_{L^{1/2}} *_t f = \left(\delta_t - \tau_0 \frac{\partial}{\partial t}\delta_t\right) *_t f = \left(I + \tau_0 \frac{\partial}{\partial t}\right) f\,.$$

The assertion $T = L^{-1}$ is then a consequence of the previous item.

5. Since K_* does not depend on $|\mathbf{x}|$ and K'_* is the kernel of D'_*, it follows that $K'_* = 0$, i.e. $D'_* \equiv 0$. The second statement is a direct consequence of Item 1 which states that $T^{1/2} = I$ for $\tau_0 = 0$.

6. Follows from (50).

□

The thermo-viscous wave equation (47) can be put in formal relation to the wave equation (28) by identifying an appropriate operator D_* as in (50):

Utilizing Item 6 of Lemma 6.1 in equation (32) and taking into account that $D'_* \equiv 0$ (cf. Item 5 of Lemma 6.1) shows that the solution of the thermo-viscous wave equation (47) with $F := L\,f$ satisfies

$$\nabla^2 p_{\mathrm{att}} - \left[D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right]^2 p_{\mathrm{att}} = \nabla^2 p_{\mathrm{att}} - \frac{1}{c_0^2} T\,\frac{\partial^2}{\partial t^2} p_{\mathrm{att}} = -f\,.$$

Conversely, the solution of equation (32) with D_* defined as in (50) satisfies the thermo-viscous wave equation (47) with $F = L\,f$.

Theorem 6.1. *Let α_* be defined as in (48). Then $\Re(\alpha_*)$ and $\Im(\alpha_*)$ satisfy the Kramers-Kronig relation, but the solution operator $\mathcal{A}$ of the thermo-viscous wave equation does not have a causal domain of influence.*

Proof. Since K_* defined as in (49) is causal, it follows that $\Re(\alpha_*)$ and $\Im(\alpha_*)$ satisfy the Kramers-Kronig relation. From [7, Theorem 7.4.3] it follows that the kernel $K := \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(-\alpha_*\,|\mathbf{x}|)\}$ is not causal and as a consequence the according solution operator of the thermo–viscous wave equation does not have a causal domain of influence. □

Remark 7. From (48) it follows that the attenuation law $\alpha = \Re(\alpha_*)$ approximates for small frequencies the frequency power law with $\gamma = 2$.

7 A causal thermo-viscous wave equation

Below we discuss a causal variant of the thermo-viscous wave equation.

Let $\alpha_1 \geq 0$. Theorem 7.1 below shows that the attenuation operator with attenuation coefficient of standard form $\beta_*(r,\omega) = \alpha_*^c(\omega) r$ and

$$\alpha_*^c(\omega) = -\frac{\alpha_1 \mathbf{i}\omega}{c_0\sqrt{1-\tau_0\mathbf{i}\omega}} \tag{55}$$

has a causal domain of influence. The operator D_* and its kernel K_* read as follows

$$\boxed{D_* := \frac{\alpha_1}{c_0} T^{1/2}\frac{\partial}{\partial t} \qquad \text{and} \qquad K_* = \frac{\alpha_1}{c_0}\frac{\partial}{\partial t} K_{T^{1/2}}\,.} \tag{56}$$

Note that $D'_* \equiv 0$, since K_* does not depend on $|\mathbf{x}|$.

For $\alpha_1 = 0$, $D_* \equiv 0$ and thus equation (28) with operator D_* defined by (56) is the standard wave equation (without attenuation). Since

$$\left(D_* + \frac{1}{c_0}\frac{\partial}{\partial t}\right)^2 = \frac{1}{c_0^2}\left[I + \alpha_1 T^{1/2}\right]^2 \frac{\partial^2}{\partial t^2} \qquad \text{and} \qquad L = T^{-1}\,,$$

(28) can be rewritten as

$$\boxed{\left(I + \tau_0\frac{\partial}{\partial t}\right)\nabla^2 p_{\mathrm{att}} - \left[\alpha_1 I + L^{1/2}\right]^2 \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} p_{\mathrm{att}} = -\left(I + \tau_0\frac{\partial}{\partial t}\right)\delta_{\mathbf{x},t}.} \tag{57}$$

Theorem 7.1. *Let α_* and α_*^c be defined as in (48) and (55), respectively. Then $\Re(\alpha_*^c)$ and $\Im(\alpha_*^c)$ satisfy the Kramers-Kronig relation and $\Re(\alpha_*) = \Re(\alpha_*^c)$. The solution operator $\mathcal{A}$ of equation (57) has a causal domain of influence.*

Proof. Since K_* defined as in (56) is causal, it follows that $\Re(\alpha_*^c)$ and $\Im(\alpha_*^c)$ satisfy the Kramers-Kronig relation. Comparison of α_* defined as in (48) and α_*^c defined as in (55) shows that $\Re(\alpha_*) = \Re(\alpha_*^c)$. From [7, Theorem 7.4.3] it follows that the kernel $K := \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(-\alpha_*^c|\mathbf{x}|)\}$ is causal and as a consequence the solution operator of equation (57) has a causal domain of influence. $\square$

Remark 8. In ultrasound imaging soft tissue is often modeled as a viscous fluid and therefore (57) is a potential model, on which thermoacoustic tomography can be based on. Moreover, the attenuation of tissue is frequently modeled as a power frequency law with $\gamma \in (1, 2)$.

8 Causal Wave Equations satisfying Frequency Power Laws for small frequencies with $\gamma \in (1, 2]$

In Example 3.1 we have shown that the frequency power law does not yield to a causal wave equation when $\gamma \geq 1$. In this section we derive causal wave equations for attenuation laws which approximate frequency power laws *for small frequencies* with exponent $\gamma \in (1, 2]$, where for $\gamma = 2$ we get the causal variant of the thermo-viscous wave equation (57).

Here we follow the notation of the previous section and introduce the following families of operators: For constants $\gamma \in (1, 2]$, $\tau_0 \geq 0$ and $\alpha_1 \geq 0$ let $T_\gamma^{1/2}$ and $L_\gamma^{1/2}$ denote time convolution operators with kernels:

$$K_{T_\gamma^{1/2}} := \frac{1}{\sqrt{2\pi}}\mathcal{F}\left\{\left(1 + (-\mathbf{i}\omega\tau_0)^{\gamma-1}\right)^{-1/2}\right\},$$
$$K_{L_\gamma^{1/2}} := \frac{1}{\sqrt{2\pi}}\mathcal{F}\left\{\left(1 + (-\mathbf{i}\omega\tau_0)^{\gamma-1}\right)^{1/2}\right\}.$$

We set $T_\gamma := \left(T_\gamma^{1/2}\right)^2$ and $L_\gamma := \left(L_\gamma^{1/2}\right)^2$. We emphasize that $T_2^{1/2} = T^{1/2}$, where $T^{1/2}$ is the operator in the thermo-viscous case. The operators $T_\gamma^{1/2}$ and $L_\gamma^{1/2}$ satisfy similar properties as the operators $T^{1/2}$ and $L^{1/2}$ in the thermo-viscous case:

The following lemma is proven analogously as Lemma 6.1.

Lemma 8.1.

- *For $\tau = 0$ we have*
$$T_\gamma^{1/2} = L_\gamma^{1/2} = I \qquad \textit{and} \qquad K_{T_\gamma^{1/2}} = K_{L_\gamma^{1/2}} = \delta_t\,.$$
- *$L_\gamma^{1/2}$ is the inverse of $T_\gamma^{1/2}$.*
- *Let $D_t^{\gamma-1}$ be the fractional derivative of order $\gamma - 1$, as defined as in (33), then*
$$L_\gamma = I + \tau_0^{\gamma-1}D_t^{\gamma-1}\,.$$

In analogy to Section 6 we consider now the standard attenuation coefficient with

$$\alpha_*(\omega) = -\frac{\alpha_1\mathbf{i}\omega}{c_0\sqrt{1 + (-\mathbf{i}\tau_0\omega)^{\gamma-1}}}\,. \tag{58}$$

Here α_1, τ_0 and c_0 are positive constants that are medium specific. The operator D_* and its kernel K_* are given by

$$\boxed{D_* := \frac{\alpha_1}{c_0} T_\gamma^{1/2} \frac{\partial}{\partial t} \qquad \text{and} \qquad K_* = \frac{\alpha_1}{c_0} \frac{\partial}{\partial t} K_{T_\gamma^{1/2}}\,.} \tag{59}$$

Moreover, the kernel K, defined by(9), reads as follows

$$K(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left\{ \exp\left(\frac{\alpha_1 \mathbf{i}\omega\, |\mathbf{x}|}{c_0 \sqrt{1+(-\mathbf{i}\tau_0\omega)^{\gamma-1}}} \right) \right\}(t) \tag{60}$$

For ω small we have

$$\alpha(\omega) \sim \alpha_0\, |\tau_0 \omega|^\gamma \qquad \text{with} \qquad \alpha_0 = \frac{\sin(\frac{\pi}{2}(\gamma-1))}{2\tau_0 c_0}.$$

Moreover, we note that the phase speed $c(\omega)$ is defined by

$$\frac{1}{c(\omega)} - \frac{1}{c_0} = -\frac{\Im(\alpha_*(\omega))}{\omega}$$

and thus from $(1+x)^{-1/2} \approx 1 - x/2$ ($|x| << 1$) and $(-\mathbf{i}\,\omega)^\gamma = |\omega|^\gamma\, e^{-\mathbf{i}\,\mathrm{sgn}(\omega)\,\pi\,\gamma/2}$, it follows for small frequencies

$$\frac{1}{c(\omega)} - \frac{1}{c_1} \approx \alpha_0\, \tan(\pi\,\gamma/2)\, |\omega|^{\gamma-1} \qquad \text{for} \qquad \alpha_0 > 0,\, \gamma \in (1,2)$$

with

$$c_1 := c_0/(1+\alpha_1) \qquad \text{and} \qquad \alpha_1\, \tau_0^{\gamma-1} := -\frac{2\, c_0\, \alpha_0\, \tan(\pi\,\gamma/2)}{\cos(\pi\,(\gamma-1)/2)} > 0\,.$$

Hence the velocity dispersion law (21) with $\omega_1 := 0$ in [27] (see also [23]) holds for small frequencies.

The wave equation (28) with D_* as in (59) reads as follows

$$\left(I + \tau_0^{\gamma-1} D_t^{\gamma-1}\right) \nabla^2 p_{\mathrm{att}} - \left[\alpha_1\, I + L_\gamma^{1/2}\right]^2 \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p_{\mathrm{att}} = -F. \tag{61}$$

In particular, for $\gamma = 2$ we recover the causal variant of the thermo-viscous wave equation (57).

Theorem 8.1. *The solution operator of equation (61) has a causal domain of influence.*

Proof. From [7, Theorem 7.4.3] it follows that K from (60) is causal and thus the solution operator of (61) has a causal domain of influence. □

9 Examples

In this section we present some calculations, highlighting the effects of non-causality. In all examples β_* is of standard form (11) and the solution operator $\mathcal{A}$ determined by β_* has the Green function

$$G(\mathbf{x},t) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} \frac{\exp\left(-\beta_*\right) \cdot \exp\left(-\mathbf{i} \cdot (t - \frac{|\mathbf{x}|}{c_0})\right)}{4\,\pi\,|\mathbf{x}|} d\omega\,.$$

We recall that the operator $\mathcal{A}$ has a causal domain of influence if and only if $\mathcal{F}^{-1}\{\exp(-\beta_*)\}$ is a causal function. In other words, non-causality can be observed if

$$\mathcal{F}^{-1}\{\exp(-\beta_*)\}(t) \neq 0 \qquad \text{for some} \qquad t < 0 \,.$$

All numerical simulations were performed in MATLAB with the fft-subroutine.

Frequency power law: Let $\alpha = \alpha_0 \left|\omega\right|^\gamma$ with some $\gamma > 0$. The extension α_* by the Kramers-Kroenig relation is given by (17). Fig. 1 shows simulations of $\mathcal{F}^{-1}\{\exp(-\beta_*)\}$, which illustrate that causality only holds for $\gamma \in [0, 1)$.

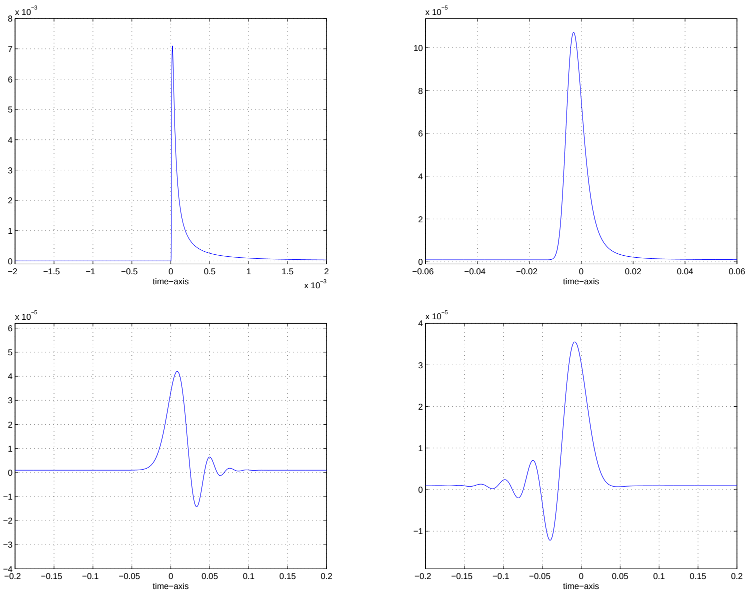

Figure 1: Simulation of $\mathcal{F}^{-1}\{\exp(-\beta_*(|\mathbf{x}|, \omega))\}$ for the frequency power law with $(\gamma, \alpha_0) \in \{(0.5, 0.1581), (1.5, 0.0316), (2.7, 0.0071), (3.3, 0.0027)\}$, $c_0 = 1$ and $|\mathbf{x}| = \frac{1}{4}$. In the first example $\gamma < 1$ and thus the function is causal. For all other cases it is non causal as predicted by the theory.

Szabos's model: Here $\alpha_*(\omega)$ is as in (38). In Fig. 2 we show simulations of $\mathcal{F}^{-1}\{\exp(-\beta_*)\}$. The numerical result confirm the mathematical considerations that causality only holds for $\gamma \in [0, 1)$.

Thermo-viscous wave equation: There α_* is as in (48). The left pictures in Fig. 3 shows a simulation of $\mathcal{F}^{-1}\{\exp(-\beta_*)\}$ for the thermo-viscous wave equation (47). Note that according to (48) and (55) the attenuation laws of the thermo-viscous wave equation and the causal variant (57) differ just by a multiplicative constant α_1. A simulation of $\mathcal{F}^{-1}\{\exp(-\beta_*)\}$ with $\alpha_1 = 1$ for the causal variant (57) of the thermo-viscous wave equation is shown in the right pictures of Fig. 3

10 Conclusions

In this paper we introduced the concept of an operator with causal domain of influence which guarantees a finite wave front speed. As a consequence these models allow for a stable numerical implementation and thus are suitable for photoacoustic imaging, where inversion techniques are required. Based on this concept, we showed that an attenuated wave described by such an operator satisfies the standard causality condition known as the *Kramers-Kronig relation*. However these relations are not sufficient to guarantee that an attenuated wave has a finite wave front speed. This is a common misunderstanding in causality theory.

We also showed that attenuated waves satisfying the frequency power law and the Kramers-Kronig relation have finite wave front speed only if $\gamma \in (0,1)$. An example of an equation where waves can propagate with infinite wave front speed is the thermo-viscous wave equation. Because of the lack of causality of standard models in the parameter range relevant for photoacoustic imaging, we developed novel equations that satisfy our causality requirement and the desired attenuation properties.

For our causality analysis all equations were formulated as inhomogeneous equations with homogeneous initial conditions, but we showed that if certain conditions are satisfied, then the attenuation problem can be formulated as a Cauchy problem with memory.

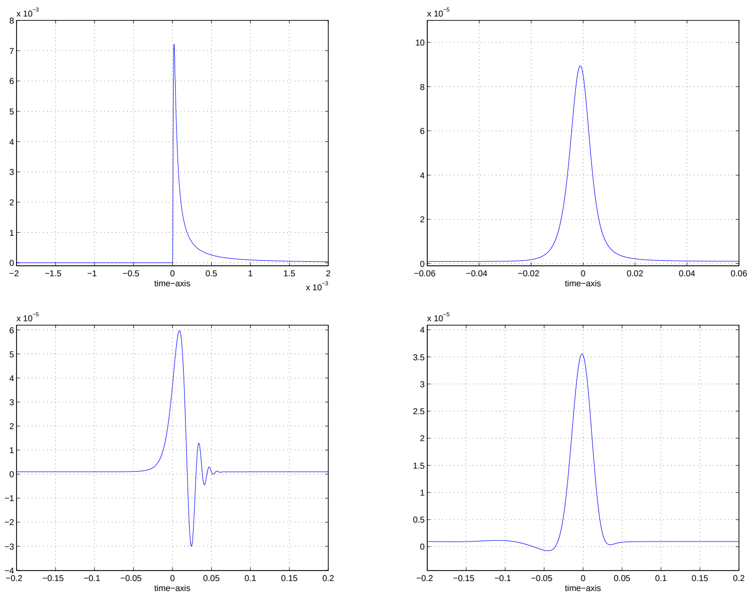

Figure 2: Simulation of $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*(|\mathbf{x}|\,,\omega)\right)\right\}$ for Szabo's frequency law with $(\gamma,\alpha_0) \in \{(0.5, 0.1581),\ (1.5, 0.0316),\ (2.7, 0.0071),\ (3.3, 0.0027)\}$, $c_0 = 1$ and $|\mathbf{x}| = \frac{1}{4}$.

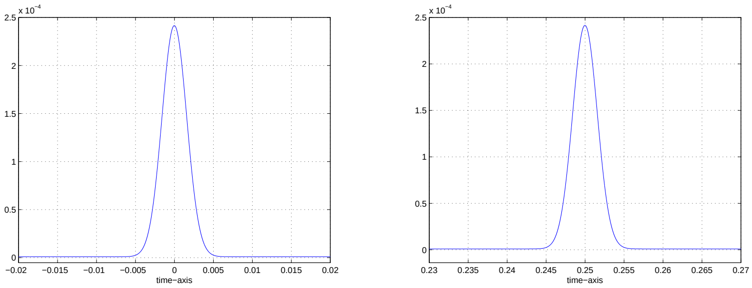

Figure 3: *Left:* $\mathcal{F}^{-1}\left\{\exp\left(-\beta_*(|\mathbf{x}|\,,\omega)\right)\right\}$ defined by the thermo-viscous wave equation (47) with $\tau_0 = 10^{-5}$, $c_0 = 1$ and fixed $|\mathbf{x}| = \frac{1}{4}$. *Right:* Causal variant (57) of the thermo-viscous wave equation with $\alpha_1 = 1$, $\tau_0 = 10^{-5}$, $c_0 = 1$ and fixed $|\mathbf{x}| = \frac{1}{4}$.

11 Appendix: Nomenclature and elementary facts

Real and Complex Numbers: $\mathbb{C}$ denotes the space of complex numbers, $\mathbb{R}$ the space of reals. For a complex number $c = a + \mathbf{i}b$ $a = \Re(c)$, $b = \Im(c)$ denote the real and imaginary parts, respectively.

Differential Operators: ∇ denotes the gradient. $\nabla\cdot$ denotes divergence, and ∇^2 denotes the Laplacian.

Product: When we write $\cdot$ between two functions, then it means a pointwise product, it can be a scaler product or if the functions are vector valued an inner product. The product between a function and a number is not explicitly stated.

Composition: The composition of operators $\mathcal{A}$ and $\mathcal{B}$ is written as $\mathcal{AB}$.

Special functions: The *signum* function is defined by

$$\mathrm{sgn} := \mathrm{sgn}(\mathbf{x}) := \frac{\mathbf{x}}{|\mathbf{x}|}\,.$$

In $\mathbb{R}^3$ it satisfies

$$\nabla \cdot \mathrm{sgn} = \frac{2}{|\mathbf{x}|}\,. \tag{62}$$

The *Heaviside* function

$$H := H(t) := \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

satisfies

$$H := \frac{1}{2}(1 + \mathrm{sgn})\,.$$

The Delta-distribution is the derivative of the Heaviside function at 0 and is denoted by $\delta_t := \delta_t(t)$. In our terminology δ_t denotes a *one*-dimensional distribution. The three dimensional Delta-distribution $\delta_{\mathbf{x}}$ is the tensor product of the three one-dimensional distributions δ_{x_i}, $i = 1, 2, 3$. Moreover,

$$\delta_{\mathbf{x},t} := \delta_{\mathbf{x},t}(\mathbf{x}, t) = \delta_{\mathbf{x}} \cdot \delta_t, \tag{63}$$

is a four dimensional distribution in space and time.

Properties related to functions: $\mathrm{supp}(g)$ denote the *support* of the function g, that is the closure of the set of points, where g does not vanish.

Derivative with respect to radial components: We use the notation

$$r := r(\mathbf{x}) = |\mathbf{x}|\,,$$

and denote the derivative of a function f, which is only dependent on the radial component $|\mathbf{x}|$, with respect to r (i.e., with respect to $|\mathbf{x}|$) by $\cdot'$.

Let $\beta = \beta(r)$, then it is also identified with the function $\beta = \beta(|\mathbf{x}|)$ and therefore

$$\nabla\beta = \frac{\mathbf{x}}{|\mathbf{x}|}\beta'\;.$$

Convolutions: Three different types of convolutions are considered: $*_t$ and $*_\omega$ denote *convolutions* with respect to time and frequency, respectively. Let f, $\hat{f}$, g and $\hat{g}$ be functions defined on the real line with complex values. Then

$$f *_t g := \int_{\mathbb{R}} f(t-t')g(t')dt', \qquad \hat{f} *_\omega \hat{g} := \int_{\mathbb{R}} \hat{f}(\omega-\omega')\hat{g}(\omega')d\omega'.$$

$*_{\mathbf{x},t}$ denotes space–time convolution and is defined as follows: Let f, g be functions defined on the Euclidean space $\mathbb{R}^3$ with complex values, then

$$f *_{\mathbf{x},t} g := \int_{\mathbb{R}^3}\int_{\mathbb{R}} f(\mathbf{x}-\mathbf{x}', t-t')g(\mathbf{x}', t')d\mathbf{x}'dt'\;.$$

Fourier transform: For more background we refer to [14, 25, 16, 7]. All along this paper $\mathcal{F}\{\cdot\}$ is the Fourier Transformation with respect to t, and the inverse Fourier transform $\mathcal{F}^{-1}\{\cdot\}$ is with respect to ω. In this paper we use the following definition of the *Fourier transform* $\mathcal{F}\{\cdot\}$ and its inverse $\mathcal{F}^{-1}\{\cdot\}$

$$\mathcal{F}\{f\}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(\mathbf{i}\omega t)f(t)dt, \quad \mathcal{F}^{-1}\left\{\hat{f}\right\}(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-\mathbf{i}\omega t)\hat{f}(\omega)d\omega\;.$$

The Fourier transform and its inverse have the following properties:

1. $$\mathcal{F}\left\{\frac{\partial}{\partial t}f\right\}(\omega) = (-\mathbf{i}\omega)\mathcal{F}\{f\}(\omega)\;.$$

2. $$\begin{aligned}\mathcal{F}\{f\cdot g\} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f\} *_\omega \mathcal{F}\{g\} \text{ and}\\ \mathcal{F}\{f\}\cdot\mathcal{F}\{g\} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f *_t g\}\,,\\ \mathcal{F}^{-1}\left\{\hat{f}\cdot\hat{g}\right\} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\hat{f}\right\} *_t \mathcal{F}^{-1}\{\hat{g}\} \text{ and}\\ \mathcal{F}^{-1}\left\{\hat{f}\right\}\cdot\mathcal{F}^{-1}\{\hat{g}\} &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{\hat{f} *_\omega \hat{g}\right\}\;.\end{aligned}$$

3. For $a \in \mathbb{R}$
$$\mathcal{F}\{f(t-a)\}(\omega) = \exp(-ia\omega)\cdot\mathcal{F}\{f(t)\}(\omega)$$
4. The Delta-distribution at $a \in \mathbb{R}$ satisfies
$$\delta_t(t-a) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{\exp(\mathbf{i}a\omega)\}(t)\ .$$
5. Let f be real and even, odd respectively, then $\mathcal{F}\{f\}$ is real and even, imaginary and odd, respectively.

The Hilbert transform for L^2-functions is defined by
$$\mathcal{H}\{f\}(t) = \frac{1}{\pi}⨍_{\mathbb{R}} \frac{f(s)}{t-s}ds\ ,$$
where $⨍_{\mathbb{R}} f(s)ds$ denotes the Cauchy principal value of $\int_{\mathbb{R}} f(s)ds$.

A more general definition of the Hilbert transform can be found in [2]. The Hilbert transform satisfies

- $\mathcal{H}\{\mathcal{F}\{f\}\}(\omega) = -i\mathcal{F}\{\mathrm{sgn} f\}(\omega)$,
- $\mathcal{H}\{\mathcal{H}\{f\}\} = -f$.

From the first of these properties the Kramers-Kronig relation can be formally derived as follows. Since $f(t)$ is a causal function if and only if $f = H\cdot f$ and $H = (1+\mathrm{sgn})/2$, it follows that $\mathcal{F}\{f\} = [\mathcal{F}\{f\}+\mathbf{i}\mathcal{H}\{\mathcal{F}\{f\}\}]/2$, which is equivalent to $\mathcal{F}\{f\} = i\mathcal{H}\{\mathcal{F}\{f\}\}$, i.e.
$$\Re(\mathcal{F}\{f\}) = -\Im(\mathcal{H}\{\mathcal{F}\{f\}\}) \qquad \text{and} \qquad \Im(\mathcal{F}\{f\}) = \Re(\mathcal{H}\{\mathcal{F}\{f\}\}).$$

The inverse Laplace transform of f is defined by
$$\mathcal{L}^{-1}\{f\}(t) = \begin{cases} 0 & \text{for} \quad t<0, \\ \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+\mathbf{i}\infty} \exp(st) f(s)ds, & \text{for} \quad t>0, \end{cases}$$
where γ is appropriately chosen.

The inverse Laplace transform satisfies (see e.g. [6]) that
$$\mathcal{L}^{-1}\{h(s-a)\}(t) = \exp(at)\mathcal{L}^{-1}\{h(s)\}(t) \text{ for all } a,t \in \mathbb{R} \tag{64}$$
and
$$\mathcal{L}^{-1}\left\{s^{-r}\right\}(t) = \frac{H(t)t^{r-1}}{\Gamma(r)} \qquad (r>0)\ . \tag{65}$$

Acknowledgement

This work has been supported by the Austrian Science Fund (FWF) within the national research network Photoacoustic Imaging in Biology and Medicine, project S10505-N20. The stay of XB in Innsbruck was partly supported by the "Frankreich-Schwerpunkt" of the University of Innsbruck.

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