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SIMULTANEOUS RECONSTRUCTIONS OF ABSORPTION DENSITY AND WAVE SPEED WITH PHOTOACOUSTIC MEASUREMENTS*

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Abstract. In this paper we propose an approach for *simultaneous* identification of the *absorption density* and the *speed of sound* using photoacoustic measurements. Experimentally our approach can be realized with sliced photoacoustic experiments. The mathematical model for such an experiment is developed and exact reconstruction formulas for both parameters are presented.

Key words. photoacoustics, parameter estimation, wave speed

AMS subject classifications. 35R30, 92C55, 65R10

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1. Introduction.

In this paper we propose an approach for *simultaneous* identification of the *absorption density* and the *speed of sound* using photoacoustic measurements.

In standard photoacoustic experiments the object is *uniformly* illuminated by a short electromagnetic impulse. Recently *sectional* photoacoustic imaging techniques [20, 25, 12, 13] and corresponding reconstruction techniques [6] have been developed. In sectional experiments thin hyperbola shaped like regions of the specimen are imaged (see Figure 1) sequentially. Experimentally, one can perform sectional photoacoustic imaging by focused illumination combined with focusing detectors. The focused illumination is achieved by using cylindrical lenses in front of the object. Moreover, contemporary focusing ultrasonic detectors have a spherical or cylindrical shape; thus the detector surface plays the role of the acoustic lens. The generated ultrasonic wave is refracted by a suitable acoustic lens such that out-of-plane (center of the hyperbola shaped regions) signals are generally weak and can be neglected. Thus essentially only signals emerging from the imaging plane are collected at the detector. This justifies that the illumination can be assumed restricted to a single plane (line in two dimensions). However, such a model requires a low scattering coefficient of the sample (which model organism specimens like the Zebra fish have). As opposed to *sectional three-dimensional photoacoustic imaging*, where stacks of complementary two-dimensional projection images are produced, the proposed approach for simultaneous imaging consists of performing overlapping sliced imaging by rotation and translation of the specimen. This also generates enough data for reconstructing the two independent parameter functions—speed of sound and absorption density (see Figure 2). Thus far the approach presented here is far from experimentally economical in the sense that a quick dimension analysis shows that we acquire far too much data—in $\mathbb{R}^3$

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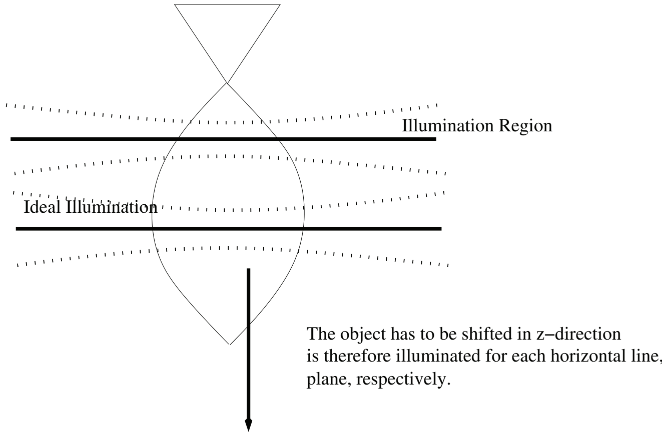

FIG. 1. *Conventional sectional photoacoustic imaging.*

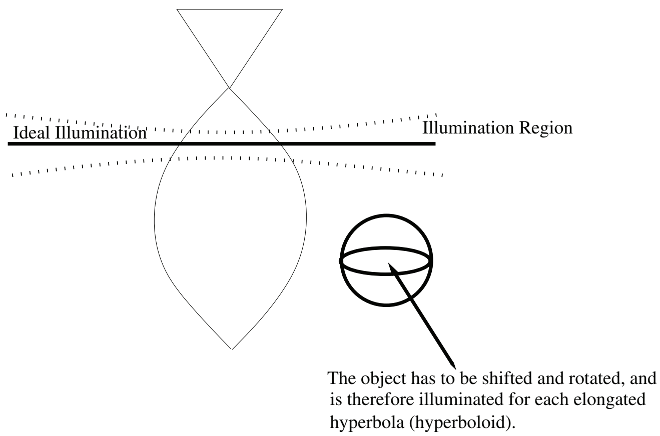

FIG. 2. *Sectional imaging in all directions produces enough data to reconstruct both imaging functions.*

we reconstruct two three-dimensional functions from eventually six-dimensional data. It is the goal of this paper, however, not to present the most economical approach, but rather to show that it is possible to derive exact reconstruction formulas for both imaging parameter functions.

The literature on reconstruction formulas and back-projection algorithms for photoacoustic imaging is vast. Wang and colleagues developed reconstruction formulas for cylindrical, spherical, and planar measurement geometries in a series of papers [30, 29, 28, 31], and recently, many more algorithms based on reconstruction formulas have been developed (see the survey [18]). As becomes apparent below, we can make use of reconstruction formulas from photoacoustics—essentially we make extensive use of inversion formulas for the spherical mean operator. Exact reconstruction formulas for the wave speed function in ultrasound reflectivity tomography are based on the *Born approximation* to the wave equation, which is valid for moderately varying wave speed. Reconstruction formulas for ultrasound reflectivity tomography have already been derived by Norton [21] and Norton and Linzer [22] and are also based on inversion formulas for the spherical mean operator. The possibility of exact inversion in both fields supports the derivation of exact inversion formulas for both parameters.

The proposed method is a hybrid and quantitative imaging method (see [2, 5, 17, 4] for some recent surveys). The terminology *quantitative* is typically reserved for estimating the diffusion and scattering parameters of the illumination source. Here, we understand quantitative in the sense that the wave speed parameter can be quantitatively estimated.

Most closely related to our approach is the work of Stefanov and Uhlmann [27], which presented a photoacoustic experiment for recovering either the sound speed or absorption density. A substitute to our work is also the recovery of the absorption density function for inhomogeneous wave speed [1, 14, 26]. In the work [3] the authors consider photoacoustic imaging with random sound speed fluctuation inhomogeneities.

The reconstruction formulas for simultaneous imaging utilize techniques from reflectivity imaging and photoacoustic imaging. In the current state of research we can provide exact reconstruction formulas, but when we try to be economical, i.e., using fewer slicing experiments, we would require reconstruction formulas, which have not been developed so far. This is highlighted in section 2.

The outline of this paper is as follows. In section 2 we discuss the mathematical model and the noneconomicality of the proposed model, which leaves room for further improvement of the results.

The practical criticism of our approach is that, due to physical constraints, sectional imaging is applicable to elongated objects. Current sectional photoacoustic imaging experiments, which, however, do not perform wave speed estimation, sample along a cylinder and do not require us to steer the sections in all angular directions; see the experimental papers [20, 25, 12, 13]. In this case, however, currently there do not exist analytical back-projection formulas for both parameters: the absorption density and wave speed function. Numerical reconstruction methods based on regularization, similar to those proposed in [33, 32], have to be implemented. For practical applications, however, we are thinking of an intermediate model, where a cylinder containing the specimen is slightly tilted (see Figure 3). This experimental suggestion does not contradict the assumptions of sectional imaging. In fact these experiments could also be used in combination with a photoacoustic spiral tomograph approach (see [8], which, however, did not elaborate on simultaneous identification of the wave speed and absorption density, but only on the latter). This paper serves as a case study for deriving more complex reconstruction formulas for advanced sampling geometries. We intend to report on numerical studies in a follow up paper. Finally, we mention that P. Elbau (RICAM, Linz, Austria) has recently discovered that one measurement in time for each section is sufficient for reconstructing both parameters. In his approach, the order of magnitude of the parameters and data recovered are the same. However, the complex steering of the experiment is still required and only the time measurements can be performed more quickly.

In section 3 we transform the model into the Fourier domain from which exact reconstruction formulas are derived in two and three dimensions in sections 3.2 and 3.1, respectively. The appendix provides the exact definitions used in this paper.

2. Model. In $\mathbb{R}^n$, we consider the following Cauchy problem for the wave equation:

$$
\begin{aligned}
\frac{1}{c^2}\,\partial_{tt}\tilde{u} \;-\; \Delta\tilde{u} &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}\,,\\
\tilde{u}(x,0) &= f(x)\,\delta_{r,\theta}(x) \quad \text{in } \mathbb{R}^n\,,\\
\partial_t \tilde{u}(x,0) &= 0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
$$

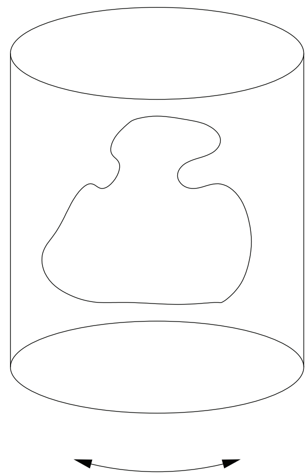

FIG. 3. *By tilting the cylinder one expects enough independent measurement data to reconstruct both parameter functions.*

where $c = c(x)$ denotes the speed of sound and $\delta_{r,\theta}(x) = \delta(\mathrm{dist}(x, E(r,\theta)))$, where $E(r,\theta)$ is the $(n-1)$-dimensional hyperplane with distance r from the origin and orientation $\theta \in S^{n-1}$. This is the photoacoustic equation with sliced illumination in the plane $E(r,\theta)$. It is our ultimate aim to reconstruct c and f from measurements of $\tilde{u}$ on some hypersurface Γ (see below).

We consider the Born approximation; that is, we expand $\tilde{u}$ formally with respect to the contrast function $q := 1/c^2 - 1$ and consider only terms of order at most q. This leads to the decomposition $\tilde{u} \approx u + v$, where $u = u^{r,\theta}$ is the solution of the wave equation

$$\begin{aligned} \partial_{tt} u \; - \; \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}\,, \\ u(x,0) &= f(x)\, \delta_{r,\theta}(x) \quad \text{in } \mathbb{R}^n\,, \\ \partial_t u(x,0) &= 0 \quad \text{in } \mathbb{R}^n\,, \end{aligned}$$

and $v = v^{r,\theta}$ solves

$$\begin{aligned} \partial_{tt} v - \; \Delta v &= -q(x)\, \partial_{tt} u \quad \text{in } \mathbb{R}^n \times \mathbb{R}_{>0}\,, \\ v(x,0) &= 0 \quad \text{in } \mathbb{R}^n\,, \\ \partial_t v(x,0) &= 0 \quad \text{in } \mathbb{R}^n\,. \end{aligned}$$

The dependence of v on r, θ is through the source function u.

It is our goal to reconstruct q and f from measurements of

$$m^{r,\theta}(x,t) \;=\; u^{r,\theta}(x,t) \;+\; v^{r,\theta}(x,t)\,, \quad (x,t) \in \Gamma \times (0,T)\,, \tag{2.1}$$

where Γ is an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$. Currently everything is fixed to complete measurements on the *whole* surface Γ.

We make the following assumptions, which are schematically represented in Figure 4:

1. $\Gamma = \partial B$ for some open and connected set $B \subset \mathbb{R}^n$ with smooth boundary. As is typical in photoacoustics, we consider B to be a sphere, a circle, or a halfplane.

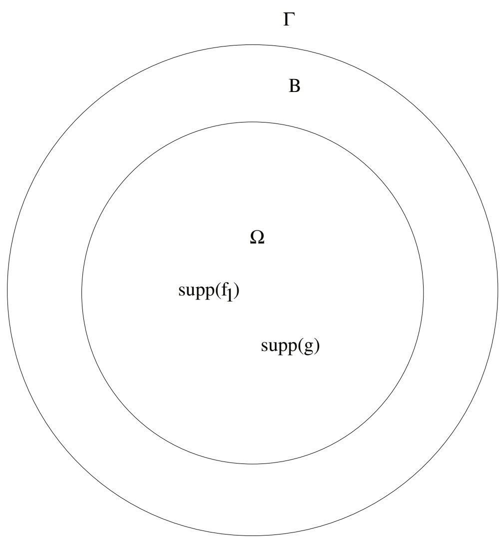

FIG. 4. *Schematic representation of the domains and supports.*

2. f is the sum of some known initial distribution f_0 with compact support and some unknown term f_1 to be determined.
3. The supports of f_1 and q are both contained in Ω for some bounded domain $\Omega \subset B$.
4. f_0, f_1, and q are smooth and $f(x) = f_0(x) + f_1(x) \neq 0$ for all $x \in \overline{B}$.

Dimensionality analysis. In the n-dimensional setting, we record measurements (cf. (2.1)) for every $(x,t) \in \Gamma \times (0,T)$ and every $(r,\theta) \in (0,\infty) \times S^{n-1}$. That is, the recorded data are $2n$-dimensional. The data to be recovered are f_1 and q, which are two n-dimensional functions. In general we think that we record too much data for the purpose of reconstructing f_1 and q. However, since we rely on Radon transforms techniques for exact inversion, we accept this disadvantage for mathematical studies.

3. Fourier reconstruction formulas. In this section we derive exact reconstruction formulas for f_1 and g. First we derive a general inversion formula, which is then evaluated differently in two and three dimensions.

In the following we omit the superscripts (r,θ) for the sake of convenience of notation. Let

$$\hat{u}(x,k) \;=\; \mathcal{F}u(x,k) \;=\; \int_0^\infty u(x,t)\, e^{\mathrm{i}kt}\, dt\,, \quad x \in \mathbb{R}^n\,,\ k \in \mathbb{R}\,,$$

be the Fourier transform. (Note that u vanishes for $t < 0$.) We assume that u is bounded in $\mathbb{R}^n \times \mathbb{R}_{>0}$ such that $u(x,\cdot) \in L^2(0,\infty)$ for every $x \in \mathbb{R}^n$. Then we note that $\hat{u}$ has the following properties (by the Paley–Wiener theorem):

(a) $\hat{u}(x,\cdot)$ has a holomorphic extension into $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ for all $x \in \mathbb{R}^n$;
(b) for every $x \in \mathbb{R}^n$ it holds that $\int_{-\infty}^\infty \big|\hat{u}(x,k_1+\mathrm{i}k_2) - \hat{u}(x,k_1)\big|^2 dk_1 \longrightarrow 0$ as k_2 tends to zero;
(c) $\hat{u}(\cdot,k)$ is bounded in $\mathbb{R}^n$ for all $k \in \mathbb{C}_+$.

We have

$$\begin{aligned}
\hat{u}(x,k) = \mathcal{F}u(x,k) \; &= \; \int_0^\infty u(x,t)\, e^{\mathrm{i}kt}\, dt \\
&= \frac{1}{\mathrm{i}k}\left[u(x,t)\, e^{\mathrm{i}kt}\big|_{t=0}^{\infty} \; - \; \int_0^\infty \partial_t u(x,t)\, e^{\mathrm{i}kt} dt \right] \\
&= -\frac{1}{\mathrm{i}k}\, u(x,0) \; - \; \frac{1}{(\mathrm{i}k)^2}\left[\partial_t u(x,t)\, e^{\mathrm{i}kt}\big|_{t=0}^{\infty} \; - \; \int_0^\infty \partial_{tt} u(x,t)\, e^{\mathrm{i}kt} dt \right] \\
&= -\frac{1}{\mathrm{i}k}\, u(x,0) \; - \; \frac{1}{k^2}\, \Delta \hat{u}(x,k)\,,
\end{aligned}$$

or, in other words,

$$\Delta \hat{u}(x,k) \; + \; k^2 \hat{u}(x,k) \; = \; \mathrm{i}k\, u(x,0)\,. \tag{3.1}$$

Moreover, we have

$$\begin{aligned}
\hat{v}(x,k) = \mathcal{F}v(x,k) \; &= \; -\frac{1}{k^2}\int_0^\infty \partial_{tt} v(x,t)\, e^{\mathrm{i}kt}\, dt \\
&= -\frac{1}{k^2}\int_0^\infty \left[\Delta v(x,t) - q(x)\, \partial_{tt} u(x,t)\right]\, e^{\mathrm{i}kt}\, dt \\
&= -\frac{1}{k^2}\, \Delta \hat{v}(x,k) \; + \; \frac{q(x)}{k^2}\int_0^\infty \partial_{tt} u(x,t)\, e^{\mathrm{i}kt} dt \\
&= -\frac{1}{k^2}\, \Delta \hat{v}(x,k) \; + \; \frac{q(x)}{k^2}\, \Delta \hat{u}(x,k)\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\Delta \hat{v}(x,k) \; + \; k^2 \hat{v}(x,k) &= q(x)\, \Delta \hat{u}(x,k) \\
&= -k^2 q(x)\, \hat{u}(x,k) \; + \; \mathrm{i}k\, q(x)\, u(x,0)\,.
\end{aligned}$$

Let Φ_k be the (radiating) fundamental solution of the Helmholtz equation $\Delta\hat{u}+k^2\hat{u} = 0$ in $\mathbb{R}^n$ for $k \in \mathbb{C}^+ := \left\{\tilde{k} \in \mathbb{C} \text{ with } \Im\tilde{k} \geq 0\right\}$; that is, in particular for $n = 2, 3$,

$$\Phi_k(x,y) \; := \; \begin{cases} \dfrac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|} & \text{for } n=3\,, \\ \dfrac{\mathrm{i}}{4}\, H_0^{(1)}(k|x-y|) & \text{for } n=2\,, \end{cases} \qquad x \neq y\,,$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind and order zero. Then *one particular* solution of (3.1) is given by

$$\begin{aligned}
\hat{u}(x,k) &= -\mathrm{i}k \int_{\mathbb{R}^n} u(y,0)\, \Phi_k(x,y)\, dy \\
&= -\mathrm{i}k \int_{y \in E(r,\theta)} f(y)\, \Phi_k(x,y)\, ds(y)\,.
\end{aligned}$$

We note that this particular solution satisfies the properties (a), (b), (c) from the beginning of this section. Any other solution is of the form $\hat{u}(\cdot,k) + w(\cdot,k)$, where w satisfies $\Delta w + k^2 w = 0$ in $\mathbb{R}^n$. The requirement that the solution satisfy the properties (a), (b), (c) from the beginning of this section implies that w vanishes. Indeed, for $k \in \mathbb{C}_+$ the function $w(\cdot, k)$ has to be bounded by property (c); therefore its Fourier transform (in the distributional sense) with respect to $x \in \mathbb{R}^n$ satisfies

$$\mathcal{F}_x w(y,k)\left[k^2 - |y|^2\right] \;=\; 0 \quad \forall\, y \in \mathbb{R}^n,$$

which implies $\mathcal{F}_x w(\cdot,k) = 0$ because $k^2 - |y|^2$ does not vanish for $k \in \mathbb{C}_+$. Therefore, as well, $w(\cdot,k) = 0$ in $\mathbb{R}^n$ for all $k \in \mathbb{C}_+$ and, by the continuity property (b), $w(\cdot,k) = 0$ in $\mathbb{R}^n$ for all $k \in \mathbb{C}^+$.

We also have

$$\begin{aligned}
\hat{v}(x,k) &= k^2 \int_{\mathbb{R}^n} q(y)\,\hat{u}(y,k)\,\Phi_k(x,y)\,dy \;-\; \mathrm{i}k \int_{\mathbb{R}^n} q(y)\,f(y)\,\delta_{r,\theta}(y)\,\Phi_k(x,y)\,dy \\
&= -\mathrm{i}k^3 \int_{\mathbb{R}^n} q(y) \int_{z\in E(r,\theta)} f(z)\,\Phi_k(y,z)\,ds(z)\,\Phi_k(x,y)\,dy \\
&\qquad - \mathrm{i}k \int_{z\in E(r,\theta)} q(z)\,f(z)\,\Phi_k(x,z)\,ds(z) \\
&= -\mathrm{i}k \int\limits_{z\in E(r,\theta)} f(z)\left[k^2 \int_{\mathbb{R}^n} q(y)\,\Phi_k(y,z)\,\Phi_k(x,y)\,dy \;+\; q(z)\,\Phi_k(x,z)\right] ds(z)\,.
\end{aligned}$$

In summary, we have

$$\begin{aligned}
\hat{m}^{r,\theta}(x,k) &= \hat{u}^{r,\theta}(x,k) \;+\; \hat{v}^{r,\theta}(x,k) \;=\; \hat{u}(x,k) \;+\; \hat{v}(x,k) \\
&= -\mathrm{i}k \int_{z\in E(r,\theta)} f(z)\Bigg[k^2 \int_{\mathbb{R}^n} q(y)\,\Phi_k(y,z)\,\Phi_k(x,y)\,dy \\
&\qquad\qquad + \;(q(z)+1)\,\Phi_k(x,z)\Bigg] ds(z) \\
&= R\big[(f(\cdot)\,L(x,\cdot,k))\big](r,\theta)\,,
\end{aligned}$$

where $R[f](r,\theta)$ is the $(n-1)$-dimensional Radon transform of f in direction (r,θ) and

$$L(x,z,k) \;=\; -\mathrm{i}k^3 \int_{\mathbb{R}^n} q(y)\,\Phi_k(y,z)\,\Phi_k(y,x)\,dy \;-\; \mathrm{i}k(q(z)+1)\,\Phi_k(x,z) \tag{3.2}$$

for $x \in \Gamma$, $z \in B$, $k \in \mathbb{R}$, $x \neq z$.

In other words, we have

$$\boxed{\mathcal{F}^{-1}\big(R^{-1}[\hat{u}+\hat{v}]\big)(x,z,t) \;=\; f(z)\,\check{L}(x,z,t)} \tag{3.3}$$

for $x \in \Gamma$, $z \in B$, $k \in \mathbb{R}$, $x \neq z$. Here, $\check{L}$ denotes the inverse Fourier transform of L.

Therefore, from the knowledge of $\hat{m}^{r,\theta}(x,k) = \hat{u}(x,k) + \hat{v}(x,k)$ for all $x \in \Gamma$, $k \in \mathbb{R}$, $r \geq 0$, and $\theta \in S^{n-1}$, we can determine

$$f(z)\,\check{L}(x,z,t) = \big(f_0(z) + f_1(z)\big)\,\check{L}(x,z,t)$$

for all $x \in \Gamma$, $z \in B$ with $x \neq z$, and $t \geq 0$.

3.1. Three-dimensional domain. For $n = 3$ we recall that

$$\Phi_k(x,y) \;=\; \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}\,, \quad x \neq y\,.$$

Thus from (3.2) it follows that

$$L(x,z,k) \;=\; -\mathrm{i}k^3 \frac{1}{16\pi^2} \int_{\mathbb{R}^3} q(y) \frac{e^{\mathrm{i}k(|y-z|+|x-y|)}}{|x-y||y-z|}\,dy \;-\; \mathrm{i}k(q(z)+1)\,\frac{e^{\mathrm{i}k|x-z|}}{4\pi|x-z|}\,.$$

Taking the inverse Fourier transform with respect to k gives

$$\begin{aligned}
\check{L}(x,z,t) &= \frac{1}{32\pi^3}\frac{(-\mathrm{i}k)^3}{(-\mathrm{i})^2}\int_{\mathbb{R}^3}\frac{q(y)}{|x-y||y-z|}\int_{-\infty}^{\infty} e^{\mathrm{i}k(|y-z|+|x-y|-t)}\,dk\,dy\\
&\quad + (q(z)+1)\,\frac{1}{8\pi^2|x-z|}\,\partial_t\left(\int_{-\infty}^{\infty} e^{\mathrm{i}k(|x-z|-t)}\,dk\right)\\
&= -\frac{1}{32\pi^3}\,\partial_{ttt}\left(\int_{\mathbb{R}^3}\frac{q(y)}{|x-y||y-z|}\int_{-\infty}^{\infty} e^{\mathrm{i}k(|y-z|+|x-y|-t)}\,dk\,dy\right)\\
&\quad + (q(z)+1)\,\frac{1}{8\pi^2|x-z|}\,\partial_t\left(\int_{-\infty}^{\infty} e^{\mathrm{i}k(|x-z|-t)}\,dk\right)\\
&= -\frac{1}{32\pi^3}\,\partial_{ttt}\left(\int_{\mathbb{R}^3}\delta(|y-z|+|y-x|-t)\frac{q(y)}{|z-y||y-x|}dy\right)\\
&\quad +\frac{q(z)+1}{8\pi^2|x-z|}\,\partial_t(\delta(|x-z|-t)).
\end{aligned}$$

Using that δ is -1 homogeneous, it follows that

$$\begin{aligned}
\check{L}(x,z,t) &= -\frac{1}{32\pi^3}\,\partial_{ttt}\left(\int_{\mathbb{R}^3}\delta(|y-z|+|y-x|-t)\frac{q(y)}{|z-y||y-x|}dy\right)\\
&\quad +\frac{q(z)+1}{8\pi^2|x-z|}\,\partial_t(\delta(t-|x-z|))\,.
\end{aligned}$$

We note that both terms on the right-hand side vanish for $t < |x-z|$. For the first term this follows from the fact that the domain of integration (with respect to y) is empty for $t < |x-z|$. Integrating this equation three times with respect to t gives

$$\begin{aligned}
\Psi(x,z,t) &:= \int_0^t\int_0^{s_2}\int_0^{s_1}\check{L}(x,z,\tau)\,d\tau\,ds_1\,ds_2\\
&= \frac{q(z)+1}{8\pi^2|x-z|}\int_0^t H(s_2-|x-z|)\,ds_2 + \mathcal{N}[q](x,z,t)\,,
\end{aligned}\tag{3.4}$$

where we used the Heaviside function $H(\tau) = 1$ for $\tau > 0$ and $H(\tau) = 0$ for $\tau < 0$. For $t > |x-z|$ we obtain

$$\Psi(x,z,t) \;=\; \frac{q(z)+1}{8\pi^2|x-z|}\,(t-|x-z|) \;+\; \mathcal{N}[q](x,z,t)\,.\tag{3.5}$$

We recall that $M(x,z,t) := f(z)\Psi(x,z,t)$ is known from measurement data for all $x \in \Gamma$, $z \in B$, and $t > 0$. First we take $z \in B \setminus \Omega$. Then $q(z) = 0$ and $f(z) = f_0(z)$; thus, $\Psi(x,z,t) = M(x,z,t)/f_0(z)$ is known and

$$\frac{M(x,z,t)}{f_0(z)} \;-\; \frac{t-|x-z|}{8\pi^2|x-z|} \;= \mathcal{N}[q](x,z,t)$$

for $t > |x-z|$. Again, the left-hand side is known. Now we let z tend to x. The limit on the right-hand side exists, and thus also on the left-hand side, and therefore for all $x \in \Gamma$ and $t > 0$ we have

$$\begin{aligned} \lim_{z\to x}\left[\frac{M(x,z,t)}{f_0(z)} - \frac{t}{8\pi^2|x-z|}\right] &= -\frac{1}{16\pi^3\, t^2}\int_{\mathbb{R}^3}\delta(|y-x|-t/2)q(y)\,ds(y) \\ &= -\frac{1}{16\pi^2}\,\mathcal{M}_2[q](x,t/2)\,, \end{aligned} \tag{3.6}$$

where $\mathcal{M}_2[q]$ is the spherical mean operator (see the appendix).

Thus the reconstruction algorithm is as follows:

1. Calculate the product $f(z)\check{L}(x,z,t)$ from (3.3) for all $x \in \Gamma$, $z \in B$, and $t > 0$. By integration (see (3.4)), this yields the knowledge of $M(x,z,t) := f(z)\Psi(x,z,t)$ for all $x \in \Gamma$, $z \in B$, and $t > 0$.
2. Solve (3.6) for q in Ω by inverting the spherical mean operator.
3. Compute $\Psi(x,z,t)$ for all $x \in \Gamma$, $z \in \Omega$, and $t > |x-z|$ from (3.5).
4. Finally, compute f from $f(z) = M(x,z,t)/\Psi(x,z,t)$ for $z \in \Omega$.

Remark 3.1. The operator $\mathcal{N}[q](x,z,t)$ is the *rotational ellipsoidal mean operator* with focal points x and z. Thus the integral equation (3.5), that is,

$$\Psi(x,z,t) \;=\; \frac{q(z)+1}{8\pi^2|x-z|}\,(t-|x-z|) \;+\; \mathcal{N}[q](x,z,t)\,,$$

can, for instance, be considered as a fixed point equation for q involving the ellipsoidal mean operator and can be solved by the fixed point iteration

$$\Psi(x,z,t) \;=\; \frac{q_n(z)+1}{8\pi^2|x-z|}\,(t-|x-z|) \;+\; \mathcal{N}[q_{n-1}](x,z,t)\,, n=1,2,\ldots.$$

Ellipsoidal mean operators have been studied in John's book [15].

3.2. Two-dimensional domain. In $\mathbb{R}^2$ we recall that the fundamental solution is given by

$$\Phi_k(x,y) \;=\; \frac{\mathrm{i}}{4}\,H_0^{(1)}(k|x-y|)\,, \quad x \neq y\,.$$

We have to compute the inverse Fourier transform of the product $\Phi_k(x,y)\,\Phi_k(z,y)$ (see (3.2)) and use the convolution theorem. First we have that

$$\frac{\mathrm{i}}{4}\,\mathcal{F}^{-1}\big(H_0^{(1)}(\cdot|x-y|)\big)(t) \;=\; \begin{cases} \dfrac{1}{2\pi\sqrt{t^2-|x-y|^2}}\,, & t > |x-y|\,, \\ 0\,, & t < |x-y|\,. \end{cases}$$

The convolution for the inverse Fourier transform is given by the formula $\mathcal{F}^{-1}(fg) = \check{f} * \check{g}$; that is,

$$\mathcal{F}^{-1}(fg)(t) \;=\; \big(\check{f} * \check{g}\big)(t) \;=\; \int_{-\infty}^{\infty}\check{f}(\tau)\,\check{g}(t-\tau)\,d\tau\,.$$

Therefore,

$$\begin{aligned} &\left(\frac{\mathrm{i}}{4}\right)^2 \mathcal{F}^{-1}\big(H_0^{(1)}(\cdot|x-y|)H_0^{(1)}(\cdot|z-y|)\big)(t) \\ &= \begin{cases} \frac{1}{(2\pi)^2}\int_{|x-y|}^{t-|y-z|}\frac{1}{\sqrt{\tau^2-|x-y|^2}}\,\frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}}\,d\tau\,, & t-|y-z| > |x-y|\,, \\ 0\,, & t-|y-z| < |x-y|\,, \end{cases} \end{aligned} \tag{3.7}$$

and thus from (3.2)

$$\begin{aligned}&\check{L}(x,z,t)\\&= -\frac{1}{4\pi^2}\,\partial_{ttt}\left(\int_{\mathbb{R}^2} q(y)\int_{|x-y|}^{t-|y-z|}\frac{1}{\sqrt{\tau^2-|x-y|^2}}\,\frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}}\,d\tau\,dy\right)\\&\quad+\frac{1}{2\pi}\,\partial_t\left(\frac{q(z)+1}{\sqrt{t^2-|x-z|^2}}H(t-|x-z|)\right),\end{aligned}$$

where H again denotes the Heaviside function. We integrate this equation three times and note that again, as in three dimensions, both terms on the right-hand side vanish for $t<|x-z|$. For $t>|x-z|$ we find

$$\begin{aligned}\Psi(x,z,t) &:= \int_0^t\int_0^{s_2}\int_0^{s_1}\check{L}(x,z,\tau)\,d\tau\,ds_1\,ds_2\\&= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} q(y)\int_{|x-y|}^{t-|y-z|}\frac{1}{\sqrt{\tau^2-|x-y|^2}}\,\frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}}\,d\tau\,dy\\&\quad+\frac{q(z)+1}{2\pi}\int_{|x-z|}^{t}\int_{|x-z|}^{s}\frac{1}{\sqrt{\tau^2-|x-z|^2}}\,d\tau\,ds\,.\end{aligned}$$

Integrating the second term on the right-hand side explicitly yields

$$\begin{aligned}\Psi(x,z,t) &= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} q(y)\int_{|x-y|}^{t-|y-z|}\frac{1}{\sqrt{\tau^2-|x-y|^2}}\,\frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}}\,d\tau\,dy\\&\quad+\frac{q(z)+1}{2\pi}\left[t\cosh^{-1}(t/|x-z|)-\sqrt{t^2-|x-z|^2}\right].\end{aligned}\tag{3.8}$$

Now we proceed as in the three-dimensional case. The function $M(x,z,t)=f(z)\Psi(x,z,t)$ is known for $x\in\Gamma$, $z\in B$, and $t>0$. For $z\in B\setminus\Omega$ we have that $q(z)=0$ and $f(z)=f_0(z)$; thus,

$$\begin{aligned}&\frac{M(x,z,t)}{f_0(z)}-\frac{1}{2\pi}\left[t\cosh^{-1}(t/|x-z|)-\sqrt{t^2-|x-z|^2}\right]\\&= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} q(y)\int_{|x-y|}^{t-|y-z|}\frac{1}{\sqrt{\tau^2-|x-y|^2}}\,\frac{1}{\sqrt{(t-\tau)^2-|z-y|^2}}\,d\tau\,dy\,.\end{aligned}\tag{3.9}$$

Letting z tend to x yields

$$\begin{aligned}\Phi(x,t) &:= \lim_{z\to x}\left[\frac{M(x,z,t)}{f_0(z)}-\frac{t\cosh^{-1}(t/|x-z|)}{2\pi}\right]-t\\&= -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} q(y)\,k(t,|x-y|)\,dy\end{aligned}\tag{3.10}$$

for $t>0$, where

$$k(t,r) := \begin{cases}\displaystyle\int_r^{t-r}\frac{1}{\sqrt{\tau^2-r^2}}\,\frac{1}{\sqrt{(t-\tau)^2-r^2}}\,d\tau\,, & t>2r>0\,,\\ 0\,, & 0<t<2r\,.\end{cases}$$

Using polar coordinates in (3.10), we therefore get

$$\begin{aligned}\int_{\mathbb{R}^2} q(y)\, k(t,|x-y|)\, dy &= \int_0^{t/2} k(t,r)\, r \int_0^{2\pi} q\left(x + r\begin{pmatrix}\cos\rho\\ \sin\rho\end{pmatrix}\right) d\rho\, dr\\ &= 2\pi \int_0^{t/2} \mathcal{M}_1[q](x,r)\, k(t,r)\, r\, dr\end{aligned}$$

with the one-dimensional circular mean operator $\mathcal{M}_1$. Using the notation

$$\tilde{k}(t,r) \ := \ r\, k(2t,r),$$

we get, for fixed x, the Volterra integral equation for $\mathcal{M}_1[q](x,\cdot)$:

$$-2\pi\, \Phi(x,2t) \ = \ \int_0^t \tilde{k}(t,r)\, \mathcal{M}_1[q](x,r)\, dr\,, \quad t \in (0,T]\,. \tag{3.11}$$

We express k as a complete elliptic integral: We introduce the variable $\phi \in (-\pi/2, \pi/2]$ by $\tau = \frac{t}{2} + \left(\frac{t}{2} - r\right)\sin\phi$. Then $d\tau = \left(\frac{t}{2} - r\right)\cos\phi\, d\phi$ and, with $t' = t/2$,

$$\begin{aligned}\tau^2 - r^2 &= (\tau - r)(\tau + r)\\ &= \left[(t'-r) + (t'-r)\sin\phi\right]\left[(t'+r) + (t'-r)\sin\phi\right],\\ (t-\tau)^2 - r^2 &= (t - \tau - r)(t - \tau + r)\\ &= \left[(t'-r) - (t'-r)\sin\phi\right]\left[(t'+r) - (t'-r)\sin\phi\right].\end{aligned}$$

For the product we conclude that

$$\begin{aligned}&\left[\tau^2 - r^2\right]\left[(t-\tau)^2 - r^2\right]\\ &= \left[(t'-r)^2 - (t'-r)^2\sin^2\phi\right]\left[(t'+r)^2 - (t'-r)^2\sin^2\phi\right]\\ &= (t'-r)^2\cos^2\phi\left[(t'+r)^2 - (t'-r)^2\sin^2\phi\right].\end{aligned}$$

Therefore,

$$\begin{aligned}k(t,r) &= \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\sqrt{(t'+r)^2 - (t'-r)^2\sin^2\phi}}\\ &= \frac{2}{t'+r}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \left(\frac{t'-r}{t'+r}\right)^2 \sin^2\phi}}\,,\end{aligned}$$

and thus

$$\tilde{k}(t,r) \ = \ \rho\, K(1-\rho) \quad \text{with} \quad \rho = \frac{2r}{t+r}\,,$$

where

$$K(\alpha) \ = \ \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha^2\sin^2\phi}}\,, \quad |\alpha| < 1\,,$$

denotes the complete elliptic integral.

Let $\delta > 0$ such that $\delta < \operatorname{dist}(\Gamma, \Omega)$. Since $\mathcal{M}_1[q](x,\tau) = 0$ for $\tau \leq \delta$ we can consider the integral equation (3.11) on $[\delta, T]$; that is,

$$\boxed{\Phi(x,t) \;=\; \int_\delta^t \tilde{k}(t,r)\,\mathcal{M}_1[q](x,r)\,dr\,, \quad t \in [\delta, T]\,.} \tag{3.12}$$

Now we note that $\tilde{k}(t,t) = K(0) = \pi/2$ and $\tilde{k} \in C^1(\Delta)$, where $\Delta = \{(t,r) \in [\delta,T]^2 : r \leq t\}$. Therefore, we can transform the equation of the first kind into one of the second kind by differentiating (3.12). This yields

$$\boxed{\partial_t \Phi(x,t) \;=\; \frac{\pi}{2}\,\mathcal{M}_1[q](x,t) \;+\; \int_\delta^t \frac{\partial}{\partial t}\tilde{k}(t,r)\,\mathcal{M}_1[q](x,r)\,dr\,, \quad t \in [\delta, T]\,.} \tag{3.13}$$

Volterra integral equations of the second kind with smooth kernel and nonvanishing diagonal have a unique solution; see, e.g., [16, section 3.3].

After calculating the spherical mean operator $\mathcal{M}_1[q]$, it can be inverted by using standard inversion formulas, which exist for various geometries. For $B = B_R^2(0) \subset \mathbb{R}^2$, analytical reconstruction formulas have been derived by Finch, Haltmeier, Rakesh [9]. For a general domain Ω, Kunyansky [19] reduced the reconstruction problem to the determination of the eigenvalues λ_k and normalized eigenfunctions u_k, $\|u_k\|_2 = 1$, of the Dirichlet Laplacian $-\Delta$ on Ω with zero boundary conditions. Recently Palamadov [24] presented a general approach leading to reconstruction algorithms for various geometries. For more details, see also the appendix.

Thus the reconstruction algorithm is as follows:

1. Calculate the product $M(x,z,t) := f(z)\check{L}(x,z,t)$ from (3.3) for all $x \in \Gamma$, $z \in B$, $t > 0$. Compute $\Phi(x,t)$ from (3.10) for all $x \in \Gamma$ and $t > 0$.
2. Solve (3.13) for calculating $\mathcal{M}_1[q]$.
3. Invert the spherical mean operator.
4. From (3.8) determine Ψ.
5. Finally use $f(z) = M(x,z,t)/\Psi(x,z,t)$ to reconstruct $f(z)$ for $z \in \Omega$.

Appendix.

Notation. $B_R^n(x)$ denotes the ball of radius R and center x in $\mathbb{R}^n$.

Bessel functions. The Bessel functions, Neumann functions, and Hankel functions of order zero are defined as

$$J_0(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}z\sin(\tau)}\,d\tau \quad \text{for } z \in \mathbb{C}\,,$$

$$Y_0(z) = \frac{2}{\pi}\left[\gamma + \ln(z/2)\right] J_0(z) \;-\; \frac{2}{\pi}\sum_{\ell=0}^{\infty} \frac{a_{\ell+1}}{(\ell!)^2}\left(\frac{z}{2}\right)^{2\ell} \quad \text{for } z \in \mathbb{C} \text{ with } \Im z \geq 0\,,$$

$$H_0^{(1)} = J_0 \;+\; \mathrm{i}\,Y_0\,,$$

where γ denotes Euler's constant, $a_1 = -\gamma$, and $a_n = -\gamma + \sum_{\ell=1}^{n-1} \ell^{-1}$ for $n \geq 2$.

From this we conclude that $H_0^{(1)}(-x) = \overline{H_0^{(1)}(x)}$ for $x \in \mathbb{R}$.

The Fourier transform. In this paper we use the following notation:

$$\hat{f}(k) \ := \ \mathcal{F}f(k) \ := \ \int_{-\infty}^{\infty} f(t)\, e^{\mathrm{i}kt}\, dt$$

denotes the Fourier transform of f and

$$\check{f}(t) \ := \ \mathcal{F}^{-1}f(t) \ := \ \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\, e^{-\mathrm{i}tk}\, dk$$

the inverse Fourier transform. For $f \in L^1(\mathbb{R})$ or f from the Schwarz space $\mathcal{S}$ these functions are defined in the classical sense, for $f \in L^2(\mathbb{R})$ they are defined by extension using the Parseval formula, and for tempered distributions they are defined by duality.

Examples:

$$\begin{aligned}
\hat{\delta}(k) = \mathcal{F}\delta(k) \ &= \ \int_{-\infty}^{\infty} e^{\mathrm{i}kt}\, \delta(t)\, dt \ \equiv \ e^{\mathrm{i}k0} \ = \ 1\,, \\
\check{\delta}(t) = \mathcal{F}^{-1}\delta(t) \ &= \ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}kt}\, \delta(k)\, dk \ \equiv \ \frac{1}{2\pi}\,.
\end{aligned}$$

This shows that

$$\hat{1}(k) \ = \ 2\pi\delta(k) \quad \text{and} \quad \check{1}(t) \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}kt}\, dk \ = \ \frac{1}{2\pi}\,\delta(t)\,.$$

Furthermore, we have (see [23])

$$\frac{\mathrm{i}}{4}\, H_0^{(1)}(ka) \ = \ \int_a^{\infty} \frac{\exp(\mathrm{i}kt)}{2\pi\sqrt{t^2-a^2}}\, dt \quad \text{for } k > 0\,.$$

From $H_0^{(1)}(-x) = \overline{H_0^{(1)}(x)}$ for all $x \in \mathbb{R}$ we conclude that this formula holds for all $k \in \mathbb{R}$ with $k \neq 0$. The right-hand side is a Fourier transform. Therefore,

$$\mathcal{F}^{-1}\left(\frac{\mathrm{i}}{4}\, H_0^{(1)}(a\cdot)\right) \ = \ \begin{cases} \dfrac{1}{2\pi\sqrt{t^2-a^2}}\,, & t > a\,, \\ 0\,, & t < a\,. \end{cases}$$

Radon transform. In $\mathbb{R}^n$, $E(r,\theta)$ denotes the $(n-1)$-dimensional hyperplane with orientation $\theta \in S^{n-1}$ and distance r from origin.

Let $f : \mathbb{R}^n \to R$ with support in Ω. Then, the $(n-1)$-dimensional Radon transform of f is defined by

$$R_{r,\theta}(f) \ = \ R[f](r,\theta) \ = \ \int_{E(r,\theta)} f(x) ds(x)\ .$$

Thus $R[f]$ is a function from $\mathbb{R} \times S^{n-1}$ into $\mathbb{R}$.

Spherical mean operator. In $\mathbb{R}^n$ the *spherical mean operator* is defined as follows (see, e.g., [7]):

$$\mathcal{M}_{n-1}[u](x,r) := \frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(x+ry)\, ds(y) \quad \text{ for } x \in \mathbb{R}^n,\ r > 0\ .$$

The spherical mean operator can be written as

$$\mathcal{M}_{n-1}[u](x,r) = \frac{1}{r^{n-1}|S^{n-1}|}\int_{\mathbb{R}^n} \delta(|x-z|-r)u(z)\,dz \qquad \text{for } x \in \mathbb{R}^n,\ r > 0\ .$$

In particular,

$$\begin{aligned}\mathcal{M}_2[u](x,r) &= \frac{1}{4\pi^2}\int_{S^2} u(x+ry)\,ds(y) = \frac{1}{4\pi r^2}\int_{\mathbb{R}^3}\delta(|x-z|-r)u(z)\,dz\,,\\ \mathcal{M}_1[u](x,r) &= \frac{1}{2\pi}\int_{S^1} u(x+ry)\,ds(y) = \frac{1}{2\pi r}\int_{\mathbb{R}^2}\delta(|x-z|-r)u(z)\,dz\ .\end{aligned}$$

There exist a variety of reconstruction formulas for averages over circular and spherical means in two and three dimensions.

In two dimensions.

- For averaged data on the circle of radius R (i.e., on $\partial\Omega = \partial B_R^2(0) \subset \mathbb{R}^2$), analytical reconstruction formulas have been derived by Finch, Haltmeier, Rakesh [9] and read as

  $$f(\xi) = \frac{1}{2\pi}\Delta_\xi\left(\int_{S^1}\int_0^{2R} r(\mathcal{M}_1[f])(R\theta,r)\log|r^2-|\xi-R\theta|^2|\,dr\,ds(\theta)\right)$$

  and

  $$f(\xi) = \frac{1}{2\pi}\int_{S^1}\int_0^{2R}\left(\partial_r r\partial_r\mathcal{M}_1[f]\right)(R\theta,r)\log|r^2-|\xi-R\theta|^2|drds(\theta).$$

- For a general domain Ω, Kunyansky [19] reduced the reconstruction problem to the determination of the eigenvalues λ_k and normalized eigenfunctions u_k, $\|u_k\|_2 = 1$, of the Dirichlet Laplacian $-\Delta$ on Ω with zero boundary conditions:

  $$\begin{aligned}\Delta u_k(\xi) + \lambda_k u_k(\xi) &= 0, \quad \xi \in \Omega,\\ u_k(\xi) &= 0, \quad \xi \in \partial\Omega.\end{aligned}$$

  Indeed, if $(\xi,\eta) \mapsto G_{\lambda_k}(|\xi-\eta|)$ is a free-space rotationally invariant Green's function of the Helmholtz equation and $n(\xi)$ denotes the outer unit normal vector of $\partial\Omega$ at $\xi \in \partial\Omega$, then

  $$f(\xi) = 2\pi\sum_{k=0}^{\infty}\tilde{M}_k u_k(\xi),$$

  where

  $$\tilde{M}_k = \int_{\partial\Omega}\int_0^\infty r\mathcal{M}_1[f](\eta,r)G_{\lambda_k}(r)\left\langle\nabla u_k(\eta),n(\eta)\right\rangle drds(\eta).$$

In three dimensions. We refer to the survey of Finch and Rakesh [11] (see also [10]). In this article, three reconstruction formulas are documented:

$$\begin{aligned}f(x) &= -\frac{1}{2\pi R_0}\int_{\partial B^3_{R_0}}\frac{\partial_{tt}\left(t^2(\mathcal{M}_2[f])(x_0,t)\right)\big|_{t=|x-x_0|}}{|x-x_0|}\,ds(x_0)\,,\\ f(x) &= -\frac{1}{2\pi R_0}\int_{\partial B^3_{R_0}}\frac{\left(\partial_t t\partial_t t(\mathcal{M}_2[f])\right)(x_0,t)\big|_{t=|x-x_0|}}{|x-x_0|}\,ds(x_0)\,,\\ f(x) &= -\frac{1}{2\pi R_0}\Delta\left(\int_{\partial B^3_{R_0}}|x-x_0|\mathcal{M}_2[f])(x_0,|x-x_0|)\,ds(x_0)\right)\ .\end{aligned}$$

Here, $x \in B^3_{R_0}$, and f is compactly supported in this set.

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