A Total Variation Based, Locally Adaptive Algorithm for Image Restoration

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Outline

1. Total Variation Regularization
   - Definition
   - Parameter Selection

2. Locally Adaptive Parameter Selection
   - A Dual Variable as Regularity Measure
   - Proposed Algorithm
   - Application to Deblurring Problems

3. Numerical Examples
   - Denoising
   - Deblurring

4. Conclusion
Denoising

Remove noise from some given image \( f : \Omega \rightarrow \mathbb{R}^d \).

Assume that

\[
f = u + n^\delta,
\]

with \( u \) a clean image and \( n^\delta \) noise.

Goal: reconstruct \( u \) from the given data \( f \).

Basic assumptions:

- Noise characterized by fast oscillations.
- Image consists of well separated uniform regions.
Assume that the noise satisfies
\[ \| n^\delta \|_2^2 = \int_{\Omega} |n^\delta(x)|^2 \, dx \approx \delta^2. \]

Minimize
\[ |Du(\Omega)| :\approx \int_{\Omega} |\nabla u(x)| \, dx \]
subject to the constraint
\[ \| u - f \|_2^2 = \int_{\Omega} (u(x) - f(x))^2 \, dx = \delta^2. \]
Equivalent formulation: Minimize

\[ T(u; \alpha) := \frac{1}{2} \int_{\Omega} (u(x) - f(x))^2 \, dx + \alpha |Du|(\Omega) \]

and choose \( \alpha > 0 \) such that \( u_\alpha := \arg \min_u T(u; \alpha) \) satisfies

\[ \|u_\alpha - f\|_2^2 = \delta^2. \]

For simplicity, the regularization parameter \( \alpha > 0 \) is often chosen a–priori.

We therefore obtain an *unconstrained* minimization problem.
Locally Adaptive TV Regularization

Example

Noisy Image

Denoised Image
Implicit assumptions:

- Noise level is known.
- Noise is identically and independently distributed.
- Significant objects in the image are of comparable scale (area times contrast per perimeter).

If assumptions are violated:

- Oversmoothing in some parts, loss of small scale structures.
- Too little smoothing in other parts, image still noisy.
Nonconstant Noise

Noisy Image

TV-Regularization, $\alpha = 20$
Nonconstant Noise

Noisy Image

TV-Regularization, $\alpha = 30$
Nonconstant Noise

Noisy Image

TV-Regularization, $\alpha = 40$
Nonconstant Noise

Noisy Image

Proposed Method
Instead of a number $\alpha \in (0, +\infty)$, use a continuous regularization function

$$\alpha : \Omega \to (0, +\infty),$$

and minimize

$$\mathcal{T}(u; \alpha) := \frac{1}{2} \int_{\Omega} (u(x) - f(x))^2 \, dx + \int_{\Omega} \alpha(x) \, d|Du|.$$ 

Problem: Definition of the regularization function.
Existing Methods with Non-constant $\alpha$


- Start with large regularization parameter $\alpha$ and compute $u_\alpha$.
- Compute the residual $u_\alpha - f$ on patches $\Omega_i \subset \Omega$.
- Test, whether $u_\alpha - f$ resembles Gaussian noise on $\Omega_i$. If not, locally decrease $\alpha$ on $\Omega_i$.
- Repeat, until $u_\alpha - f$ everywhere resembles noise.

Similar ideas in Frigaard & Scherzer (2006), and in Gilboa, Sochen & Zeevi (2006) for texture preserving TV-regularization.
Several methods with anisotropic total variation term:

Berkels, Burger et al. (2006): TV-regularization with rotating $\ell^1$-TV term.


Existing Methods with Non-constant $\alpha$


Parameter choice based on scale recognizing properties of total variation regularization:

$$\text{Scale} = \frac{\text{Contrast} \times \text{Area}}{\text{Perimeter}}.$$ 

Fix target scale $S$ and iteratively adapt the regularization parameter, until no features of scale less than $S$ are left.

Only method able to deal with unknown noise level.
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Minimizing $\mathcal{T}(\cdot; \alpha)$ is equivalent to solving the dual problem:

$$\mathcal{J}(V) := \frac{1}{2} \int_{\Omega} (\text{div} \, V(x) + f(x))^2 \, dx \rightarrow \min,$$

$V \in L^\infty(\Omega; \mathbb{R}^n \times \mathbb{R}^d), \quad \|V/\alpha\|_\infty \leq 1, \quad V \cdot \nu = 0 \text{ on } \partial \Omega.$

Here $\text{div} \, V$ and the equation $V \cdot \nu = 0$ are understood componentwise in case $d > 1$.

That is,

$V_\alpha$ solves the dual problem $\iff u_\alpha := f + \text{div} \, V_\alpha \in \arg \min \mathcal{T}(\cdot; \alpha).$
Interpretation

The dual variable $V_\alpha$ corresponds to the direction of $\nabla u_\alpha$,

$$V_\alpha(x) = \alpha(x) \frac{\nabla u_\alpha(x)}{|\nabla u_\alpha(x)|}.$$  

Variations of $V_\alpha/\alpha \leftrightarrow$ oscillations of $u_\alpha$.

Goal of parameter adaptation:
Use the regularity of $V_\alpha/\alpha$ as a measure of the regularity of $u_\alpha$.
Iteratively adapt $\alpha$ until the variations of $V_\alpha/\alpha$ look uniform over the whole image.
Consider smooth convolution kernel $\eta : \mathbb{R}^n \to \mathbb{R}$.

Define local mean $M_\eta(V_\alpha/\alpha) : \Omega \to \mathbb{R}^{n \times d}$:

$$M_\eta(V/\alpha) := \eta \ast (V_\alpha/\alpha) .$$

Define local variance $\text{Var}_\eta(V_\alpha/\alpha) : \Omega \to [0, 1]$:

$$\text{Var}_\eta(V_\alpha/\alpha) := \eta \ast (\|M_\eta(V_\alpha/\alpha) - V_\alpha/\alpha\|^2) .$$

Iteratively adapt $\alpha$ until $\text{Var}_\eta(V_\alpha/\alpha) \approx \text{const.}$
Example for Local Variance

Denoised Image

Local Variance
Choose some final target variation $0 < \theta < 1$ and update $\alpha$ until $\text{Var}_\eta (V_\alpha / \alpha) \approx \theta$:

If $\text{Var}_\eta (V_\alpha / \alpha)(x) < \theta$: decrease $\alpha(x)$.
If $\text{Var}_\eta (V_\alpha / \alpha)(x) > \theta$: increase $\alpha(x)$.

Chosen update:

$$\alpha_{\text{new}}(x) := \alpha(x) \cdot \frac{\text{Var}_\eta (V_\alpha / \alpha)(x) + \theta}{2\theta}.$$ 

For better stability: convolve $\alpha_{\text{new}}$ with some smooth kernel $\rho$. 
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Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.

   The smaller $\theta$, the smoother the output image:
   - $\theta = 1$ will result in $f$ as output.
   - $\theta = 0$ will result in a constant image.
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$. 

Locally Adaptive TV Regularization
Locally Adaptive Parameter Selection
Proposed Algorithm

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Locally Adaptive TV Regularization
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$.
3. Compute

$$V_i := \arg \min_{V} \| f + \text{div } V \|^2_2$$

subject to the constraints

$$\| V/\alpha_i \|_\infty \leq 1 \quad \text{and} \quad V \cdot \nu = 0 \text{ on } \partial \Omega .$$

4. If

$$\| \text{div } V_i - \text{div } V_{i-1} \| < \varepsilon$$

stop the iteration, else

$$i \rightarrow i + 1$$

and go to 3.

5. Define the smoothed image $u := f + \text{div } V_i$. 
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$.
3. Compute $V_i = \arg \min_V \| f + \text{div} \ V \|^2_2$ subject to $\| V/\alpha_i \|_\infty \leq 1$.
4. Compute

$$\text{Var}_\eta(V_i/\alpha_i) = \eta \ast (| \eta \ast (V_\eta/\alpha) - V_\alpha/\alpha|^2) .$$
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$.
3. Compute $V_i = \arg \min_V \| f + \text{div} V \|_2^2$ subject to $\| V / \alpha_i \|_\infty \leq 1$.
4. Compute $\text{Var}_\eta (V_i / \alpha_i)$.
5. Compute the update of $\alpha_i$ as

$$\tilde{\alpha}_{i+1} := \alpha_i \cdot \frac{\text{Var}_\eta (V_i / \alpha_i) + \theta}{2\theta}$$

and

$$\alpha_{i+1} := \rho \ast \tilde{\alpha}_{i+1}.$$
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$.
3. Compute $V_i = \arg \min_V \| f + \operatorname{div} V \|_2^2$ subject to $\| V / \alpha_i \|_\infty \leq 1$.
4. Compute $\operatorname{Var}_\eta(V_i/\alpha_i)$.
5. Compute $\alpha_{i+1} := \rho \ast (\alpha_i(\operatorname{Var}_\eta(V_i/\alpha_i) + 1)) / 2\theta$.
6. If $\| \operatorname{div} V_i - \operatorname{div} V_{i-1} \| < \varepsilon$ stop the iteration, else $i \mapsto i + 1$ and go to 3.
Proposed Algorithm

1. Choose target variation $0 < \theta < 1$.
2. Choose initial regularization function $\alpha_1 : \Omega \to \mathbb{R}_{>0}$, set $i = 1$.
3. Compute $V_i = \arg\min_V \| f + \text{div } V \|_2^2$ subject to $\| V/\alpha_i \|_\infty \leq 1$.
4. Compute $\text{Var}_\eta(V_i/\alpha_i)$.
5. Compute $\alpha_{i+1} := \rho \ast (\alpha_i(\text{Var}_\eta(V_i/\alpha_i) + 1)) / 2\theta$.
6. If $\|\text{div } V_i - \text{div } V_{i-1}\| < \varepsilon$ stop the iteration, else $i \mapsto i + 1$ and go to 3.
7. Define the smoothed image $u := f + \text{div } V_i$. 
Computation of $V_i = \arg\min_V \mathcal{J}$ can be carried out by an iterative projected gradient method:

- Start with $V_i^{(0)} := V_{i-1} / \max\{1, |V_{i-1}|/\alpha_i\}$.
- Alternatingly compute (with $0 < \tau < 1/4$)
  \[
  \tilde{V}_i^{(k+1)} := V_i^{(k)} + \tau \nabla (f + \text{div } V_i^{(k)}),
  \]
  and project back:
  \[
  V_i^{(k+1)} := \frac{\tilde{V}_i^{(k+1)}}{\max\{1, |\tilde{V}_i^{(k+1)}|/\alpha_i\}}.
  \]
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4. Conclusion
Let $X$ be a Hilbert space $K : L^2(\Omega) \rightarrow X$ a (compact) bounded linear operator. For given noisy data $f \in X$, solve the equation

$$Ku = f$$

by minimizing

$$\mathcal{T}(u; \alpha) := \frac{1}{2} \| Ku - f \|_2^2 + \int_{\Omega} \alpha(x) d|Du| .$$

Application of the proposed method is also possible here.
Locally Adaptive TV Regularization

Locally Adaptive Parameter Selection

Application to Deblurring Problems

Iterative Solution Method for Standard TV

Minimize

$$T(u; \alpha) := \frac{1}{2} \| Ku - f \|_2^2 + \int_{\Omega} \alpha(x) \, d|Du|$$

by the iteration

$$w_k = u_k + \mu K^* (f - Ku_k),$$

$$u_{k+1} = \arg \min_u \left( \frac{1}{2} \| u - w_k \|_2^2 + \int_{\Omega} \mu \alpha(x) \, d|Du| \right),$$

where $\mu \| K^* K \| < 1$. 

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Locally Adaptive TV Regularization
Modified Algorithm

1. Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1 : \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^* f$. 
Modified Algorithm

1. Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1: \Omega \to \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^*f$.

2. Compute

$$V_i := \arg \min_{V} \|w_i + \text{div } V\|_2^2$$

subject to the constraints

$$\|V/\mu \alpha_i\|_{\infty} \leq 1 \quad \text{and} \quad V \cdot \nu = 0 \text{ on } \partial \Omega.$$ 

Again, a projected gradient descent method is used.
Modified Algorithm

1. Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1 : \Omega \to \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^* f$.

2. Compute $V_i = \arg \min_V ||w_i + \text{div } V||_2^2$ subject to $\|V/\mu \alpha_i\|_\infty \leq 1$.

3. Compute

$$\text{Var}_\eta(V_i/\mu \alpha_i) = \eta \ast \left( |\eta \ast (V_\eta/\mu \alpha) - V_\alpha/\mu \alpha|^2 \right).$$
Locally Adaptive TV Regularization

Locally Adaptive Parameter Selection

Application to Deblurring Problems

Modified Algorithm

1. Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1 : \Omega \to \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^* f$.

2. Compute $V_i = \arg \min_V \| w_i + \text{div } V \|_2^2$ subject to $\| V/\mu \alpha_i \|_\infty \leq 1$.

3. Compute $\text{Var}_\eta(V_i/\mu \alpha_i)$.

4. Compute the update of $\alpha_i$ as

   $$\tilde{\alpha}_{i+1} := \alpha_i \cdot \frac{\text{Var}_\eta(V_i/\mu \alpha_i) + \theta}{2\theta}$$

   and

   $$\alpha_{i+1} := \rho \cdot \tilde{\alpha}_{i+1}.$$
Locally Adaptive TV Regularization

Locally Adaptive Parameter Selection

Application to Deblurring Problems

Modified Algorithm

1. Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1: \Omega \rightarrow \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^* f$.

2. Compute $V_i = \arg \min_V \| w_i + \text{div } V \|_2^2$ subject to $\| V / \mu \alpha_i \|_\infty \leq 1$.

3. Compute $\text{Var}_\eta(V_i / \mu \alpha_i)$.

4. Compute $\alpha_{i+1} := \rho * (\alpha_i(\text{Var}_\eta(V_i / \mu \alpha_i) + 1)) / 2\theta$.

5. Define

$$u_{i+1} = w_i + \text{div } V_i, \quad w_{i+1} = u_{i+1} + \mu K^*(f - K u_{i+1}) .$$
Choose target variation $0 < \theta < 1$, choose initial regularization function $\alpha_1: \Omega \to \mathbb{R}_{>0}$, set $i = 1$ and $w_1 := \mu K^* f$.

2. Compute $V_i = \arg \min_V \| w_i + \text{div} \, V \|_2^2$ subject to $\| V/\mu \alpha_i \|_\infty \leq 1$.

3. Compute $\text{Var}_\eta(V_i/\mu \alpha_i)$.

4. Compute $\alpha_{i+1} := \rho \star \left( \alpha_i(\text{Var}_\eta(V_i/\mu \alpha_i) + 1) \right)/2\theta$.

5. Define

$$u_{i+1} = w_i + \text{div} \, V_i, \quad w_{i+1} = u_{i+1} + \mu K^* (f - K u_{i+1}) .$$

6. If $\| u_i - u_{i+1} \| < \varepsilon$ stop the iteration, else $i \mapsto i + 1$ and go to 3.
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Gray Level Image

Noise free Image

Regularization with $\theta = 0.7$
Locally Adaptive TV Regularization

Numerical Examples

Denoising

Gray Level Image

Gaussian noise, $\sigma = 10$

Regularization with $\theta = 0.7$
Gray Level Image

Gaussian noise, $\sigma = 30$  
Regularization with $\theta = 0.7$
Gaussian noise, $\sigma = 50$

Regularization with $\theta = 0.7$
Locally Adaptive TV Regularization

Numerical Examples

Denoising

Colour Image

Noise free Image

Regularization with $\theta = 0.8$
Gaussian noise, $\sigma = 30$  

Regularization with $\theta = 0.8$
Denoising

Colour Image

Gaussian noise, $\sigma = 50$

Regularization with $\theta = 0.8$
Locally Adaptive TV Regularization

Numerical Examples

Denoising

Colour Image

Gaussian noise, $\sigma = 100$

Regularization with $\theta = 0.8$
Non-constant Noise

Noisy Image

Regularization with $\theta = 0.7$
Application to Deconvolution

Noise free, blurred image

Regularization with $\theta = 0.7$
Application to Deconvolution

Blurred, noisy image, $\sigma = 2$  
Regularization with $\theta = 0.7$
Application to Deconvolution

Blurred, noisy image, $\sigma = 5$  
Regularization with $\theta = 0.7$
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Summary

- Variations of a dual variable as regularity measure for the minimizer of TV regularization.
- Iterative adaptation of the regularization function $\alpha$, until a uniform smoothness of the solution is reached.
- Ability to deal with unknown noise levels, varying within an image.
- Extension to deblurring problems (and general inverse problems).