After having explained how curves can be described by integral invariants, which in this § will be assumed to be simply functions $I(s)$ of a parameter $s$, we address the problem of comparing two shapes by means of such an integral invariant function. As such we should assign a measure of distance between 2 shapes based on their 2 respective functions $I_1 = I_1(s)$ and $I_2 = I_2(s)$.

We first remark that this distance should be zero if the 2 functions $I_1$ and $I_2$ are the same up to a translation of the parameter $s$:

We thus search for a reparametrisation correspondence which is described by the disparity function $d(s)$:

\[ s - d(s) \leftrightarrow s + d(s) \]

Parametrisation for first curve

Parametrisation for second curve
The "matching" or "correspondence" between the point \( s - d(s) \) left and \( s + d(s) \) right should be considered as good if the integral invariants at corresponding points are approximately equal, which means that

\[
\int \| I_1(s - d(s)) - I_2(s - d(s)) \|^2 \, ds \tag{13-1}
\]

is small.

Now we also want to assign a "cost" to the bending or stretching of the correspondence.

Indeed, if we would not do so, even the two shapes represented by

\[ I_1 \quad \text{and} \quad I_2 \]

would become zero in the sense that there does exist a correspondence between the parametric domains whose disparity function yields makes (13-1) zero.

The remedy is to consider the functional

\[
E(I_1, I_2, d) = \int \| I_1(s - d(s)) - I_2(s + d(s)) \|^2 \, ds + \alpha \int \| d(s) \|^2 \, ds
\]

for a certain \( \alpha > 0 \).

The "coupling constant" \( \alpha \) determines the "cost for reparametrization".

Finally, a shape distance is defined as

\[
D(I_1, I_2) = \min_d E(I_1, I_d) \tag{13-2}
\]
(In the article it is more or less assumed that this minimum exists and is unique, but this need not be the case.)

The function \( d^* \) where this minimum is achieved determines correspondence or shape matching.

(This means, \( s - d(1) \) on curve 1 should be similar to \( s + d(1) \) on curve 2.)

Let us now consider two shapes with the following integral invariants:

\[
\begin{align*}
I_1 & \\
I_2
\end{align*}
\]

For various values of \( \alpha \) the optimal correspondence can be found and represented as follows:

\[
\begin{align*}
& I_1 \\
& I_2
\end{align*}
\]

\( \alpha \) large

\[
\begin{align*}
& I_1 \\
& I_2
\end{align*}
\]

\( \alpha \) small (reparameterization is "cheap")

We mention finally that the shape distance \( D \), defined in (13-2), satisfies

\[
D(I_1, I_2) = D(I_2, I_1)
\]
In [4], the following has been described: for a planar curve \( y = (x(t), y(t)) \), we want to represent the values of \( t \) which correspond to an inflection point on the curve, taking into account only information of the curve on a certain level of detail \( \sigma \).

Thus for a certain \( \sigma \) one convolves the co-ordinate functions \( x, y \) with Gaussian kernels of distribution \( \sigma \), resulting in a smoother version of the curve, and then traces the inflection points of this curve. (An inflection point of the curve is a point where its curvature vanishes.)

As an example one takes the coast of Africa.

\[ \sigma = \text{small} \]

\[ \sigma = \text{large} \]
Typically one sees that, upon taking $\sigma$ larger, the number of inflection points decreases (they "cancel each other"): 

$\sigma$ small 

$\sigma$ large 

Then, one represents the figure by the map of all inflection points at all levels of detail $\sigma$:

scale

and calls this scale space image ".
In such a picture we typically see how, when \( \tau \) increases, 2 inflection points approach and then cancel out. Particularly, the parameter value of the inflection points depends on the scale parameter \( \tau \).

In the article under discussion of Manwy et al. this idea is borrowed and adapted to the local area integral invariant \( I_r \), where \( r \) plays the role of level of detail.

Particularly, one can draw e.g. the critical points of \( I_r \) (as a function of \( \tau \)) and this for different values of \( r \).

Here it is claimed that these values are not dependent on \( r \), i.e., the picture would be static:
§7. A few remarks about implementation and experimental results.

We now discuss the implementation of the concepts of §5.

Particularly: for two given functions $I_1$, $I_2$,

both of them known in a discrete # of points,

find such an optimal correspondence.

This can be translated into a graph search problem.

Let $I_1$ be known in points $0, A_1, \ldots, M A_1, = 1$

$I_2$

$0, A_2, \ldots, N A_2, = 2.$

Represent $I_1$ parameter horizontally

$I_2$ vertically

A possible correspondence between these parameter domain looks as:

```
\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{chart.png}
\end{figure}
```

Thus a path from $(0,0)$ to $(M,N)$. (Here we assume end- and begin-
points of the 2 curves to match.)

To obtain a discrete version of a $(1,1)$ correspondence only
steps to the right or up are allowed.

on diagonal?
Such a discrete path is thus a sequence of vertices
\[ p = (v_0, v_1, v_2, \ldots, v_L) \quad \begin{cases} v_0 = (0, 0) \\ v_L = (M, N) \end{cases} \]

The discrete version of the functional we minimise can be written as
\[ w(p) = \sum_{t=0}^{L-1} w(v_t, v_{t+1}) \]

where \( w(v_t, v_{t+1}) \) is a certain weight assigned to the edge.

The can be solved (for global minimisers) using Dijkstra's algorithm

Roughly speaking, this algorithm starts from the initial node with calculating:

1. First only the "start" node is marked.
2. Set it as "current", i.e., "on the boundary of our calculation".
3. Next we consider all neighbours of the current node and write down the least distance of them to the start.
4. Then we consider the neighbours of the neighbours, etc.
At every stage during runtime, we can graphically represent our calculations as:

where for each point in the calculated domain the minimal distance to the start (within this domain) is known, as well as how to realise it.

If the entire graph is finished the solution is known.

Experimental results.

* Fig 10 occluded finger
+ Shape matching via area integral invariant is much more robust than with curvature! Fig 11
Additional References.

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[3] H. Pottmann, J. Wallner, Q. Huang, Y.-L. Yang,

[4] J. Mokhtarian and A. Mackworth,
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