1. Consider the symmetric matrix

\[
A = \begin{pmatrix}
3 & 2 & 1 \\
2 & 3 & 2 \\
1 & 2 & 3
\end{pmatrix}.
\]

(a) Show that the matrix \( A \) is positive definite.

(b) Show that the Jacobi method for solving the linear system \( Ax = b \) for any choice of \( b \) and staring vector \( x^{(0)} \) diverges.

2. Consider the Jacobi method to solve the linear equation \( Ax = b \), for a given diagonal dominant matrix \( A = (a_{ij})_{i,j=1}^n \) of the form

\[
A = \begin{pmatrix}
1 & 0 & \alpha \\
\beta & 1 & 0 \\
0 & \gamma & 1
\end{pmatrix}, \quad \alpha, \beta, \gamma \in (0, 1)
\]

and a vector \( b \in \mathbb{R}^n \). Instead, we can equivalently consider the equation

\[ PAx = Pb, \]

where \( P \) is an invertible matrix of the form

\[
P = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\ell_2 & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
-\ell_{n-1} & \vdots & \ddots & 1 & 0 \\
-\ell_n & 0 & \cdots & 0 & 1
\end{pmatrix}, \quad \ell_i = \frac{a_{ii}}{a_{11}}, \quad i = 2, \ldots, n.
\]

Show that the convergence of the Jacobi method is improved using this so-called preconditioning for the given matrix \( P \).

Hint: The smaller the spectral radius of the iteration matrix is, the faster the iteration method converges.
3. Let
\[ A = \begin{pmatrix} 4 & 1/2 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}. \]
Examine if the Gauss-Seidel method converges to the solution of \( Ax = b, \ b \in \mathbb{R}^3 \) for every starting vector \( x^{(0)} \in \mathbb{R}^3 \).

4. Consider the matrix
\[ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \]
Show that the iterative methods, Jacobi, Gauss-Seidel and SOR converge to the solution of the linear equation \( Ax = b, \ b \in \mathbb{R}^2 \). Compare the speed of convergence of the three methods.

*Hint: For the SOR method, consider the optimal value of the over-relaxation parameter \( \omega \) which occurs when the two eigenvalues of the iteration matrix are equal.*

5. Consider a matrix \( A \in \mathbb{R}^{n \times n} \) with positive diagonal elements and the matrix
\[ \tilde{A} = D^{-1/2} AD^{-1/2}, \quad \text{where} \quad D = \text{diag}(a_{11}, \ldots, a_{nn}). \]
Then, show that the above scaling does not affect the spectral radii of the Jacobi and the SOR methods, this means,
\[ \rho(M_J) = \rho(\tilde{M}_J) \quad \text{and} \quad \rho(M_{SOR}) = \rho(\tilde{M}_{SOR}), \]
where \( M_J, \tilde{M}_J \) are the iteration matrices of the Jacobi method for \( A \) and \( \tilde{A} \), respectively. Equivalently, \( M_{SOR}, \tilde{M}_{SOR} \) are the iteration matrices of the SOR method for \( A \) and \( \tilde{A} \), respectively.

6. Implement in MATLAB the Arnoldi iterative method (Lecture notes: Algorithm 6).

7. Write a MATLAB-Program, that if the input matrix is symmetric and positive definite approximates the solution \( x \in \mathbb{R}^n \) of the linear system \( Ax = b \), for a given vector \( b \in \mathbb{R}^n \), using the conjugate gradient method, otherwise it returns an error message.