Exercise Sheet 2

1. Consider the symmetric matrix

\[
A = \begin{bmatrix}
4 & 1 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 \\
0 & 1 & 4 & 1 & 0 \\
0 & 0 & 1 & 4 & 1 \\
0 & 0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

Localize the spectrum of the inverse matrix \( A^{-1} \), using the Gershgorin circle theorem.

2. A matrix \( A \in \mathbb{R}^{n \times n} \) is diagonalizable if it has \( n \) distinct eigenvalues. Apply the Gershgorin circle theorem to show that the following matrix is diagonalizable,

\[
A = \begin{bmatrix}
-20 & 0 & 1 & 0 & 1 \\
2 & -10 & 0 & 3 & 0 \\
0 & 4 & 0 & 4 & 1 \\
0 & 3 & 0 & 10 & 2 \\
2 & 0 & 1 & 0 & 20 \\
\end{bmatrix}
\]

3. Let

\[
A = \begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2 \\
\end{bmatrix}
\]

Approximate the biggest eigenvalue of \( A \) using the Power method. Perform three steps of the method with initial vector \( x^{(0)} = (1, 1, 1)^T \).

4. Consider the QR-algorithm for a matrix \( A \): Let \( A_0 = A \), then

\[A_{k+1} = R_k Q_k, \quad k \in \mathbb{N},\]

where $A_k = Q_kR_k$ is the QR-decomposition of $A_k$, $Q_k$ is unitary matrix and $R_k$ is right triangular matrix. Show that:

(a) $A_{k+1} = Q_k^*A_kQ_k$

(b) $A_{k+1} = (Q_0Q_1\cdots Q_k)^*A(Q_0Q_1\cdots Q_k)$

(c) $A^k = (Q_0Q_1\cdots Q_k)(R_kR_{k-1}\cdots R_0)$

5. Write a MATLAB-PROGRAM that for a given matrix $A \in \mathbb{C}^{n \times n}$ results in a graph that represents the Gershgorin circles for the matrices $A$ and $A^*$ in the complex plane.

6. Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $n$ real eigenvalues $|\lambda_1| > ... > |\lambda_n|$ and corresponding eigenvectors which form an orthonormal basis $\{v_1, ..., v_n\}$, (with respect to the $\|\cdot\|_2$ norm). Then, the matrix $B$ given by

$$B = A - \lambda_k v_kv_k^T,$$

for a given $k = 1, ..., n$

admits the eigenvalues $\lambda_1, ..., \lambda_{k-1}, 0, \lambda_{k+1}, ..., \lambda_n$. Implement in MATLAB, a function that approximates all the eigenvalues of $A$, considering the Power method and the given matrix $B$. 