1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $u \in C([0, \infty) \times \overline{\Omega})$ be a classical solution of the parabolic problem

$$\frac{\partial u}{\partial t}(t, x) - \text{div}_x(\sigma(x) \nabla_x u(t, x)) + c(x)u(t, x) = f(x), \quad t \in (0, T), \ x \in \Omega,$$

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

$$u(t, x) = 0, \quad t \in (0, T), \ x \in \partial \Omega,$$

for some given functions $u_0 \in \mathbb{C}^2(\overline{\Omega})$, $f, c \in \mathbb{C}(\overline{\Omega})$ and $\sigma \in \mathbb{C}^1(\overline{\Omega})$ with $\sigma(x) > 0$ for all $x \in \overline{\Omega}$, so that the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous.

Moreover, let $v \in H^1_0(\Omega)$ be a weak solution of the elliptic problem

$$-\text{div}(\sigma \nabla v)(x) + c(x)v(x) = f(x), \quad x \in \Omega.$$

Show that

$$\|u(t, \cdot) - v\|_{L^2(\Omega)} \to 0 \quad (t \to \infty).$$

2. We consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x), \quad t > 0, \ x \in (0, \pi)$$

with initial data

$$u(0, x) = \sum_{j=1}^{N} c_j \sin(jx), \quad x \in (0, \pi),$$

for some $N \in \mathbb{N}$ and $(c_j)_{j=1}^{N} \subset \mathbb{R}$, and boundary data

$$u(t, 0) = u(t, \pi) = 0, \quad t > 0.$$

(a) Determine the solution $u \in C^\infty([0, \infty) \times [0, \pi])$ of this problem.

(b) We approximate for given step-size $h > 0$ the solution $u(ih, x)$ numerically with $u_i(x), \ i \in \mathbb{N}$, by using an $s$-stage Runge-Kutta method $(A, b, c)$:

$$\eta_j(x) = u_i(x) + h \sum_{k=1}^{s} A_{ij} \eta^*_k(x), \quad x \in (0, \pi), \ j = 1, \ldots, s,$$

$$u_{i+1}(x) = u_i(x) + h \sum_{k=1}^{s} b_k \eta^*_k(x), \quad x \in (0, \pi),$$

for $i \in \mathbb{N}$. The coefficients are determined by the Butcher tableau:

$$A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1s} \\
A_{21} & A_{22} & \cdots & A_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
A_{s1} & A_{s2} & \cdots & A_{ss}
\end{bmatrix},$$

$$b = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_s
\end{bmatrix},$$

where $A_{ij}$ and $b_k$ are given for $i, j = 1, \ldots, s$. The matrices $A$ and $b$ determine the Runge-Kutta method.
where we impose for every $\eta_k$ the boundary conditions

$$\eta_k(0) = \eta_k(\pi) = 0, \quad k = 1, \ldots, s.$$ 

Show that the solution of this boundary value problem is analytically given by

$$u_i(x) = \sum_{j=1}^{N} R(-hj^2)^i c_j \sin(jx), \quad x \in (0, \pi), \quad i \in \mathbb{N},$$

where $R : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ denotes the stability function of the Runge-Kutta method $(A, b, c)$, and compare this result with the analytical solution of the heat equation.

(c) Write a program that solves this problem with the Crank-Nicolson method.

3. (a) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $a, c \in C([0, T] \times \Omega)$ and $b \in C([0, T] \times \Omega; \mathbb{R}^2)$ be given functions with $a(t, x) > 0$ for all $t \in [0, T]$ and $x \in \Omega$.

We consider a function $u \in C([0, T] \times \Omega)$, $(t, x) \mapsto u(t, x)$, so that the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous. Moreover, assume that $u$ fulfills the inequalities

$$\frac{\partial u}{\partial t}(t, x) - a(t, x) \Delta u(t, x) + \langle b(t, x), \nabla_x u(t, x) \rangle + c(t, x)u(t, x) \leq 0, \quad t \in (0, T), \quad x \in \Omega,$$

$$u(0, x) \leq 0, \quad x \in \Omega,$$

$$u(t, x) \leq 0, \quad t \in (0, T), \quad x \in \partial \Omega.$$

Show that $u(t, x) \leq 0$ for all $t \in (0, T)$ and $x \in \Omega$.

*Hint:* Consider the function $v(t, x) = e^{-\gamma t}u(t, x)$ for suitable $\gamma \in \mathbb{R}$.

(b) Let $\Omega$ be again a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Use this result to show that the parabolic problem

$$\frac{\partial u}{\partial t}(t, x) - \text{div}_x(\sigma \nabla_x u)(t, x) + c(t, x)u(t, x) = f(t, x), \quad t \in (0, T), \quad x \in \Omega,$$

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

$$u(t, x) = 0, \quad t \in (0, T), \quad x \in \partial \Omega,$$

has for given functions $u_0 \in C^2(\Omega)$, $f, c \in C([0, T] \times \Omega)$, and $\sigma \in C^1([0, T] \times \Omega)$ with $\sigma(t, x) > 0$ for all $(t, x) \in [0, T] \times \Omega$ at most one classical solution $u \in C([0, T] \times \Omega)$ whose partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous.

4. Write a program that approximates the solution $u \in C^1([0, \infty) \times \mathbb{R})$ of the transport equation

$$\frac{\partial u}{\partial t}(t, x) + a \frac{\partial u}{\partial x}(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R},$$
for some given constant $a \in \mathbb{R}$ and given initial data $u_0 \in C^1_c(\mathbb{R})$,

$$u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

by using the finite difference method.