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# Exercise Sheet 10

1. Let $\Omega$ be bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $u \in C([0,\infty) \times \bar{\Omega})$ be a classical solution of the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \operatorname{div}_x(\sigma(x)\nabla_x u(t,x)) + c(x)u(t,x) = f(x), &\quad t \in (0,T),\ x \in \Omega, \\ u(0,x) = u_0(x), &\quad x \in \Omega, \\ u(t,x) = 0, &\quad t \in (0,T),\ x \in \partial\Omega, \end{aligned}$$

for some given functions $u_0 \in C^2(\bar{\Omega})$, $f, c \in C(\bar{\Omega})$ and $\sigma \in C^1(\bar{\Omega})$ with $\sigma(x) > 0$ for all $x \in \bar{\Omega}$, so that the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous.

Moreover, let $v \in H_0^1(\Omega)$ be a weak solution of the elliptic problem

$$-\operatorname{div}(\sigma\nabla v)(x) + c(x)v(x) = f(x), \quad x \in \Omega.$$

Show that

$$\|u(t,\cdot) - v\|_{L^2(\Omega)} \to 0 \quad (t \to \infty).$$

2. We consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \quad t > 0,\ x \in (0,\pi)$$

with initial data

$$u(0,x) = \sum_{j=1}^{N} c_j \sin(jx), \quad x \in (0,\pi),$$

for some $N \in \mathbb{N}$ and $(c_j)_{j=1}^N \subset \mathbb{R}$, and boundary data

$$u(t,0) = u(t,\pi) = 0, \quad t > 0.$$

   (a) Determine the solution $u \in C^\infty([0,\infty) \times [0,\pi])$ of this problem.

   (b) We approximate for given step-size $h > 0$ the solution $u(ih,x)$ numerically with $u_i(x)$, $i \in \mathbb{N}$, by using an $s$-stage Runge-Kutta method $(A,b,c)$:

$$\begin{aligned} \eta_j(x) &= u_i(x) + h\sum_{k=1}^{s} A_{ij}\eta_k''(x), \quad x \in (0,\pi),\ j = 1,\ldots,s, \\ u_{i+1}(x) &= u_i(x) + h\sum_{k=1}^{s} b_k\eta_k''(x), \quad x \in (0,\pi), \end{aligned}$$

where we impose for every $\eta_k$ the boundary conditions

$$\eta_k(0) = \eta_k(\pi) = 0, \quad k = 1, \ldots, s.$$

Show that the solution of this boundary value problem is analytically given by

$$u_i(x) = \sum_{j=1}^{N} R(-hj^2)^i c_j \sin(jx), \quad x \in (0, \pi),\ i \in \mathbb{N},$$

where $R : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ denotes the stability function of the Runge-Kutta method $(A, b, c)$, and compare this result with the analytical solution of the heat equation.

(c) Write a program that solves this problem with the Crank-Nicolson method.

3. (a) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $a, c \in C([0,T] \times \bar{\Omega})$ and $b \in C([0,T] \times \bar{\Omega}; \mathbb{R}^2)$ be given functions with $a(t,x) > 0$ for all $t \in [0,T]$ and $x \in \bar{\Omega}$.

   We consider a function $u \in C([0,T] \times \bar{\Omega})$, $(t,x) \mapsto u(t,x)$, so that the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous. Moreover, assume that $u$ fulfils the inequalities

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &- a(t,x)\Delta u(t,x) \\ &+ \langle b(t,x), \nabla_x u(t,x)\rangle + c(t,x)u(t,x) \le 0, \quad t \in (0,T),\ x \in \Omega, \\ &\qquad u(0,x) \le 0, \quad x \in \Omega, \\ &\qquad u(t,x) \le 0, \quad t \in (0,T), x \in \partial\Omega. \end{aligned}$$

   Show that $u(t,x) \le 0$ for all $t \in (0,T)$ and $x \in \Omega$.
   *Hint:* Consider the function $v(t,x) = \mathrm{e}^{-\gamma t} u(t,x)$ for suitable $\gamma \in \mathbb{R}$.

   (b) Let $\Omega$ be again a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Use this result to show that the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) - \mathrm{div}_x(\sigma \nabla_x u)(t,x) + c(t,x)u(t,x) &= f(t,x), \quad t \in (0,T),\ x \in \Omega, \\ u(0,x) &= u_0(x), \quad x \in \Omega, \\ u(t,x) &= 0, \qquad t \in (0,T),\ x \in \partial\Omega, \end{aligned}$$

   has for given functions $u_0 \in C^2(\bar{\Omega})$, $f, c \in C([0,T] \times \bar{\Omega})$, and $\sigma \in C^1([0,T] \times \bar{\Omega})$ with $\sigma(t,x) > 0$ for all $(t,x) \in [0,T] \times \bar{\Omega}$ at most one classical solution $u \in C([0,T] \times \bar{\Omega})$ whose partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist for all $i, j = 1, \ldots, n$ and are continuous.

4. Write a program that approximates the solution $u \in C^1([0,\infty) \times \mathbb{R})$ of the transport equation

$$\frac{\partial u}{\partial t}(t,x) + a\frac{\partial u}{\partial x}(t,x) = 0, \quad t > 0,\ x \in \mathbb{R},$$

for some given constant $a \in \mathbb{R}$ and given initial data $u_0 \in C_{\mathrm{c}}^1(\mathbb{R})$,

$$u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

by using the finite difference method.

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