Exercise Sheet 5

1. We introduce for some fixed $n \in \mathbb{N}$ the discrete Fourier transform matrix

$$F_n = \left( \frac{1}{\sqrt{n}} e^{-\frac{2\pi i}{n} jk} \right)_{j,k=0}^{n-1}.$$ 

(a) Prove that the matrix $F_n$ is unitary.

(b) Show that the discrete Fourier transform diagonalises the discrete second derivative of a periodic function, i.e. show that the $n \times n$ matrix

$$F_n^* \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ -1 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} F_n$$

is diagonal and calculate its entries.

(c) Find the eigenvalues of the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ -1 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix},$$

which corresponds to the discrete second derivative of a function which vanishes at the boundary.

2. (a) Show that the stability function $R$ of every isometry preserving Runge-Kutta method fulfils

$$\lim_{|\zeta| \to \infty} |R(\zeta)| = 1.$$ 

(b) Prove that if the $s$-stage Runge-Kutta method $(A, b, c)$ fulfils that $A \in \text{GL}(s, \mathbb{R})$ and that $a_{sj} = b_j$ for all $j = 1, \ldots, s$, then

$$\lim_{|\zeta| \to \infty} R(\zeta) = 0.$$ 

(An $A$-stable Runge-Kutta method with this property is called L-stable.)
3. We consider the initial value problem

\[ y'(t) = -\frac{1}{2} \left( \frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1} \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} \right) y(t), \quad t > 0, \]

\[ y(0) = y_0 \]

for the function \( y \in C^1([0, \infty); \mathbb{R}^2) \) for some \( y_0 \in \mathbb{R}^2 \) and \( \lambda_2 > \lambda_1 > 0 \).

(a) Calculate the explicit solution \( y \) of this initial value problem and show that in the limit \( \lambda_2 \to \infty \), we have (for fixed \( \lambda_1 > 0 \)) the behaviour

\[ y(t) = \frac{y_{0,1} - y_{0,2}}{2} e^{-\lambda_1 t} \left(1 - \frac{1}{-1}\right) + O(e^{-\lambda_2 t}). \]

(b) Show that if \( (y_j)_{j=0}^\infty \) is the solution obtain with an arbitrary explicit Runge-Kutta method and an arbitrary fixed step size \( h > 0 \) for this initial value problem, then the same limit always leads to

\[ y_j = \frac{y_{0,1} + y_{0,2}}{2} c^j \lambda_2^s \left(1 - \frac{1}{1}\right) + O\left(\frac{1}{\lambda_2}\right) \]

for some constants \( c \in \mathbb{R} \setminus \{0\} \) and \( s \in \mathbb{N}, j \in \mathbb{N} \).

4. The Padé approximant of order \((m,n)\), \( m,n \in \mathbb{N} \), is defined as the rational function

\[ R : \mathbb{C} \to \mathbb{C} \cup \{\infty\}, \quad R(z) = \frac{P(z)}{Q(z)}, \]

with polynomials \( P \) of degree \( m \) and \( Q \) of degree \( n \) with \( Q(0) = 1 \), whose Taylor polynomial of degree \( m + n \) is equal to the Taylor polynomial of the exponential function:

\[ R(z) - e^z = O(z^{m+n+1}) \quad (z \to 0). \]

(a) Verify that the Padé approximant of every order exists and is unique.

(b) Prove that the stability function of an \( s \)-stage Runge-Kutta-Gauß method is a Padé approximant of order \( (s,s) \).

5. (a) Write a program that calculates the coefficients \((A,b,c)\) of an \( s \)-stage Runge-Kutta-Gauß method for arbitrary \( s \in \mathbb{N} \).

(b) Use this program to get the Padé approximants of order \((n,n)\) and compare them for some values \( n \in \mathbb{N} \) with the Taylor polynomial of degree \( 2n \).