# Exercise Sheet 5

1. We introduce for some fixed $n \in \mathbb{N}$ the discrete Fourier transform matrix
$$F_n = \left(\frac{1}{\sqrt{n}} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n} jk}\right)_{j,k=0}^{n-1}.$$
   (a) Prove that the matrix $F_n$ is unitary.
   (b) Show that the discrete Fourier transform diagonalises the discrete second derivative of a periodic function, i.e. show that the $n \times n$ matrix
$$F_n^* \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} F_n$$
   is diagonal and calculate its entries.
   (c) Find the eigenvalues of the $n \times n$ matrix
$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix},$$
   which corresponds to the discrete second derivative of a function which vanishes at the boundary.

2. (a) Show that the stability function $R$ of every isometry preserving Runge-Kutta method fulfils
$$\lim_{|\zeta| \to \infty} |R(\zeta)| = 1.$$
   (b) Prove that if the $s$-stage Runge-Kutta method $(A, b, c)$ fulfils that $A \in \mathrm{GL}(s, \mathbb{R})$ and that $a_{sj} = b_j$ for all $j = 1, \dots, s$, then
$$\lim_{|\zeta| \to \infty} R(\zeta) = 0.$$
   (An A-stable Runge-Kutta method with this property is called L-stable.)

3. We consider the initial value problem

$$\begin{aligned} y'(t) &= -\frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_2 & \lambda_2-\lambda_1 \\ \lambda_2-\lambda_1 & \lambda_1+\lambda_2 \end{pmatrix} y(t), \quad t>0, \\ y(0) &= y_0 \end{aligned}$$

for the function $y \in C^1([0,\infty);\mathbb{R}^2)$ for some $y_0 \in \mathbb{R}^2$ and $\lambda_2 > \lambda_1 > 0$.

(a) Calculate the explicit solution $y$ of this initial value problem and show that in the limit $\lambda_2 \to \infty$, we have (for fixed $\lambda_1 > 0$) the behaviour

$$y(t) = \frac{y_{0,1} - y_{0,2}}{2} \mathrm{e}^{-\lambda_1 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O(\mathrm{e}^{-\lambda_2 t}).$$

(b) Show that if $(y_j)_{j=0}^{\infty}$ is the solution obtain with an arbitrary explicit Runge-Kutta method and an arbitrary fixed step size $h > 0$ for this initial value problem, then the same limit always leads to

$$y_j = \frac{y_{0,1} + y_{0,2}}{2} c^j \lambda_2^{js} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O\Big(\frac{1}{\lambda_2}\Big) \right)$$

for some constants $c \in \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{N}$, $j \in \mathbb{N}$.

4. The Padé approximant of order $(m,n)$, $m,n \in \mathbb{N}$, is defined as the rational function

$$R : \mathbb{C} \to \mathbb{C} \cup \{\infty\}, \quad R(z) = \frac{P(z)}{Q(z)},$$

with polynomials $P$ of degree $m$ and $Q$ of degree $n$ with $Q(0) = 1$, whose Taylor polynomial of degree $m+n$ is equal to the Taylor polynomial of the exponential function:

$$R(z) - \mathrm{e}^z = O(z^{m+n+1}) \quad (z \to 0).$$

(a) Verify that the Padé approximant of every order exists and is unique.

(b) Prove that the stability function of an $s$-stage Runge-Kutta-Gauß method is a Padé approximant of order $(s,s)$.

5. (a) Write a program that calculates the coefficients $(A,b,c)$ of an $s$-stage Runge-Kutta-Gauß method for arbitrary $s \in \mathbb{N}$.

(b) Use this program to get the Padé approximants of order $(n,n)$ and compare them for some values $n \in \mathbb{N}$ with the Taylor polynomial of degree $2n$.

—