Exercise Sheet 3

1. We consider the bilinear map

$$L : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad L(v,w) = v \times w,$$

given by the cross product on $\mathbb{R}^3$, and write it in components as

$$L(v,w)_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} v_j w_k, \quad i = 1,2,3,$$

for some coefficients $\varepsilon_{ijk} \in \mathbb{R}$, $i,j,k \in \{1,2,3\}$.[1]

(a) Calculate the coefficients ε_{ijk}, $i,j,k \in \{1,2,3\}$.

(b) Show that these coefficients fulfil the relation

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}.$$

(c) Prove with this the vector identity

$$(u \times v) \times w = \langle u, w\rangle\, v - \langle v, w\rangle\, u$$

for all $u, v, w \in \mathbb{R}^3$.

2. We define the differential $\mathrm{d}f(x) : \mathbb{R}^n \to \mathbb{R}^m$ of a function $f \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ at the position $x \in \mathbb{R}^n$ as the map given by

$$\mathrm{d}f(x)(v) = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h} \quad \text{for all} \quad v \in \mathbb{R}^n.$$

Recursively, we define for $f \in C^k(\mathbb{R}^n;\mathbb{R}^m)$ the kth derivative $\mathrm{d}^k f(x) : (\prod_{j=1}^k \mathbb{R}^n) \to \mathbb{R}^m$, $k > 1$, at the position $x \in \mathbb{R}^n$ by

$$\mathrm{d}^k f(x)(v^{(1)}, \dots, v^{(k)}) = \lim_{h\to 0} \frac{\mathrm{d}^{k-1} f(x + hv^{(k)})(v^{(1)}, \dots, v^{(k-1)}) - \mathrm{d}^{k-1} f(x)(v^{(1)}, \dots, v^{(k-1)})}{h}$$

for all $v^{(j)} \in \mathbb{R}^n$, $j = 1, \dots, k$.

Let now $k \in \mathbb{N}$, $f \in C^k(\mathbb{R}^n;\mathbb{R}^m)$ and $x \in \mathbb{R}^n$ be arbitrary.

[1] According to the Einstein summation convention, we could leave out the summation sign and automatically take the sum over the whole range of all indices which appear twice. For clarity, we write the summation sign greyed out.

(a) Check that $\mathrm{d}^k f(x)$ is a multilinear map (i.e. is linear in every argument).

(b) We write the map $\mathrm{d}^k f(x)$ in components:

$$\mathrm{d}^k f(x)(v^{(1)}, \dots, v^{(k)}) = \sum_{i_1,\dots,i_k=1}^{n} a_{i_1 \dots i_k}(x) v_{i_1}^{(1)} \cdots v_{i_k}^{(k)}.$$

Calculate the coefficients $a_{i_1 \dots i_k}(x) \in \mathbb{R}$ for all $i_1, \dots, i_k \in \{1, \dots, n\}$.

(c) Verify that the Taylor approximation of kth order around the point $x_0 \in \mathbb{R}^n$ can be written as

$$f(x) = f(x_0) + \sum_{j=1}^{k} \frac{1}{j!} \mathrm{d}^j f(x_0)(x - x_0, \dots, x - x_0) + o(\|x - x_0\|^k).$$

3. Show that the classical Runga-Kutta method defined by the step

$$y_{i+1} = y_i + h \sum_{j=1}^{4} b_j f(t_i + c_j h, \eta_j),$$
$$\eta_j = y_i + h \sum_{k=1}^{4} a_{jk} f(t_i + c_k h, \eta_k), \quad j = 1, \dots 4,$$

with the coefficients $A = (a_{jk})_{j,k=1}^4$, $b = (b_j)_{j=1}^4$ and $c = (c_j)_{j=1}^4$ given by

$$\begin{array}{c|c} c & A \\ \hline & b^{\mathrm{T}} \end{array} = \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

is of forth order.

4. (a) Prove that if the one-step method

$$y_{i+1} = y_i + h \sum_{j=1}^{s} b_j f(t_i + c_j h, \eta_j), \quad \sum_{j=1}^{s} b_j = 1,$$

has for some choice of $(\eta_j)_{j=1}^s$ the order q, then the quadrature formula

$$\sum_{j=1}^{s} b_j g(c_j) \approx \int_0^1 g(x)\,\mathrm{d}x$$

is exact for all polynoms g of degree less than q.

(b) Conclude that every s-stage one-step method has at most order $2s$.

5. Prove that an explicit s-stage Runge-Kutta method is at most of order s.

   Hint: Apply the Runge-Kutta method to the differential equation $y' = y$ and compare the first step y_1 with the exact solution $y(h)$.

6. Write a program that implements the classical Runga-Kutta algorithm defined in Exercise 3.

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