Exercise Sheet 2

1. Let \( f \in C^n([0,T] \times \mathbb{R}^d; \mathbb{R}^d) \), \( n \in \mathbb{N} \). Show that the initial problem
\[
y'(t) = f(t, y(t)), \quad y(0) = y_0
\]
has for every initial value \( y_0 \in \mathbb{R}^d \) a unique solution \( y \in C^{n+1}([0, T_0] \times \mathbb{R}^d; \mathbb{R}^d) \) for some \( T_0 \in (0, T] \).

2. We consider the differential equation
\[
y'(t) = -y(t), \quad t > 0,
\]
with the initial condition \( y(0) = 1 \). We solve the equation with the implicit Euler method for some fixed step size \( h > 0 \) and obtain the recurrence relation
\[
y_{i+1} = \frac{1}{1 + h} y_i + \varepsilon, \quad i \in \mathbb{N}_0,
\]
with \( y_0 = 1 \). Here, \( \varepsilon > 0 \) shall model the rounding error. Show that the approximation error is bounded by
\[
|y(ih) - y_i| \leq \frac{1}{(1 + h)^i} + \frac{1 + h}{h} \varepsilon \quad \text{for all} \quad i \in \mathbb{N}_0.
\]
How does this result change if we use the explicit Euler method instead?

3. Let us consider the recursive step
\[
Y = y + hf(t, Y)
\]
appearing in the implicit Euler method for some fixed step size \( h > 0 \) at some time \( t \in (0, T) \). We assume that the function \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) is continuously differentiable and fulfills the one-sided Lipschitz condition
\[
\langle f(t, x) - f(t, z), x - z \rangle \leq l\|x - z\|^2 \quad \text{for all} \quad x, z \in \mathbb{R}^d
\]
for some constant \( l \in \mathbb{R} \), so that the equation has a unique solution \( Y \in \mathbb{R}^d \) if \( hl < 1 \). We additionally assume that \( f \) fulfills the Lipschitz condition
\[
\|f(t, x) - f(t, z)\|_2 \leq L\|x - z\|_2 \quad \text{for all} \quad x, z \in \mathbb{R}^d
\]
with a Lipschitz constant \( L > 0 \).
(a) Show that the fixed point iteration

\[ Y_{k+1} = y + hf(t, Y_k), \quad k \in \mathbb{N}_0, \]

converges for every initial value \( Y_0 \in \mathbb{R}^d \) to the solution \( Y \) if \( hL < 1 \).

(b) Give an example of such a function \( f \) where the fixed point iteration does not converge for \( hL = 1 \).

4. Let \( G \in \mathbb{R}^{d \times d} \) be a symmetric and positive definite matrix. We define the inner product \( \langle x, z \rangle_G = x^T G z \) and the corresponding norm \( \| x \|_G = \sqrt{\langle x, x \rangle_G} \) on \( \mathbb{R}^d \).

We further pick a function \( f \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) which fulfills the one-sided Lipschitz condition

\[ \langle f(t, x) - f(t, z), x - z \rangle_G \leq l \| x - z \|_G^2 \quad \text{for all} \quad t \in [0, T], \; x, z \in \mathbb{R}^d \]

with respect to this inner product for some Lipschitz constant \( l < 0 \).

We now construct with the implicit Euler method a sequence \((y_i)_{i=0}^{\infty} \subset \mathbb{R}^d \) approximating the solution \( y \in C^2([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \) of the initial value problem

\[ y'(t) = f(t, y(t)), \quad y(0) = y_0 \]

for an arbitrary \( y_0 \in \mathbb{R}^d \):

\[ y_{i+1} = y_i + hf(t_i, y_i), \quad t_i = ih, \; i \in \mathbb{N}_0, \]

for some step size \( h > 0 \). Show that we have the error estimate

\[ \| y_i - y(t_i) \|_G \leq \frac{h}{2 \| l \|_{C^1([0, T])}} \| y''(t) \|_G, \quad i \in \mathbb{N}_0. \]

5. Let \((x_i)_{i=0}^l \) be a mesh on an interval \([-a, b] \).

(a) Find a set \((p_j)_{j=0}^l \) of polynomials of degree \( l \) with the property

\[ p_j(x_i) = \delta_{ij} \quad \text{for all} \quad i, j \in \{0, \ldots, l\}. \]

(These polynomials are called Lagrange polynomials.)

(b) Write a program that calculates for an arbitrary function \( f : [a, b] \to \mathbb{R} \) the interpolation polynomial \( p \) of degree \( l \) of \( f \), which is defined by the property

\[ p(x_i) = f(x_i) \quad \text{for all} \quad i \in \{0, \ldots, l\}. \]

(c) We choose a uniform mesh \((x_i)_{i=0}^l \) on the interval \([-1, 1] \) and pick as function

\[ f : [-1, 1] \to \mathbb{R}, \quad f(x) = \frac{1}{1 + 25x^2}. \]

Compare the shape of the interpolation polynomial of \( f \) with the approximation of \( f \) by its natural interpolating cubic spline for the values \( l = 5, 10, 20 \).