Exercise 27. Let \( k \in L^2(G \times G) \) be such that the associated integral operator \( K : L^2(G) \to L^2(G) \) is self adjoint, and let \( (\lambda_i, x_i)_{i \in \mathbb{N}} \) be an eigensystem of \( K \). Show that the family

\[
\left( (s, t) \mapsto x_i(s) \overline{x_j(t)} \right)_{(i,j) \in \mathbb{N}^2}
\]

is an orthonormal basis of a closed subspace \( W \) of \( L^2(G \times G) \), and that \( k \in W \).

Exercise 28. Show that a compact self adjoint operator \( K : H \to H \) is positive semi-definite, if and only if all eigenvalues are non-negative.

Exercise 29. Let \( k \in C(G \times G) \) and let the associated integral operator \( K : L^2(G) \to L^2(G) \) be positive semi-definite. Show that \( k(s, s) \geq 0 \) for all \( s \in G \).