Exercises Inverse Problems, SS 2010

Exercise Sheet 3 (Radon Transform)

Markus Haltmeier

We will make use of the following definitions:

(i) By $C_0^\infty(\mathbb{R}^n)$ we denote the space of all infinitely times differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support in $\mathbb{R}^n$. Analogously, $C_0^\infty(S^{n-1} \times \mathbb{R})$ denotes the space of all infinitely times differentiable functions $g : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ with compact support in $S^{n-1} \times \mathbb{R}$. (Here S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.)

(ii) The Radon transform $\mathbf{R}\, f : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ of a function $f \in C_0^\infty(\mathbb{R}^n)$ is defined by

$$(\mathbf{R}_\mathbf{n}\, f)(r) := (\mathbf{R}\, f)(\mathbf{n}, r) := \int_{E(\mathbf{n})} f(r\mathbf{n} + y) dS(y)\,, \quad (\mathbf{n}, r) \in S^{n-1} \times \mathbb{R}\,.$$

Here $E(\mathbf{n}) := \{y \in \mathbb{R}^n : y \cdot \mathbf{n} = 0\}$ denotes the orthogonal complement of the line $\mathbb{R}\mathbf{n}$ with $\cdot$ being the standard scalar product, and dS denotes the standard surface measure.

The function $\mathbf{R}_\mathbf{n}\, f$ is called projection orthogonal to $\mathbf{n}$.

(iii) The backprojection $\mathbf{R}^\sharp\, g : \mathbb{R}^n \to \mathbb{R}$ of a function $g \in C_0^\infty(S^{n-1} \times \mathbb{R})$ is defined by

$$(\mathbf{R}^\sharp\, g)(x) := \int_{S^{n-1}} g(\mathbf{n}, \mathbf{n} \cdot x) dS(\mathbf{n})\,, \quad x \in \mathbb{R}^n\,.$$

Moreover, the 1D backprojection (orthogonal to $\mathbf{n}$) of a function $h \in C_0^\infty(\mathbb{R})$ is defined by $(\mathbf{R}_\mathbf{n}^\sharp\, h)(x) := h(\mathbf{n} \cdot x)$.

(iv) The Fourier transform $\mathbf{F}\, f : \mathbb{R}^n \to \mathbb{C}$ of a function $f \in C_0^\infty(\mathbb{R}^n)$ is defined by

$$(\mathbf{F}\, f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx\,, \quad \xi \in \mathbb{R}^n\,.$$

(v) The convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ of two functions $f, g \in C_0^\infty(\mathbb{R}^n)$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y) dy\,, \quad x \in \mathbb{R}^n\,.$$

(vi) If $f, g \in C_0^\infty(S^{n-1} \times \mathbb{R})$, then $f *_r g$ denotes the convolution of f and g with respect to the second variable r only, that is,

$$(f *_r g)(\mathbf{n}, r) := \big(f(\mathbf{n}, \cdot) * g(\mathbf{n}, \cdot)\big)(r)\,.$$

Moreover we define

$$(\mathbf{F}_r\, f)(\mathbf{n}, \rho) := \big((\mathbf{F}\, f)(\mathbf{n}, \cdot)\big)(\rho)\,, \quad \rho \in \mathbb{R}\,.$$

as the one dimensional Fourier transform of f with respect to the second variable r.

Exercise 6. Let $f \in C_0^\infty(\mathbb{R}^n)$, $g \in C_0^\infty(S^{n-1} \times \mathbb{R})$ and $h \in C_0^\infty(\mathbb{R})$.

(a) Illustrate the definitions of $\mathbf{R_n}\, f$, $\mathbf{R}\, f$, $\mathbf{R}_\mathbf{n}^\sharp\, h$ and $\mathbf{R}^\sharp\, g$ (draw pictures for the cases $n = 2$ and $n = 3$).

(b) Show that $\mathbf{R}\, f \in C_0^\infty(S^{n-1} \times \mathbb{R})$ and $\mathbf{R_n}\, f \in C_0^\infty(\mathbb{R})$ for fixed $\mathbf{n} \in S^{n-1}$.

(c) Show that $\mathbf{R}\, f(\mathbf{n}, r) = \mathbf{R}\, f(-\mathbf{n}, -r)$.

Exercise 7. Let $f \in C_0^\infty(\mathbb{R}^n)$, $h \in C_0^\infty(\mathbb{R})$ and $\mathbf{n} \in S^{n-1}$. Show that

$$\int_{\mathbb{R}} (\mathbf{R_n}\, f)(r) h(r)\, dr = \int_{\mathbb{R}^n} f(x) (\mathbf{R}_\mathbf{n}^\sharp\, h)(x)\, dx\,.$$

Exercise 8. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $g \in C_0^\infty(S^{n-1} \times \mathbb{R})$. Show that

$$\int_{S^{n-1}} \int_{\mathbb{R}} (\mathbf{R}\, f)(\mathbf{n}, r) g(\mathbf{n}, r)\, dr dS(\mathbf{n}) = \int_{\mathbb{R}^n} f(x) (\mathbf{R}^\sharp\, g)(x)\, dx\,.$$

Exercise 9. Let $f \in C_0^\infty(\mathbb{R}^n)$. Show that

$$(\mathbf{F}_r\, \mathbf{R}\, f)(\mathbf{n}, \rho) = (\mathbf{F}\, f)(\mathbf{n}\rho)\,, \quad (\mathbf{n}, \rho) \in S^{n-1} \times \mathbb{R}\,.$$

Exercise 10. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, a vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, and a function $f \in C_0^\infty(\mathbb{R}^n)$ define $\partial_x^\alpha f(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$, take $\theta^\alpha := \prod_i \theta_i^{\alpha_i}$ and set $|\alpha| := \sum_i \alpha_i$. Show that

$$\big(\mathbf{R}\, \partial_x^\alpha f\big)(\mathbf{n}, r) = \mathbf{n}^\alpha \big(\partial_r^{|\alpha|}\, \mathbf{R}\, f\big)(\mathbf{n}, r)\,, \quad (\mathbf{n}, r) \in S^{n-1} \times \mathbb{R}\,.$$

(Verify first the case that $\partial_x^\alpha = \partial/\partial x_i$ for some i.)

Exercise 11. Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Show that

$$(\mathbf{R}\, f) *_r (\mathbf{R}\, g) = \mathbf{R}(f * g)\,.$$

Exercise 12. Let $f \in C_0^\infty(\mathbb{R}^n)$ and $g \in C_0^\infty(S^{n-1} \times \mathbb{R})$. Show that

$$(\mathbf{R}^\sharp\, g) * f = \mathbf{R}^\sharp(g *_r \mathbf{R}\, f)\,.$$

Exercise 13. Let $g \in C_0^\infty(S^{n-1} \times \mathbb{R})$. Show that

$$(\mathbf{F}\, \mathbf{R}^\sharp\, g)(\rho\mathbf{n}) = \rho^{-1}\big((\mathbf{F}_r\, g)(\mathbf{n}, \rho) + (\mathbf{F}_r\, g)(-\mathbf{n}, -\rho)\big)\,, \quad (\mathbf{n}, \rho) \in S^{n-1} \times \mathbb{R}\,.$$