1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying $f(a) \cdot f(b) < 0$. We define the Bisection method, to approximate the solution of the equation $f(x) = 0$, as:

**Algorithm (Bisection method)**

For $k = 1, 2, \ldots$

- Set
  
  $$x_k = a + \frac{b - a}{2}$$

- If $f(x_k) = 0$
  then $x_k$ is the solution.
  end if.

- If $f(a) \cdot f(x_k) > 0$
  
  $$a = x_k$$

  else
  
  $$b = x_k$$

end if.

end for.

(a) Given that the sequence $\{x_k\}_{k=1}^{\infty}$ converges to the solution $x^* \in (a, b)$ of $f(x) = 0$, show that

$$|x_k - x^*| \leq \frac{b - a}{2^k}, \quad k = 1, 2, \ldots$$
(b) Determine the number of iteration steps required for approximating \( x^* \) with tolerance \( 10^{-\alpha} \).

2. Consider the function \( f(x) = x^3 - 2x - 5 \). Approximate the solution of the equation \( f(x) = 0 \), using the first three steps of the:

(a) Bisection method at the Interval \([2, 3]\),
(b) Secant method with \( x_0 = 3 \) and \( x_1 = 3.5 \),
(c) Newton method with \( x_0 = 3 \).

3. To approximate the solutions of the equation \( x^2 - x - 2 = 0 \) we can rewrite it in two different forms:

(a) \( x = x^2 - 2 := \phi_1(x) \),
(b) \( (x^2 - x - 2)/x = 0 \) \( \Rightarrow \ x = 1 + 2/x := \phi_2(x), \ x \neq 0 \)

and we consider the fixed-point method for \( j = 1, 2 \),
\[ x_{n+1} = \phi_j(x_n), \quad n = 0, 1, ... \]

Setting \( x_0 = -3 \), perform the first four steps for both iteration functions \( \phi_j \) and analyse the convergence of the method.

4. Consider the following system of equations
\[ f(x, y) := \begin{pmatrix} xy \\ xy^2 + x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
Perform the first two steps of the Newton method with initial vector \((x_0, y_0) = (1/2, 1)\).

5. Let \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = \sqrt{1 + x^2} \). Show that the Newton method for the equation \( f'(x) = 0 \) converges to the exact solution \( x^* = 0 \) if the initial guess satisfies \( |x^{(0)}| < 1 \).

6. Assume \( f : \mathbb{R} \to \mathbb{R} \) to be a three times continuously differentiable function and \( x^* \) one of its zeros. Consider the following iteration method,
\[ x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad \text{where} \quad g(x) = \frac{f(x + f(x)) - f(x)}{f(x)} \]
to approximate \( x^* \). Implement in MATLAB the above method for solving the equation \( e^{-x} - \sin(x) = 0 \).
7. Consider the system of equations

\[
f(x, y, z) := \begin{pmatrix}
xy - z^2 - 1 \\
xyz - x^2 + y^2 + 2 \\
e^x - e^y + z - 3
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Implement in MATLAB the Broyden’s method to approximate the solution of the above system with initial guess \(x_0 = (x_0, y_0, z_0) = (1, 1, 1)^T\) and the exact Jacobian \(B_0 = J_f(x_0)\).