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Exercise Sheet 1

1. Consider the symmetric matrix

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

   Localize the spectrum of the inverse matrix A^{-1}, using the Gershgorin circle theorem.

2. A matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if it has n distinct eigenvalues. Apply the Gershgorin circle theorem to show that the following matrix is diagonalizable,

$$A = \begin{bmatrix} -20 & 0 & 1 & 0 & 1 \\ 2 & -10 & 0 & 3 & 0 \\ 0 & 4 & 0 & 4 & 1 \\ 0 & 3 & 0 & 10 & 2 \\ 2 & 0 & 1 & 0 & 20 \end{bmatrix}.$$

3. Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

   Approximate the biggest eigenvalue of A using the Power method. Perform three steps of the method with initial vector $x^{(0)} = (1,1,1)^T$.

4. Consider the QR-algorithm with shift for a matrix A : Let $A_0 = A$, then

$$A_{k+1} = R_k Q_k + \mu_k I, \quad k \in \mathbb{N},$$

where $A_k - \mu_k I = Q_k R_k$ is the QR-decomposition of $A_k - \mu_k I$, Q_k is unitary matrix and R_k is right triangular matrix. Show that:

$$\prod_{j=0}^{k}(A_0 - \mu_j I) = (Q_0 Q_1 \cdots Q_k)(R_k R_{k-1} \cdots R_0).$$

5. Let $A, B \in \mathbb{C}^{n\times n}$ be symmetric and B positive definite. Then, the generalized eigenvalue problem is to find $\lambda \in \sigma(A, B)$ and a non-zero $x \in \mathbb{C}^n$ such that

$$Ax = \lambda Bx.$$

   Show that the pair (A, B) has real eigenvalues and linearly independent eigenvectors and that there exist a non-singular matrix $H \in \mathbb{R}^{n\times n}$ such that

$$H^T AH = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n), \quad H^T BH = I,$$

   where λ_j, $j = 1, ..., n$ are the eigenvalues of (A, B) and $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

6. Write a Matlab-Program that for a given matrix $A \in \mathbb{C}^{n\times n}$ results in a graph that represents the Gershgorin circles for the matrices A and A^* in the complex plane.

7. Consider a symmetric matrix $A \in \mathbb{R}^{n\times n}$ with n real eigenvalues

$$|\lambda_1| > ... > |\lambda_n|$$

   and corresponding eigenvectors which form an orthonormal basis $\{v_1, ..., v_n\}$, (with respect to the $\|\cdot\|_2$ norm). Then, the matrix B given by

$$B = A - \lambda_k v_k v_k^T, \qquad \text{for a given} \quad k = 1, ..., n$$

   admits the eigenvalues $\lambda_1, ..., \lambda_{k-1}, 0, \lambda_{k+1}, ..., \lambda_n$. Implement in Matlab, a function that approximates all the eigenvalues of A, considering the Power method and the given matrix B.