Exercise Sheet 6

1. Let $x_0 = -1$, $x_1 = 0$ and $x_2 = 2$. Compute the Lagrange basis polynomials associated to these points and use them to interpolate the function
$$f : [-1, 2] \to \mathbb{R}, \quad \text{with} \quad f(x) = x^4,$$
with a polynomial p of degree 2. Give a bound for the error
$$\max_{x \in [-1,2]} |f(x) - p(x)|.$$

2. Let $f \in C^1[a, b]$. Consider the polynomial p of the form
$$p(x) = \sum_{j=0}^{n} K_j(x) f(x_j) + \sum_{j=0}^{n} M_j(x) f(x_j),$$
with
$$K_j(x) = \left[1 - 2(x - x_j) L_j'(x_j)\right] L_j^2(x), \quad M_j(x) = (x - x_j) L_j^2(x), \text{ for } j = 0, ..., n,$$
where L_j are the Lagrange polynomials, and L_j' their derivatives with respect to x.

   Show that

   (a) $p \in \mathbb{P}_{2n+1}$

   (b) $p(x_j) = f(x_j), \quad p'(x_j) = f'(x_j), \quad \text{for} \quad j = 0, ..., n$

   (c) p is unique. *Hint: Assume that there exist another $q \in \mathbb{P}_{2n+1}$ satisfying (b) and derive a contradiction.*

3. Compute the polynomial $p : \mathbb{R} \to \mathbb{R}$ of Exercise 2 that interpolates the function $f(x) = 1/x$, at the points $x_j = j + 1, \ j = 0, 1, 2$.

4. Show that a linear polynomial of the form
$$p_j(x) = f(x_j) + \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}(x - x_j), \quad j = 0, ..., n$$
satisfies
$$p_j(x) = s(x), \quad \text{for} \quad x \in [x_j, x_{j+1}],$$
where $s \in S^1(\Delta)$ is the linear spline that interpolates f at some partition Δ of the interval $[a, b]$.

5. Let $f(x) = x^2,\ x \in [0, 3]$. Determine the linear spline that interpolates f at the points $x_j = j,\ j = 0, 1, 2, 3$.

6. Consider the function

$$f(x) = \frac{1}{1 + 20x^2}, \quad x \in [-1, 1].$$

Verify Runge's phenomenon in Matlab (or Octave) by implementing polynomials p_n (Lagrange) interpolating f at the equidistant points

$$x_j^{(n)} = -1 + \frac{2j}{n}, \quad j = 0, ..., n$$

for increasing number n of points. Compare the quality of the approximation for grid points of the form

$$x_j^{(n)} = \cos\left(\frac{\pi}{2}\frac{2j+1}{n+1}\right), \quad j = 0, ..., n.$$

7. Create a Matlab (or Octave) function that implements linear spline interpolation. (i) It takes as input a set of data points $(x_i, y_i) \in \mathbb{R}^2$, $i = 0, \ldots, n$, (ii) computes the coefficients of all first degree polynomials and (iii) produces a plot of the spline, which also highlights the data points.