1. Show that the Cholesky factorization of a symmetric and positive definite (SPD) matrix $A \in \mathbb{R}^{n \times n}$ is unique.

   *Hint:* Assume that there exist two different upper triangular matrices $R_1, R_2 \in \mathbb{R}^{n \times n}$ with positive diagonal elements satisfying $A = R_1^T R_1 = R_2^T R_2$ and use matrix operations to derive a contradiction.

2. Compute the Cholesky factorization of the matrix
   $$
   \begin{bmatrix}
   2 & 1 & 0 & 0 \\
   1 & 2 & 1 & 0 \\
   0 & 1 & 2 & 1 \\
   0 & 0 & 1 & 2
   \end{bmatrix}.
   $$

3. Solve, using the Cholesky factorization, the following linear system
   
   $x_1 + x_2 + 2x_3 = 1$
   $x_1 + 2x_2 + 2x_3 = 3$
   $2x_1 + 2x_2 + 8x_3 = -2$

4. Compute the reduced QR factorization of the matrix
   $$
   A = \begin{bmatrix}
   3 & 7 \\
   0 & 12 \\
   4 & 1
   \end{bmatrix},
   $$

   using Gram-Schmidt orthogonalization. Then extend it to a full one.

5. Let $w \in \mathbb{K}^n$ be a nonzero vector. The Householder reflection
   $$
   H = I - 2 \frac{ww^*}{w^*w} \in \mathbb{K}^{n \times n}
   $$

   reflects every point across the $(n - 1)$-dimensional subspace orthogonal to $w$.

   (a) Show that $H$ is self-adjoint, orthogonal and involutory ($H^2 = I$).

   (b) Determine the eigenvalues, determinant and singular values of $H$. 
6. Let \( w = (2, -2, 1)^\top \), \( x = (3, -\sqrt{3}, \sqrt{13})^\top \) and \( y = (0, 1, 2)^\top \). Compute

(a) the Householder reflection \( H \in \mathbb{R}^{3 \times 3} \) reflecting across the two-dimensional plane orthogonal to \( w \),
(b) the norm \( \|Hx\|_2 \),
(c) the vector \( H(Hx) \),
(d) the inner product \( (Hw)^\top Hy \).

7. Use the Householder method to compute a full QR factorization of the matrix from Exercise 4.

8. A matrix \( A \in \mathbb{R}^{n \times n} \) has a Cholesky factorization, if and only if it is SPD. Write a Matlab (or Octave) function that implements Cholesky factorization and terminates (with an appropriate error message) if \( A \) is not SPD.