Exercise Sheet 5

1. Show that the Cholesky factorization of a symmetric and positive definite (SPD) matrix $A \in \mathbb{R}^{n\times n}$ is unique.

   Hint: Assume that there exist two different upper triangular matrices R_1, $R_2 \in \mathbb{R}^{n\times n}$ with positive diagonal elements satisfying $A = R_1^\top R_1 = R_2^\top R_2$ and use matrix operations to derive a contradiction.

2. Compute the Cholesky factorization of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

3. Solve, using the Cholesky factorization, the following linear system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 3 \\ 2x_1 + 2x_2 + 8x_3 &= -2 \end{aligned}$$

4. Compute the reduced QR factorization of the matrix

$$A = \begin{bmatrix} 3 & 7 \\ 0 & 12 \\ 4 & 1 \end{bmatrix},$$

   using Gram-Schmidt orthogonalization. Then extend it to a full one.

5. Let $w \in \mathbb{K}^n$ be a nonzero vector. The *Householder reflection*

$$H = I - 2\frac{ww^*}{w^*w} \in \mathbb{K}^{n\times n}$$

   reflects every point across the $(n-1)$-dimensional subspace orthogonal to w.

   (a) Show that H is self-adjoint, orthogonal and involutory ($H^2 = I$).

   (b) Determine the eigenvalues, determinant and singular values of H.

6. Let $w = (2, -2, 1)^\top$, $x = (3, -\sqrt{3}, \sqrt{13})^\top$ and $y = (0, 1, 2)^\top$. Compute

   (a) the Householder reflection $H \in \mathbb{R}^{3\times 3}$ reflecting across the two-dimensional plane orthogonal to w,

   (b) the norm $\|Hx\|_2$,

   (c) the vector $H(Hx)$,

   (d) the inner product $(Hw)^\top Hy$.

7. Use the Householder method to compute a full QR factorization of the matrix from Exercise 4.

8. A matrix $A \in \mathbb{R}^{n\times n}$ has a Cholesky factorization, *if and only if* it is SPD. Write a Matlab (or Octave) function that implements Cholesky factorization and terminates (with an appropriate error message) if A is not SPD.