Exercise Sheet 1

1. (a) Check if the columns of the following matrices are linearly independent subsets of $\mathbb{R}^3$:

$$\begin{bmatrix} 2 & 1 & 2 \\ 5 & 1 & 6 \\ 3 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 5 & 1 & 6 \\ 2 & 1 & 2 \\ 5 & 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 5 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 & 5 \\ 4 & -1 & 2 \\ 5 & 2 & 10 \end{bmatrix}.$$

   (b) Check if the columns of the following matrices are linearly independent subsets of $\mathbb{C}^2$:

$$\begin{bmatrix} i & 1 \\ i-1 & 1+i \end{bmatrix}, \quad \begin{bmatrix} 1+i & i \\ -i & 1-i \end{bmatrix}.$$

2. Show that if $A \in \mathbb{K}^{m\times n}$, then $\operatorname{ran} A$ is a subspace of K^m and $\ker A$ is a subspace of $\mathbb{K}^n$.

3. Let A be a square real matrix satisfying $A^\top = -A$. Show that the matrix $B = (I-A)^{-1}(I+A)$, where I is the identity matrix, is an orthogonal matrix.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

   (a) Show that the columns of A form a basis of $\mathbb{R}^3$.

   (b) Construct a set of orthonormal vectors $v_1, v_2, v_3 \in \mathbb{R}^3$, using the Gram-Schmidt algorithm, starting with the columns of A.

5. Let $\|\cdot\|$ be a vector norm and A be an arbitrary regular matrix. Show that $\|x\|_A := \|Ax\|$ defines a vector norm.

6. Given the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix},$$

   compute their norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_F$.

7. Let A be an arbitrary 4×4 matrix. We apply to A the following operations:

   (a) double the values of the 1st column.

   (b) halve the values of the 3rd row.

   (c) add the 3rd row to the 1st row.

(d) interchange the 1st and the 4th columns.

(e) subtract the 2nd row from all the others.

The above operations can be summarized and written as a matrix product LAR. Specify the square matrices L and R.

8. Consider the matrix

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 5 & 3 & 1 \end{bmatrix}.$$

Find a matrix U, such that $UA = R$, where R is an upper triangular matrix with ones on the diagonal.