Tikhonov and Iterative Regularization Methods for Embedded Inverse Problems

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Abstract

In this paper we suggest two novel classes of regularization techniques for systems of nonlinear ill–posed tomographic problems. We analyze variational regularization method as well as iterative regularization techniques analytically. The later turns out to be of Landweber-Kaczmarz type. We discuss new stopping criteria for such iterative methods and present a subtle convergence analysis. The stopping criterion is favourable (both analytically and practically) to existing stopping stratgies.

1 Introduction

In this paper we investigate a new concept of regularization methods for solving *nonlinear ill–posed operator equations* of the form

$$a(p,u) = g(p)\,, \qquad p \in \mathbb{S} \tag{1}$$

where $a(p,\cdot) : D \subseteq X \to Y$ are operators between separable Hilbert spaces X and Y, which are parameterized on the one dimensional sphere $\mathbb{S} := \mathbb{R}/(2\pi\mathbb{Z})$. Many *inverse problems*, such as impedance tomography [4] and computerized tomography [17, 7] can be formulated in such a way.

The basic idea of this paper is to rewrite (1) into a system of equations on the space of Bochner integrable periodic functions $H^s(\mathbb{S}, X)$, $s > 0$

$$a(p,u(p)) = g(p)\,, \qquad p \in \mathbb{S}\,, \tag{2}$$

$$\sqrt{\lambda \frac{1}{2\pi}\int_{\mathbb{S}} \left\| \frac{\partial^s u}{\partial p^s} \right\|_X^2 dp} = 0 \qquad \text{with some fixed } \lambda > 0\,. \tag{3}$$

If u is a solution of (1) then $\{p \mapsto u(p) \equiv u\}$ is a solution of (2), (3), respectively, and vice versa. That means that the systems of equations (1) and (2), (3) are equivalent.

In this paper we develop and analyze Tikhonov and iterative regularization methods for the solution of embedded equations (2), (3).

In Section 2 we derive and motivate variational and iterative regularization techniques for solving (2), (3). Since the equation (1) is embedded into a system of equations on a larger function space we call the resulting regularization techniques *embedded regularization* methods.

In Section 3 we analyze variational embedded regularization techniques. In particular, we show that the corresponding regularization techniques are well posed, convergent and stable. In Section 4 we show that the iterative regularization techniques are of Landweber-Kacmarcz form (cf. [12]) and present convergence results for these modified Landweber-Kaczmarz iterations.

Here, the Kacmarcz iteration schemes are combined with a reliable and efficient discrepancy principle which consists in enforcing the residual of *each* experiment (i.e. for each p) to be below a certain threshold. This termination criteria has several advantages (both analytically and numerically) over stopping criteria previously proposed in the literature. We remark that this stopping criterion also applies to already analyzed methods, like the Landweber-Kacmarcz iteration.

2 Embedded Regularization Methods

There are at least two basic concepts for solving systems of equations of the form (1): *iterative regularization methods* (cf. e.g. [13, 1, 5, 10, 6, 2]) and *Tikhonov type regularization* methods [20, 19, 18, 6].

In order to motivate our approach and indicate differences to existing methods we introduce the *function valued* operator

$$\begin{aligned} \mathcal{A} : D \subseteq X &\to S(\mathbb{S}, Y)\,, \\ u &\mapsto \{p \mapsto a(p,u)\} \end{aligned}$$

and the data set $\mathcal{G} = \{p \mapsto g(p)\}$, where $S(\mathbb{S}, Y)$ is a space of Bochner measurable functions.

We can state that $u \in X$ is a solution of (1) if it is a minimizer of

$$\|\mathcal{A}(u) - \mathcal{G}\|^2_{S(\mathbb{S},Y)}\,, \tag{4}$$

where $\|\cdot\|_{S(\mathbb{S},Y)}$ is a norm on the space $S(\mathbb{S}, Y)$, which in the the sequel (for the clarity of presentation) we always take the space $L^2(\mathbb{S}, Y)$ with the associated norm

$$\|\mathcal{G}\|^2_{L^2(\mathbb{S},Y)} := \frac{1}{2\pi}\int_{\mathbb{S}} \|g(p)\|^2_Y \, dp\,. \tag{5}$$

Then, standard Tikhonov regularization consists in approximating a minimizer of (4) by a minimizer of the functional

$$\|\mathcal{A}(u) - \mathcal{G}\|^2_{L^2(\mathbb{S},Y)} + \alpha\|u - u^*\|^2_X\,, \tag{6}$$

where $u^* \in D$ is some initial guess of the solution to be recovered and α is a regularization parameter.

Gradient type regularization methods aim to approximate the minimizer of (4) via an iterative procedure. Exemplarily we consider the Landweber iteration, which reads as follows:

$$u^{n+1} = u^n - \mathcal{A}'(u^n)^*(\mathcal{A}(u^n) - \mathcal{G}), \tag{7}$$

where $\mathcal{A}'(u)^*$ denotes the adjoint of the Fréchet-derivative of the operator $\mathcal{A}$ with respect to the spaces $L^2(\mathbb{S}, Y)$ and X.

Standard Tikhonov regularization for solving (2), (3) consists in calculating a minimizer of

$$\int_{\mathbb{S}} \|a(p, u(p)) - g(p)\|_Y^2 \, dp + \lambda \int_{\mathbb{S}} \left\| \frac{\partial^s u}{\partial p^s} \right\|_X^2 dp + \alpha \int_{\mathbb{S}} \|u(p) - u^*\|_X^2, \tag{8}$$

with a regularization parameter $\alpha > 0$.

In practical applications one is committed to a finite number of experiments (represented mathematically by one equation of the system) instead of an infinitesimal uncountable number as used in the above motivation. However, mostly it is possible to identify each experiment with a position on the sphere in a straight foward manner.

For instance in tomography one could identify the position on the sphere with the angle of emmitted X-rays. In this situation we use the operator

$$A(u) := \big(a_i(u)\big)_{i=0}^{N-1} := \big(a(p_i, u)\big)_{i=0}^{N-1},$$

where p_i denotes the index of the measurement. Standard iterative methods for the solution of the operator equation

$$A(u) = G := (g(p_i))_{i=0}^{N-1} \tag{9}$$

become inefficient if the evaluations of $a_i(u)$ and $a_i'(u)^*$ are expensive or if N is large. In this case typically Kaczmarz-type iterative methods are used. For instance, the Landweber-Kaczmarz method (cf. [12]) is defined by

$$u^{n+1} = u^n - a_i'(u^n)^* \left(a_i(u^n) - g(p_i)\right), \tag{10}$$

with i satisfying $n = \lfloor n/N \rfloor N + i$ (here $\lfloor x \rfloor$ is the largest integer less or equal x).

In this paper we also investigate an *embedded* Landweber-Kaczmarz type iteration for the solution of (2) and (3), which we define by:

$$U^{(n+1/2)} = U^{(n)} - \Big(a_i'(u_i^{(n)})^*(a_i(u_i^{(n)})) - g(p_i))\Big)_{i=0,\dots,N-1} \tag{11}$$

$$U^{(n+1)} = U^{(n+1/2)} - L(U^{(n+1/2)}), \tag{12}$$

where

$$U := (u_i)_{i=0,\dots,N-1}, \text{ with } u_i \in X.$$

and L corresponds to a discretized steepest descent direction of the continuous functional

$$\{p \mapsto u(p)\} \mapsto \lambda \frac{1}{2\pi} \int_{\mathbb{S}} \left\| \frac{\partial^s u}{\partial p^s} \right\|_X^2 dp \tag{13}$$

on $H^s(\mathbb{S}, X)$. This scheme is called of Kacmarcz type since the two "blocks" of equations are considered successively.

In the following we first analyze the embedded Tikhonov regularization and later on we provide a convergence analysis of the Landweber-Kacmarcz type iteration.

3 Embedded Tikhonov Regularization

A common approach for a stable solution of (1) is to minimize the functional (6) over D. As mentioned in the introduction we apply Tikhonov regularization to the operator-equations (3), (2), that is we minimize the functional (8) over $\mathcal{D}^s := \{u \in H^s(\mathbb{S}, X) : u(p) \in D, p \in \mathbb{S}\}$.

Convergence and stability of Tikhonov regularization for the solution of (1) is on the hand if $\mathcal{A}$ is continuous and weakly sequentially closed. We show that we can apply standard Tikhonov regularization as well as *embedded Tikhonov regularization* for a stable solution of (1) if the following assumptions hold true:

T1. The mapping $a : \mathbb{S} \times D \to Y$ is continuous and weakly continuous on bounded subsets.

T2. The domain D of the operators $a(p, \cdot)$ is weakly closed.

T3. The (exact) data $\mathcal{G}$ is in $L^2(S, Y)$ and (1) has a solution $u^\dagger$ in D.

Here a is called *weakly continuous on bounded subsets*, if $a|_{\mathbb{S} \times B_X}$ is continuous for all bounded $B_X \subseteq X$, when $\mathbb{S} \times B_X$ is considered with the product topology of the usual metric topology on $\mathbb{S}$ and the weak topology on X restricted to B_X, and Y is considered with the weak topology.

3.1 Notations and technical results

First we introduce some notations and general results on equi-continuity and Bochner Spaces (see, e.g., Yosida [22, Sections V.4, V.5]).

Equi-continuity

Let E and F be locally convex spaces, $B_E \subseteq E$ and $\mathcal{M} \subseteq C(B_E, F)$.

$\mathcal{M}$ is called *equi-continuous* on B_E if for every $e_0 \in B_E$ and every zero neighborhood $V \subseteq F$ there is a zero neighborhood $U \subseteq E$ such that $A(e_0) - A(e) \in V$ for all $A \in \mathcal{M}$ and all $e \in B_E$ with $e - e_0 \in U$. $\mathcal{M}$ is called *uniformly equi-continuous* if for every zero neighborhood $V \subseteq F$ there is a zero neighborhood $U \subseteq E$ such that $A(e) - A(f) \in V$ for all $A \in \mathcal{M}$ and all $e, f \in B_E$ with $e - f \in U$. If B_E is compact, then equi-continuity

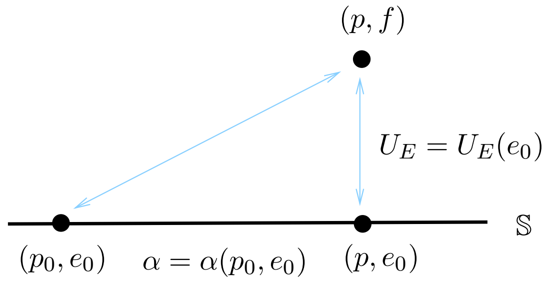

Figure 1: The equi-continuity of $\{a(p,\cdot) : p \in \mathbb{S}\}$ in e_0, the continuity of $A(\cdot, e_0)$ in p_0 and the triangle inequality imply the continuity of A in (p_0, u_0).

and uniform equi-continuity are equivalent. If E and F are Hilbert spaces then $\mathcal{M}$ is called *strongly* (resp. *weakly*) *equi-continuous* if E and F are considered with the strong (resp. weak) topology. Finally, a family (resp. sequence) $(A_\lambda)_{\lambda\in\Lambda} \subseteq C(B_E, F)^\Lambda$ is called (uniformly) equi-continuous if the set $\{A_\lambda : \lambda \in \Lambda\} \subseteq C(E, F)$ is (uniformly) equi-continuous.

If (E, d_E) is a metric space, analogous definitions hold where statements like $e - e_0 \in U$ have to be replaced by statements like $d_E(e, e_0) < \alpha$.

Proposition 3.1. *Let $B_E \subseteq E$ and $a : \mathbb{S} \times B_E \to F$. If $\mathcal{M}_1 := \{a(p,\cdot) : p \in \mathbb{S}\} \subseteq C(E, F)$, $\mathcal{M}_2 := \{a(\cdot, e) : e \in E\} \subseteq C(\mathbb{S}, F)$ and $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is equi-continuous, then a is continuous.*

Conversely, if a is continuous then $\mathcal{M}_1$ is equi-continuous and if additionally B_E is compact then $\mathcal{M}_2$ is equi-continuous, too.

Proof. Let $(\varphi_\lambda)_{\lambda\in\alpha}$ be a family of semi-norms generating the topology of F. Assume $\mathcal{M}_1$ being equi-continuous and let $(p_0, e_0) \in \mathbb{S} \times B_E$, $\lambda \in \Lambda$ and $\varepsilon > 0$. Since $\mathcal{M}_1$ is equi-continuous in e_0 there is a zero neighborhood U_E such that $e - e_0 \in U_E$ and $p \in \mathbb{S}$ implies $\varphi_\lambda(a(p, e) - a(p, e_0)) < \varepsilon/2$. Since $a(\cdot, e_0)$ is continuous in p_0 there is $\eta > 0$ such that $|p - p_0| < \eta$ implies $\varphi_\lambda(a(p, e_0) - a(p_0, e_0)) < \varepsilon/2$. Hence for all $(p, e) \in B_\eta(p_0) \times (e_0 + U_E)$

$$\varphi_\lambda(a(p, e) - a(p_0, e_0)) \leq \varphi_\lambda(a(p, e) - a(p, e_0)) + \varphi_\lambda(a(p, e_0) - a(p_0, e_0)) < \varepsilon\,.$$

This shows that a is continuous in (e_0, b_0). To proof that a is continuous when $\mathcal{M}_2$ is equi-continuous is analogous.

Now assume that a is continuous and let $e_0 \in E$, $\lambda \in \Lambda$ and $\varepsilon > 0$. Since $a(\cdot, e_0)$ is continuous there exist $\alpha(p) > 0$ and zero neighborhoods $U_E(p)$ such that $\varphi_\lambda(a(q, e) - a(p, e_0)) < \varepsilon/2$ for all $(q, e) \in B_{\alpha(p)}(p) \times (e_0 + U_E(p))$. Hence

$$\varphi_\lambda(a(q, e) - a(q, e_0)) \leq \varphi_\lambda(a(q, e) - a(p, e_0)) + \varphi_\lambda(a(p, e_0) - a(q, e_0)) < \varepsilon \tag{14}$$

for all $q \in B_{\alpha(p)}(p)$ and all $e \in e_0 + U(p)$. Since $(B_{\alpha(p)}(p))_{p\in\mathbb{S}}$ is an open covering of $\mathbb{S}$ and $\mathbb{S}$ is compact there is a finite subcovering $\{B_{\alpha(p_1)}(p_1), \ldots, B_{\alpha(p_K)}(p_K)\}$ of $\mathbb{S}$. Define $U_E := \bigcap_{k=1}^K U_E(p_k)$, let $e \in e_0 + U_E$ and $q \in \mathbb{S}$. Then there exists $k \in \{1, \ldots, K\}$

with $q \in B_{\alpha(p_k)}(p_k)$ and from (14) it follows that $\varphi_\lambda(a(q,e) - a(q,e_0)) < \varepsilon$. This shows that $\mathcal{M}_1$ is equi-continuous in e_0. The proof for the equi-continuity of $\mathcal{M}_2$ can be done analogously. $\square$

Spaces of Bochner integrable functions

Let H be a separabel Hilbert space.

A step function $\mathcal{U} : \mathbb{S} \to H$ is called measurable if $\mathcal{U}^{-1}(x)$ is measurable for all $x \in H$. A function $\mathcal{U} : \mathbb{S} \to H$, which we will often write in the form $\mathcal{U} = \{p \mapsto u(p)\}$, is called measurable if there is a sequence $\mathcal{U}_n$ of step functions such that $u_n(p) \to u(p)$ in H for a.e. $p \in \mathbb{S}$. It is called (Bochner-)integrable, if additionally $\lim_{n\to\infty} \int_{\mathbb{S}} \|u_n(p) - u(p)\|_H \, dp = 0$, in which case

$$\int_{\mathbb{S}} u(p)\, dp := \lim_{n\to\infty} \int_{\mathbb{S}} u_n(p)\, dp := \lim_{n\to\infty} \sum_{x \in H} \lambda(\mathcal{U}_n^{-1}(x))x$$

is called the integral of $\mathcal{U}$.

A Function $\mathcal{U}$ is measurable if and only if it is weakly measurable, i.e., $p \mapsto \langle u(p), \xi\rangle$ is measurable for all $\xi \in H$ (Pettis 1938). If $\mathcal{U},\mathcal{U}' : \mathbb{S} \to H$ are measurable then $p \mapsto \|u(p)\|_H$ and $p \mapsto \langle u(p), u(p)'\rangle_H$ are measurable. A measurable function $\mathcal{U}$ is integrable if and only if $p \mapsto \|u(p)\|_H$ is integrable (Bochner 1933). Finally $L^2(\mathbb{S}, H)$ denotes the space of (equivalence classes of) measurable functions $\mathcal{U}$ with

$$\|\mathcal{U}\|^2_{L^2(\mathbb{S},H)} := \frac{1}{2\pi} \int_{\mathbb{S}} \|u(p)\|_H^2 \, dp \; < \; \infty \, .$$

Here and in the following two functions are identified if they differ only on a set of measure zero.

The space $L^2(\mathbb{S}, H)$ with the scalar product

$$\langle \mathcal{U},\mathcal{U}'\rangle_{L^2(\mathbb{S},H)} := \frac{1}{2\pi} \int_{\mathbb{S}} \langle u(p), u'(p)\rangle_H \, dp$$

is a Hilbert space with the associated norm $\|\mathcal{U}\|_{L^2(\mathbb{S},H)} = \sqrt{\langle \mathcal{U},\mathcal{U}'\rangle_{L^2(\mathbb{S},H)}}$.

For $\mathcal{U} \in L^2(\mathbb{S}, H)$ we define the *Fourier transform* $\hat{\mathcal{U}} := (\hat{u}(k))_{k\in\mathbb{Z}}$ of $\mathcal{U}$ by

$$\hat{u}(k) := \frac{1}{2\pi} \int_{\mathbb{S}} \exp(-ikp)\, u(p)\, dp \, . \tag{15}$$

Since $\{p \mapsto \exp(-ikp)u(p)\}$ is weakly measurable and $L^2(\mathbb{S}, \mathbb{R}) \subseteq L^1(\mathbb{S}, \mathbb{R})$ the *Fourier coefficients* in (15) are well defined. It is easy to see, that $\langle \mathcal{U},\mathcal{U}'\rangle_{L^2(\mathbb{S},H)} = \sum_{k\in\mathbb{Z}} \langle \hat{u}(k), \hat{u}'(k)\rangle_{H_{\mathbb{C}}}$. Here $H_{\mathbb{C}} := H \oplus iH$ denotes the complexification of H and $\langle \cdot, \cdot\rangle_{H_{\mathbb{C}}}$ the associated inner product. Finally $H^s(\mathbb{S}, H)$ is defined as the space of all (equivalence classes of) Bochner measurable functions $\mathcal{U} \in L^2(\mathbb{S}, H)$ with $\|\mathcal{U}\|_s^2 := \sum_{k\in\mathbb{Z}} (1 + |k|^s)^2 \|\hat{u}(k)\|^2_{H_{\mathbb{C}}} < \infty$. Note that $H^s(\mathbb{S}, H)$ is a Hilbert space with the scalar product

$$\langle \mathcal{U},\mathcal{U}'\rangle_s := \sum_{k\in\mathbb{Z}} (1 + |k|^s)^2 \langle \hat{u}(k), \hat{v}(k)\rangle_{H_{\mathbb{C}}} \tag{16}$$

and the associated norm $\|\cdot\|_s$.

Lemma 3.2. *Let $s > 1/2$. Each element $\mathcal{U} \in H^s(\mathbb{S}, H)$ has a continuous representative and the mapping $i_s : H^s(\mathbb{S}, H) \hookrightarrow C(\mathbb{S}, H)$ is continuous. Let $\mathcal{U} \in H^s(\mathbb{S}, H)$, $x \in H$ and define $\langle \mathcal{U}, x\rangle_H := \{p \mapsto \langle u(p), x\rangle_H\}$. Then $\langle \mathcal{U}, x\rangle_H \in H^s(\mathbb{S})$ and $\|\langle \mathcal{U}, x\rangle\|_{H^s(\mathbb{S})} \leq \|\mathcal{U}\|_s \|x\|_H$.*

Proof. Let $n < m \in \mathbb{N}$. The Cauchy-Schwartz inequality shows

$$\left\| \sum_{|k|=n}^{m} \hat{u}(k) \exp(ikp) \right\|_H \leq \sum_{|k|=n}^{m} \|\hat{u}(k)\|_{H_{\mathbb{C}}} (1 + k^s)/(1 + k^s)$$
$$\leq \|\mathcal{U}\|_s \left(2 \sum_{k=n}^{\infty} 1/(1 + k^s)^2 \right)^{1/2} .$$

Since $s > 1/2$ the series $\sum_{k=0}^{\infty} 1/(1+k^s)^2$ is convergent and hence $\sum_{|k|\leq n} \hat{u}(k) \exp(ikp)$ is uniformly convergent with limit $\sum_{k\in\mathbb{Z}} \hat{u}(k) \exp(ikp)$. Since $\mathcal{U} \mapsto (\hat{u}(k))_{k\in\mathbb{Z}}$ in one to one it follows that $u(p) = \sum_{k\in\mathbb{Z}} \hat{u}(k) \exp(ikp)$ and that u is continuous. Using (17) for $n = 0$ and taking the limit $m \to \infty$ shows that i_s is bounded. Now let $\mathcal{U} \in H^s(\mathbb{S}, H)$ and $x \in H$. Then the Fourier coefficients of $\langle \mathcal{U}, x\rangle_H$ are given by

$$\frac{1}{2\pi} \int_{\mathbb{S}} \langle u(p), x\rangle_H e^{-ikp}\, dp = \langle \frac{1}{2\pi} \int_{\mathbb{S}} e^{-ikx} u(p)\, dp, x\rangle_{H_{\mathbb{C}}} = \langle \hat{u}(k), x\rangle_{H_{\mathbb{C}}} .$$

From the Cauchy-Schwartz inequality it follows

$$\begin{aligned} \|\langle \mathcal{U}, x\rangle\|_{H^s(\mathbb{S})}^2 &= \sum_{k\in\mathbb{Z}} (1 + k^s)^2 |\langle \hat{u}(k), x\rangle_{H_{\mathbb{C}}}| \\ &\leq \sum_{k\in\mathbb{Z}} (1 + k^s)^2 \|\hat{u}(k)\|_{H_{\mathbb{C}}}^2 \|x\|_H^2 \\ &= \|\mathcal{U}\|_s^2 \|x\|_H^2 < \infty . \end{aligned}$$

This shows $\langle \mathcal{U}, x\rangle_H \in H^s(\mathbb{S})$ and $\|\langle \mathcal{U}, x\rangle\|_{H^s(\mathbb{S})} \leq \|\mathcal{U}\|_s \|x\|_H$. □

3.2 Tikhonov Regularization for the solution of $\mathcal{A}(u) = \mathcal{G}$

Assumption T1 implies that $a(\cdot, u)$ is continuous for all $u \in D$, therefore $a(\cdot, u) \in L^2(\mathbb{S}, Y)$ and hence

(17)
$$\begin{aligned} \mathcal{A} : D \subseteq X &\to L^2(\mathbb{S}, Y) , \\ u &\mapsto \{p \mapsto a(p, u)\} \end{aligned}$$

is well defined. By assumption T3 we have $\mathcal{G} \in L^2(\mathbb{S}, Y)$ and hence (1) is equivalent to the single operator-equation

(18)
$$\mathcal{A}(u) = \mathcal{G} .$$

In order to apply Tikhonov regularization to (18) we use proposition 3.1 to prove the following lemma:

Lemma 3.3. *The mapping $\mathcal{A} : D \to L^2(\mathbb{S}, Y)$ is continuous and the restricted mapping $\mathcal{A}|_{B_X} : D \cap B_X \to L^2(\mathbb{S}, Y)$ is weakly continuous for all bounded $B_X \subseteq X$.*

Proof. Let $u_0 \in D$ and $\varepsilon > 0$. By assumption T1 a is continuous and from Proposition 3.1 it follows that $\mathcal{M}_1 = \{a(p, \cdot) : p \in \mathbb{S}\}$ is strongly equi-continuous in u_0. Hence there is $\alpha > 0$ such that $\sup\{\|a(p, u) - a(p, u_0)\|_Y : p \in \mathbb{S}\} < \varepsilon$ for all $u \in D \cap B_\alpha(u_0)$. Hence

$$\|\mathcal{A}(u_0) - \mathcal{A}(u)\|^2_{L^2(\mathbb{S},Y)} = \frac{1}{2\pi} \int_{\mathbb{S}} \|a(p, u_0) - a(p, u)\|^2_Y \, dp < \varepsilon^2$$

for all $u \in D \cap B_\alpha(u_0)$. This shows that $\mathcal{A}$ is continuous at u_0.

Now let $B_X \subseteq X$ be bounded. Since $a|_{\mathbb{S}\times B_X}$ is weakly continuous and D is weakly closed the image $a|_{\mathbb{S}\times B_X}$ is bounded. This implies that

$$\int_{\mathbb{S}} \|a(p, u)\|^2_Y \, dp \leq R^2$$

for $u \in B_X \cap D$. Hence $\mathrm{Im}(\mathcal{A}|_{B_X})$ is bounded and the weak topology on $\mathrm{Im}(\mathcal{A}|_{B_X})$ is generated by the semi-norms $|\langle \cdot, g \otimes f\rangle|$ with $g \in Y$, $f \in L^2(\mathbb{S})$. To verify that $\mathcal{A}|_{B_X}$ is weakly continuous it is sufficient to show that for all $u_0 \in D \cap B_X$, $g \in Y$, $f \in L^2(\mathbb{S})$ and $\varepsilon > 0$ there is a weak neighborhood U of u_0 such that $\langle \mathcal{A}(u) - \mathcal{A}(u_0), g \otimes f\rangle_{L^2(\mathbb{S},Y)} < \varepsilon$ for all $u \in U$.

Since $\{a(p, \cdot) : p \in \mathbb{S}\}$ is weakly equi-continuous on B_X there is a weak zero neighborhood U such that $|\langle a(p, u) - a(p, u_0), g\rangle_Y| < \varepsilon/\|f\|_{L^2(\mathbb{S})}$ for all $u \in D \cap B_X$ and $p \in S^1$. Hence, from the Cauchy inequality on $L^2(\mathbb{S})$, it follows that

$$\begin{aligned}
&4\pi^2 \left|\langle \mathcal{A}(u) - \mathcal{A}(u_0), g \otimes f\rangle_{L^2(\mathbb{S},Y)}\right|^2 \\
=&\left|\int_{\mathbb{S}} \langle a(p, u) - a(p, u_0), (g \otimes f)(p)\rangle_Y \, dp\right|^2 \\
=&\left|\int_{\mathbb{S}} f(p) \cdot \langle a(p, u) - a(p, u_0), g\rangle_Y \, dp\right|^2 \\
\leq&2\pi\|f\|^2_{L^2(\mathbb{S})} \int_{\mathbb{S}} |\langle a(p, u) - a(p, u_0), g\rangle_Y|^2 dp < 4\pi^2\varepsilon^2 \, .
\end{aligned}$$

This shows that $\mathcal{A}|_{B_X}$ is weakly continuous in u_0. □

If we have only *noisy data* $\mathcal{G}^\delta := \{p \mapsto g^\delta(p)\}$ with $\|\mathcal{G} - \mathcal{G}^\delta\| < \delta$ and (1) is ill posed then

$$a(p, u) = g^\delta(p) \, , \qquad \text{for } p \in \mathbb{S} \tag{19}$$

may not have a solution, and even if a solution u^δ of (19) exists $\|u^* - u^\delta\|_X$ can be arbitrary large (see, e.g., [6, Chapter 3]). Therefore (1) has to be regularized. Hence instead of trying to solve (19), one minimizes the Tikhonov functional

$$J_{\alpha,\mathcal{G}^\delta}(u) := \|\mathcal{A}(u) - \mathcal{G}^\delta\|^2_{L^2(\mathbb{S},Y)} + \alpha\|u - u^*\|^2_X \tag{20}$$

over D, where $u^* \in D$ is some initial guess and $\alpha > 0$ is a regularization parameter.

Since X is separable, the weak topology on bounded subsets of X is metrizable. Therefore weak sequential-continuity and sequential-continuity on bounded subsets are equivalent. Hence Lemma 3.3 guaranties that the following results for Tikhonov regularization hold true (cf. [6, Chapter 10]):

1. *Well-posedness.* Let $\alpha > 0$ and $\mathcal{G}^\delta \in L^2(\mathbb{S}, Y)$. Then the functional $K_{\alpha,\mathcal{G}^\delta}$ attains a minimizer over $\mathcal{D}^s$.
2. *Stability.* Let $\mathcal{G}_n$ be a sequence in $L^2(\mathbb{S}, Y)$ with $\mathcal{G}_n \to \mathcal{G}^\delta$ and let u_n be a minimizer of $J_{\alpha,\mathcal{G}_n}$. Then there is a convergent subsequence of u_n and the limit of any convergent subsequence of u_n is a minimizer of $J_{\alpha,\mathcal{G}^\delta}$.
3. *Convergence.* Let $\hat{\alpha}(\delta)$ satisfy

$$\lim_{\delta\to 0} \hat{\alpha}(\delta) = \lim_{\delta\to 0} \delta^2/\hat{\alpha}(\delta) = 0\,.$$

   Assume that $(\delta_n)_{n\in\mathbb{N}}$ is a sequence converging to zero and assume that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{S}, Y)$ satisfying $\|\mathcal{G}_n - \mathcal{G}\|_{L^2(\mathbb{S},Y)} < \delta_n$. Let u_n be a minimizer of $J_{\alpha_n,\mathcal{G}_n}$ with $\alpha_n := \hat{\alpha}(\delta_n)$.

   Then, $(u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence. Moreover, each weakly convergent subsequence is strongly convergent and the limit is an u^*-minimal norm solution of (18). If (18) has a unique u^*-minimal norm solution $u^\dagger$, then $u_n \to u^\dagger$.

Convergence Rates

Convergence rates for Tikhonov Regularization can be derived if the solution satisfies a *source wise representation* (cf. [6]). are obtained under the so called *source conditions.* Let us assume for the rest of this Section that the following additional condition holds true:

T4. D is convex, $\{a(p,\cdot) : p \in \mathbb{S}\}$ is Fréchet equi-differentiable and $D_2 a : \mathbb{S} \times D \to B(X, Y)$ is continuous.

Here $\{a(p,\cdot) : p \in \mathbb{S}\}$ is called Fréchet equi-differentiable if, for all $u_0 \in D$, all $\varepsilon > 0$, there is $\eta > 0$ such that

$$\sup_{p\in\mathbb{S}} \|a(p, u_0 + h) - a(p, u_0) - D_2 a(p, u_0)(h)\|_Y < \varepsilon \|h\|_X\,, \tag{21}$$

for $\|h\|_X < \eta$, where $D_2 a(p, u_0) \in B(X, Y)$ denotes the Fréchet derivative of $a(p,\cdot)$ at u_0.

Lemma 3.4. *The operator $\mathcal{A}$ is Fréchet differentiable and the Fréchet derivative $\mathcal{A}'(u_0) \in B(X, L^2(\mathbb{S}, Y))$ at $u_0 \in D$ is given by*

$$\mathcal{A}'(u_0)(h)(p) = D_2 a(p, u_0)(h)\,. \tag{22}$$

Proof. Let $u_0 \in D$, define $\mathcal{A}'(u_0)$ by (22), and let $h \in X$. Since $D_2 a(\cdot, u_0) : \mathbb{S} \to B(X,Y)$ is continuous and $\mathbb{S}$ is compact, we have $\sup_{p\in\mathbb{S}} \|D_2 a(p,u_0)\|_{B(X,Y)} =: C < \infty$. Hence

$$2\pi\|\mathcal{A}'(u_0)(h)\|_{L^2(\mathbb{S},Y)} = \int_{\mathbb{S}} \|D_2 a(p,u_0)(h)\|_Y^2 \, dp \le 2\pi C\|h\|_s \,,$$

and consequently $\mathcal{A}'(u_0) : X \to L^2(\mathbb{S}, Y)$ is a bounded linear operator.

Now let $\varepsilon > 0$. Since $\{a(p,\cdot) : p \in \mathbb{S}\}$ is Fréchet equi-differentiable, there exists $\eta > 0$ such that

$$\sup_{p\in\mathbb{S}} \|a(p, u_0 + h) - a(p, u_0) - D_2 a(p, u_0)(h)\|_Y < \varepsilon\|h\|_X \,, \tag{23}$$

for $\|h\|_X < \eta$. Hence

$$\begin{aligned}
&2\pi\|\mathcal{A}(u_0 + h) - \mathcal{A}(u_0) - \mathcal{A}'(u_0)(h)\|_{L^2(\mathbb{S},Y)}^2 \\
&= \int_{\mathbb{S}} \|a(p, u_0 + h) - a(p, u_0) - D_2 a(p, u_0)(h)\|_Y^2 \, dp \\
&\le \int_{\mathbb{S}} \varepsilon^2 \|h\|_X^2 \, dp < 2\pi\varepsilon^2\|h\|_X^2 \,.
\end{aligned}$$

This shows, that $\mathcal{A}$ is Fréchet differentiable at u_0 and its derivative is given by $\mathcal{A}'(u_0)$. □

From Lemma 3.4 it follows that (see, e.g., [6, Theorem 10.7])

Theorem 3.5. *Let $u^\dagger$ be an u^*-minimum norm solution of (1) and let $\mu \in [1/2, 1]$. Assume that there exist $\gamma > 0$ and $\rho > 2\|u^* - u^\dagger\|$ such that*

$$\|\mathcal{A}'(u^\dagger) - \mathcal{A}'(u)\| \le \gamma\|u^\dagger - u\| \,, \quad u \in B_\rho(u^\dagger)$$

If there exits $w \in X$ satisfying $\gamma\|w\| < 1$ and

$$u^\dagger - u^* = (\mathcal{A}'(u^\dagger)^* \mathcal{A}'(u^\dagger))^\mu w \,,$$

and if $\alpha \sim \delta^{2/(2\mu+1)}$, then $\|u_\alpha^\delta - u^\dagger\| = O(\delta^{2\mu/(2\mu+1)})$.

In Section 3.4 we prove similar statements for the embedding approach. Note, that the continuity in p is not needed for Tikhonov regularization, where it would be sufficient to assume that $A(\cdot, u)$ is measurable for all $u \in D$ and $\{a(p,\cdot) : p \in \mathbb{S}\}$ is equi-continuous and weakly equi-continuous on bounded sets (cf. proof of Lemma 3.3). Nevertheless this continuity is necessary for the embedding approach (cf. proof of Proposition 3.10). However the continuity in p seems to be a natural assumption and hold true for many practical problems.

3.3 Examples

Example 3.6 (Computerized Tomography)**.** Let $\Omega := B_1(0) \subseteq \mathbb{R}^2$ denote the unit ball in $\mathbb{R}^2$ and let $p \in \mathbb{S}$. For $\varphi \in C_0^\infty(\Omega)$ and s in $(-1,1)$ we define

$$r(p,\varphi)(s) := \int_{L(p,s)} \varphi(g)\,dy := \int_{-1}^{1} \varphi(sp + tp^\perp)\,dt\,, \tag{24}$$

i.e., the integral of φ over the line $L(s,p) := sp + \mathbb{R}p^\perp$ (see, e.g., [17]). By applying the Cauchy Schwartz inequality one can see that $\|r(p,\varphi)\|^2_{L^2(-1,1)} \leq 2\|\varphi\|^2_{L^2(\Omega)}$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ there is a unique linear mapping

$$r(p,\cdot) : L^2(\Omega) \to L^2(-1,1) \;:\; u \mapsto g(p) = r(p,u)\,,$$

such that $r(p,\varphi)$ is given by (24) for all $\varphi \in C_0^\infty(\Omega)$ and $\|r(p,\cdot)\|^2 \leq 2$. It is easy to see that this estimate is sharp, i.e., $\|r(p,\cdot)\|^2 = 2$. The function $r(p,u)$ is called the projection of u in direction p. The Radon transform $R : L^2(\Omega) \to L^2(\mathbb{S}, L^2(-1,1)) \equiv L^2(\mathbb{S}\times(-1,1),\mathbb{R})$ is defined by $\mathcal{G} = R(u) = \{p \mapsto r(p,u)\}$.

The aim in Computerized Tomography is to reconstruct $u \in L^2(\Omega)$ from the measured noisy projections $\mathcal{G}^\delta = \{g_p^\delta\}_{p\in\mathbb{S}}$ with $\|\mathcal{G}^\delta - \mathcal{G}\|_{L^2(\mathbb{S},L^2(-1,1))} < \delta$. Due to Lemma 3.1 Computerized Tomography fits in our general framework with $D = X = L^2(\Omega)$, $Y = L^2(-1,1)$ and $a(p,\cdot) = r(p,\cdot)$ if we can show that $\{r(p,\cdot) : p \in S^1\}$ is equi-continuous, $r(p,\cdot)|_{B_X}$ is weakly continuous on bounded sets and $\{r(\cdot,\varphi) : \varphi \in B_X\}$ is weakly equi-continuous for all bounded sets $B_X \subseteq X$.

Since $r(p,\cdot)$ is bounded and linear it is weakly continuous for all $p \in \mathbb{S}$. Since $\{r(p,\cdot) : p \in S^1\}$ is equi-bounded there is a constant $c > 0$ such that $\|r(p,\cdot)\|_Y \leq C$ for all $p \in \mathbb{S}$. If $\varepsilon > 0$ and $\|u - u_0\|_X < \varepsilon/C$ then

$$\|r(p,u_0) - r(p,u)\|_Y \leq \|r(p,\cdot)\|\|u_0 - u\|_X < \varepsilon\,,$$

for all $p \in \mathbb{S}$. This shows, that $\{r(p,\cdot)\}$ is equi-continuous. Finally we show that $\{r(\cdot,\varphi) : \varphi \in B_X\}$ is uniformly weakly equi-continuous. Let $\psi \in C_0^\infty(-1,1)$ and $\varepsilon > 0$ and $d := \max\{\|\varphi\|_X : x \in B_X\}$. Then there is $\alpha > 0$ such that $|s-t| < \alpha$ implies $|\psi(s) - \psi(t)|^2 < \varepsilon^2/(d^2\pi)$. If $|p-q| \leq \alpha$, then $|\langle p,x\rangle - \langle q,x\rangle| < \alpha$ for all $x \in B_1(0)$ and therefore

$$|\langle \varphi, r(p,\cdot)^*\psi - r(q,\cdot)^*\psi\rangle_X|^2 \leq \|\varphi\|_X^2 \int_{B_1(0)} (\psi(\langle p,x\rangle) - \psi(\langle q,x\rangle))^2\,dx < d^2\pi\varepsilon^2/(d^2\pi) = \varepsilon^2$$

for all $\varphi \in X$. This shows that $\{r(\cdot,\varphi) : \varphi \in B_X\}$ is uniformly weakly equi-continuous.

Example 3.7 (Electrical Impedance Tomography (EIT))**.** Let $\Omega \subseteq \mathbb{R}^2$ be a bounded simply connected domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and outer normal ν and $m > 0$ a positive real number. Further let $L_m^\infty(\Omega) := \{u \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega(u) \geq m\}$, $L_\diamond^2(\partial\Omega) := \{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f(s)\,ds = 0\}$, $u \in L_m^\infty(\Omega)$ and $f \in L_\diamond^2(\partial\Omega)$. In EIT the function u is interpreted as the *conductivity* inside some *body* Ω and f as the applied *boundary current* (see [4, 3, 9, 16]).

Using standard techniques for second order elliptic differential equations (see, e.g., [21]) it can be shown that the Neumann problem

$$\nabla \cdot (u \nabla \varphi) = 0 \,, \quad \text{in } \Omega \,, \tag{25}$$

$$u \nabla \frac{\partial \varphi}{\partial \nu} = f \,, \quad \text{on } \partial\Omega \tag{26}$$

attains an unique (weak) solution $\varphi = L(f,u)$ in

$$H^1_\diamond(\Omega) := \{\varphi \in H^1(\Omega) : T(u) \in L^2_\diamond(\partial\Omega)\} \,,$$

which can be interpreted as the attuning *electrical potential*. Here $T : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace operator (which is bounded linear with norm $\|T\| < \infty$). Note that $H^1_\diamond(\Omega)$ is a closed subspace of $H^1(\Omega)$ and from the Poincaré inequality follows that $\langle \varphi, \psi \rangle_\diamond := \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx$ defines a inner product on $H^1_\diamond(\Omega)$ with associated norm $\|\varphi\|_\diamond := \sqrt{\langle \varphi, \psi \rangle_\diamond}$ and $\|\cdot\|_\diamond \sim \|\cdot\|_{H^1(\Omega)}$. The stability estimates

$$\|L(f,u) - L(f',u)\|_\diamond \leq \frac{\|T\|}{m} \|f - f'\|_{L^2_\diamond(\partial\Omega)} \,, \tag{27}$$

$$\|L(f,u) - L(f,u')\|_\diamond \leq \frac{\|T\|}{m^2} \|f\|_{L^2_\diamond(\Omega)} \|u - u'\|_{L^\infty(\Omega)} \,, \tag{28}$$

hold for all $u, u' \in L^\infty_m(\Omega)$ and $f, f' \in L^2_\diamond(\partial\Omega)$.

Now let $\{p \mapsto f(p)\}$ be a family of applied boundary currents uniformly bounded in $L^2_\diamond(\partial\Omega)$ and depending smoothly on p.

$$H^2_m(\Omega) := \{u \in H^2(\Omega) : \inf_\Omega u \geq m\} \subseteq H^2(\Omega) \tag{29}$$

is closed and convex and thus weakly closed. Since $\mathrm{i} : H^2(\Omega) \to L^\infty(\Omega)$ is linear and compact (and thus bounded with norm $\|\mathrm{i}\|$) the mappings

$$a(p,\cdot) : H^2_m(\Omega) \to L^2_\diamond(\partial\Omega) : u \mapsto a(p,u) := g(p) := (T \circ L)(f(p), \mathrm{i}(u))$$

are continuous and weakly continuous on bounded sets. The aim in EIT is to reconstruct $u \in L^2(\Omega)$ from the measured noisy boundary voltages $\{p \mapsto g^\delta(p)\}$. Hence EIT fits in our general framework with $X = H^2(\Omega)$, $D = H^2_m(\Omega)$ and $Y = L^2_\diamond(\mathbb{S})$ if we can show that $\{a(p,\cdot) : p \in S^1\}$ is equi-continuous, $\{a(p,\cdot)|_{B_X} : p \in \mathbb{S}\}$ is weakly equi-continuous and $a(\cdot,\varphi)$ is continuous for all $\varphi \in D$. From (27) and (29) follows immediately that $a(\cdot,\varphi)$ is continuous. If we introduce the mappings

$$\tilde{a}(p,\cdot) : L^\infty_m(\Omega) \to L^2_\diamond(\partial\Omega) : u \mapsto a(p,u) := g(p) := (T \circ L)(f(p), u)$$

then we have $a(p,\cdot) = \tilde{a}(p,\cdot) \circ \mathrm{i}$. Since $\{p \mapsto f(p)\}$ is uniformly bounded it follows immediately from (28) that $\{\tilde{a}(p,\cdot) : p \in \mathbb{S}\}$ and hence $\{a(p,\cdot) : p \in \mathbb{S}\}$ is strongly equi-continuous. The weak equi-continuity of $\{a(p,\cdot)|_{B_X} : p \in \mathbb{S}\}$ for bounded $B_X \subseteq X$ will be a consequence of the following lemma.

Lemma 3.8. *Let $(E, \|\cdot\|_E)$ be a Banach space, $K : X \to E$ linear and compact, $B_X \subseteq X$ bounded, $B_E \subseteq E$ and $K(B_X) \subseteq B_E$. If $\tilde{\mathcal{M}} \subseteq C(B_E, Y)$ is strongly equi-continuous, then $\mathcal{M} := \{\tilde{a} \circ K|_{B_X} : \tilde{a} \in \tilde{\mathcal{M}}\}$ is weakly equi-continuous.*

Proof. Since K is linear and compact it is completely sequentially continuous. Since the weak topology on B_X is metrizable the restriction $K|_{B_X}$ is completely continuous. Let $u_0 \in B_X$ and $\varepsilon > 0$. Since $\tilde{\mathcal{M}}$ is equi-continuous in $K(u_0)$ there is $\alpha > 0$ such that $\|e - K(u_0)\|_E < \alpha$ implies $\|\tilde{a}(e) - \tilde{a}(K(u_0))\|_Y < \varepsilon$ for all $\tilde{a} \in \tilde{\mathcal{M}}$. Since $K|_{B_X}$ is completely continuous in u_0 there is a weak neighborhood $U \subseteq D \cap B_X$ of u_0 such that $u \in U$ implies $\|K(u) - K(u_0)\|_B < \alpha$ and hence $\|a(u) - a(u_0)\|_Y = \|\tilde{a}(Ku) - \tilde{a}(K(u_0))\|_Y < \varepsilon$. This shows that $\mathcal{M}|_{B_X}$ is weakly equi-continuous in u_0. □

Remark 3.9 (Inverse Doping Profile)**.** An application closely related to Example 3.7 is the identification of doping profiles in semiconductor devices. The main difference between this identification problem and EIT is the fact that the boundary condition in (26) is substituted by mixed boundary conditions: $\varphi = f$ on $\partial\Omega_0$, $\varphi = 0$ on $\partial\Omega_1$, $u\nabla\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega\backslash(\partial\Omega_0 \cup \partial\Omega_1)$, i.e. the input corresponds to dirichlet data (voltage). The output is the Neumann trace measured at $\partial\Omega_1$ and corresponds to $u\nabla\frac{\partial\varphi}{\partial\nu}$ (current). Here $\partial\Omega_0$ and $\partial\Omega_1$ represent the Ohmic contacts of the semiconductor, and $\partial\Omega - (\partial\Omega_0 \cup \partial\Omega_1)$ are the insulating surfaces.

The unknown coefficient u models the doping concentration, which is produced by diffusion of different materials into the silicon crystal and by implantation with an ion beam. For a detailed exposition of this inverse problem we refer the reader to [15] and also to the review paper [14] and the references therein.

3.4 Analysis of Embedded Tikhonov Regularization

Here we consider the solution of (2) and (3) with a Tikhonov regularization method. For the analysis of this methods we require that $s > 1/2$, which we assume in the sequel to hold.

We define

$$\begin{aligned} \mathcal{F} : \mathcal{D}^s \subseteq H^s(\mathbb{S}, X) &\to L^2(\mathbb{S}, Y)\,, \\ \{p \mapsto u(p)\} &\mapsto \{p \mapsto a(p, u(p))\} \end{aligned} \tag{30}$$

Note that $\mathcal{F}$ now plays the role of $\mathcal{A}$ but acts on a family of elements $\mathcal{U} = \{p \mapsto u(p)\}$ (pointwise). For a function in $H^s(\mathbb{S}, X)$ which is not dependent on p we do not use caligraphic letters.

Proposition 3.10. *The operator $\mathcal{F}$ is well defined, continuous and (uniformly) weakly continuous on bounded sets $\mathcal{B} \subseteq \mathcal{D}^s$.*

Proof. Lemma 3.2 states that the embedding $H^s(\mathbb{S}, X) \hookrightarrow C(\mathbb{S}, X)$ is well defined and continuous. Hence, for the continuity of $\mathcal{F}$, it is sufficient to show that $\mathcal{F}$ maps $C(\mathbb{S}, X)$ continuously into $L^2(\mathbb{S}, Y)$. Let $\mathcal{U} \in C(\mathbb{S}, X)$. Since a is continuous (assumption T1) we conclude that $\mathcal{F}(\mathcal{U}) = \{p \mapsto a(p, u(p))\}$ is continuous, hence it is weakly measurable, $\{p \mapsto \|\mathcal{F}(\mathcal{U})(p)\|_Y^2\}$ is bounded, and thus $\mathcal{F}(\mathcal{U}) \in L^2(\mathbb{S}, Y)$, proving the first statement.

Now let $\varepsilon > 0$. Since $\{a(p,\cdot) : p \in \mathbb{S}\}$ is uniformly equi-continuous, there is $\eta > 0$ such that for all $u, u' \in D$ with $\|u - u'\|_X < \eta$, and all $p \in \mathbb{S}$, we have $\|a(p,u) - a(p,u')\|_Y^2 < \varepsilon^2$. Hence

$$\|\mathcal{F}(\mathcal{U}) - \mathcal{F}(\mathcal{U}')\|^2 = \frac{1}{2\pi}\int_{\mathbb{S}} \|a(p,u(p)) - a(p,u'(p))\|_Y^2 \, dp < \varepsilon^2 .$$

for all $\mathcal{U},\mathcal{U}' \in C(\mathbb{S}, X)$ with $\sup\{\|u(p) - u'(p)\|_X : p \in \mathbb{S}\} < \eta$. This shows that $\mathcal{F}$ is continuous.

Now assume $\mathcal{B} \subseteq \mathcal{D}^s$ being bounded.

1. Since $H^s(\mathbb{S}, X) \hookrightarrow C(\mathbb{S}, X)$ is bounded, $\mathcal{U} \in \mathcal{B}$ implies that $\{u(p) : p \in \mathbb{S}\}$ is uniformly bounded. Let $\rho \geq \sup\{\|u(p)\|_X : p \in \mathbb{S}\}$, $f \in L^2(\mathbb{S})$, $g \in Y$ and $\varepsilon > 0$. Since $\{a(p,\cdot) : p \in \mathbb{S}\} \subseteq C(D, Y)$ is uniformly weakly equi-continuous on $B_\rho(0)$, there are $x_1, \ldots, x_N \in X$ and $\eta > 0$ such that $|\langle a(p,u) - a(p,u'), g\rangle_Y| < \varepsilon/\|f\|_{L^2(\mathbb{S})}$ for all $p \in \mathbb{S}$ and u, $u' \in B_\rho(0)$ with $\max\{|\langle u - u', x_n\rangle_X| : n = 1, \ldots, N\} < \eta$.

2. Since $\langle \mathcal{U}, x_n\rangle_X \in H^s(\mathbb{S})$ and $\|\langle \mathcal{U}, x_n\rangle\|_{H^s(\mathbb{S})} \leq \|\mathcal{U}\|_s \|x_n\|_X$ (cf. Lemma 3.2) there is a closed bounded ball $\bar{B} \subseteq H^s(\mathbb{S})$ with $\langle \mathcal{U}, x_n\rangle_X \in \bar{B}$ for all $n \in \{1, \ldots, N\}$ and $\mathcal{U} \in \mathcal{B}$. Since $H^s(\mathbb{S})$ is compactly embedded in $C(\mathbb{S})$ and $\bar{B}$ is compact in the weak topology of $H^s(\mathbb{S})$ there are $f_{n,1}, \ldots, f_{n,M(n)} \in H^s(\mathbb{S})$ and $\zeta_n > 0$ such that $\|f\|_{C(\mathbb{S})} < \eta$ for all $f \in \bar{B}$ with $\max\{|\langle f, f_{n,m}\rangle_{H^s(\mathbb{S})}| : m = 1, \ldots, M(n)\} < \zeta_n$.

3. Define the weak zero neighborhood $U := \bigcap_{n=1}^N U_n$ with

   $$U_n := \{\mathcal{U} \in H^s(\mathbb{S}, X) : |\langle \mathcal{U}, x_n \otimes f_{n,m}\rangle_s| \leq \zeta_n \,,\ m = 1, \ldots, M(n)\} .$$

   Let $\mathcal{U},\mathcal{U}' \in \mathcal{B}$ and $\mathcal{U} - \mathcal{U}' \in U$. Hence for all $n = 1, \ldots, N$ and $m = 1, \ldots, M(n)$

   $$\begin{aligned} |\langle\langle \mathcal{U} - \mathcal{U}', x_n\rangle_X, f_{n,m}\rangle_{H^s(\mathbb{S})}| &= \left|\sum_{k\in\mathbb{Z}} (1 - k^s)^2 \langle \hat{u}(k) - \hat{u}'(k), x_n\rangle_X \hat{f}_{n,m}(k)\right| \\ &= |\langle \mathcal{U} - \mathcal{U}', x_n \otimes f_{n,m}\rangle_s| < \zeta_n . \end{aligned}$$

Hence $\mathcal{U},\mathcal{U}' \in \mathcal{B}$, $\mathcal{U} - \mathcal{U}' \in U$ implies (cf. step 2) $\|\langle \mathcal{U} - \mathcal{U}', x_n\rangle_X\|_{C(\mathbb{S})} < \eta$ and from the Cauchy Schwartz inequality on $L^2(\mathbb{S})$ and step 1 it follows that

$$\begin{aligned} 4\pi^2 |\langle \mathcal{F}(\mathcal{U}) - \mathcal{F}(\mathcal{U}'), g \otimes f\rangle_{L^2(\mathbb{S},Y)}|^2 &= \left|\int_{\mathbb{S}} f(p) \cdot \langle a(p,u(p)) - a(p,u'(p)), g\rangle_Y \, dp\right|^2 \\ &\leq 2\pi \|f\|_{L^2(\mathbb{S})}^2 \int_{\mathbb{S}} |\langle a(p,u(p)) - a(p,u'(p)), g\rangle_Y|^2 \, dp < 4\pi^2\varepsilon^2. \end{aligned}$$

The last inequality shows that $\mathcal{F}|_{\mathcal{B}}$ is uniformly weakly continuous. □

In the following we consider the case where we have only perturbed data $\mathcal{G}^\delta$ with $\|\mathcal{G} - \mathcal{G}^\delta\|_{L^2(\mathbb{S},Y)} < \delta$. As discussed in the introduction we apply Tikhonov regularization to the system of equations (3) and (2). Hence we minimize the functional

$$K_{\alpha,\mathcal{G}^\delta}(\mathcal{U}) := \left(\|\mathcal{F}(\mathcal{U}) - \mathcal{G}\|_{L^2(\mathbb{S},Y)}^2 + \lambda |\mathcal{U}|_s^2\right) + \alpha \|\mathcal{U} - u^*\|_s^2 \,, \tag{31}$$

over D, where $u^* \in D$ is some initial guess and α a regularization parameter. Since $\|\cdot\|_s^2 = |\cdot|_s^2 + \|\cdot\|_{L^2(\mathbb{S},X)}^2$ and $|u^*|_s^2 = 0$ we have

$$K_{\alpha,\mathcal{G}^\delta}(\mathcal{U}) := \|\mathcal{F}(\mathcal{U}) - \mathcal{G}\|_{L^2(\mathbb{S},Y)}^2 + (\lambda+\alpha)|\mathcal{U}|_s^2 + \alpha\|\mathcal{U} - u^*\|_{L^2(\mathbb{S},X)}^2\,, \tag{32}$$

We call the corresponding regularization technique *embedded Tikhonov regularization* for solving (1). The following theorem states that embedded Tikhonov regularization is in fact a regularization technique for the solution of (1):

Theorem 3.11. *Let $s > 1/2$.*

1. (Well-posedness of embedded Tikhonov regularization) *For $\alpha > 0$ and $\mathcal{G}^\delta \in L^2(\mathbb{S}, Y)$, the functional $K_{\alpha,\mathcal{G}^\delta}$ attains a minimizer on $\mathcal{D}^s$.*
2. (Stability of embedded Tikhonov regularization) *Assume that $\mathcal{G}_n$ is a sequence in $L^2(\mathbb{S}, Y)$ with $\mathcal{G}_n \to \mathcal{G}^\delta$ and let $\mathcal{U}_n$ be a minimizer of $K_{\alpha,\mathcal{G}_n}$. Then there exists a convergent subsequence of $\mathcal{U}_n$ and the limit of any convergent subsequence of $\mathcal{U}_n$ is a minimizer of $K_{\alpha,\mathcal{G}^\delta}$.*
3. (Convergence of embedded Tikhonov regularization) *Let $\hat\alpha(\delta)$ satisfy*
$$\lim_{\delta\to 0} \hat\alpha(\delta) = \lim_{\delta\to 0} \delta^2/\hat\alpha(\delta) = 0\,.$$
Assume that $(\delta_n)_{n\in\mathbb{N}}$ is a sequence converging to zero. Moreover, we assume that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{S}, Y)$ satisfying $\|\mathcal{G}_n - \mathcal{G}\|_{L^2(\mathbb{S},Y)} < \delta_n$. Let $\mathcal{U}_n$ be a minimizer of $K_{\alpha_n,\mathcal{G}_n}$ with $\alpha_n := \hat\alpha(\delta_n)$. Then, $(\mathcal{U}_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence and each weakly convergent subsequence is strongly convergent to $\mathcal{U}^\dagger = u^\dagger$ which is an u^*-minimal norm solution of (1) (and therefore constant with respect to p). If $u^\dagger$ is unique, then $\mathcal{U}_n \to u^\dagger$.

Proof. From Proposition 3.10 it follows that the operator

$$(\mathcal{F}, \sqrt{\lambda}|\cdot|_s)^T : \mathcal{D}^s \subseteq H^s(\mathbb{S}, X) \to L^2(\mathbb{S}, Y) \times \mathbb{R}$$

is continuous and weakly continuous on bounded sets. Hence, it follows that minimizing $K_{\alpha,\mathcal{G}^\delta}$ is well posed and stable. Moreover, under the requirements of 3, the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ has a weak convergent subsequence, and each weak convergent subsequence of $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is strong convergent with limit $\mathcal{U}^\dagger$, where $\mathcal{U}^\dagger$ is a u^*-minimum norm solution of $\mathcal{F}(\mathcal{U}) = g$, $\lambda|\mathcal{U}|_s^2 = 0$. Therefore $\mathcal{U}^\dagger = u^\dagger$, where $u^\dagger$ is a u^*-minimal norm solution of (1). If $u^\dagger$ is unique then $\mathcal{U}^\dagger$ is unique, and consequently $\mathcal{U}_n \to u^\dagger$. $\square$

Convergence Rates

Let us assume for the rest of this Section that condition T4 holds true.

Lemma 3.12. *The operator $\mathcal{F}$ is Fréchet differentiable and the Fréchet derivative $\mathcal{F}'(u_0) \in B(H^s(\mathbb{S}, X), L^2(\mathbb{S}, Y))$ at $\{p \mapsto u_0(p)\} =: \mathcal{U}_0 \in \mathcal{D}^s$ in direction $\{p \mapsto v(p)\} =: \mathcal{V} \in H^s(\mathbb{S}, X)$ is given by*

$$\mathcal{F}'(\mathcal{U}_0)(\mathcal{V})(p) = D_2 a(p, u_0(p))(v(p))\,. \tag{33}$$

Proof. Let $\mathcal{U}_0 \in \mathcal{D}^s$ and $\mathcal{V} \in H^s(\mathbb{S}, X)$. Define $\Theta(\mathcal{U}_0) : \mathcal{V} \mapsto D_2 a(p, u_0(\cdot))(v(\cdot))$. Since $D_2 a$ is continuous and $i_s : H^s(\mathbb{S}, X) \hookrightarrow C(\mathbb{S}, X)$ is bounded linear, $\Theta(\mathcal{U}_0)(\mathcal{V})$ is continuous and hence measurable. Since $\mathbb{S}$ is compact and $D_2 a$ is continuous, $p \mapsto D_2 a(p, u_0(p))$ is bounded, say $\|D_2 a(p, u_0(p))\| \leq M$, for all $p \in \mathbb{S}$. Hence

$$2\pi \|\Theta(\mathcal{U}_0)(\mathcal{V})\|_{L^2(\mathbb{S},Y)} \leq \int_{\mathbb{S}} \|D_2 a(p, u_0(p))(v(p))\|_Y^2 \, dp \leq 2\pi M \|i_s\| \|\mathcal{V}\|_s \,.$$

This shows that $\Theta(\mathcal{U}_0) \in B(H^s(\mathbb{S}, X), L^2(\mathbb{S}, Y))$. Now let $\varepsilon > 0$. Since $\{A(p, \cdot) : p \in \mathbb{S}\}$ is Fréchet equi-differentiable, there is $\alpha > 0$ such that

$$\|A(p, u_0(p)) - A(p, u_0(p) + v(p)) - D_2 a(p, u_0(p))(v(p))\|_Y^2 < \varepsilon^2 \|v(p)\|_X^2 \,,$$

for all $p \in \mathbb{S}$ and $\|v(p)\|_X < \alpha$. Since $H^s(\mathbb{S}, X) \hookrightarrow C(\mathbb{S}, X)$ there exists $\beta > 0$ such that $\|\mathcal{V}\|_s < \beta$ implies $\|v(p)\|_X < \alpha$ for all $p \in \mathbb{S}$. Thus, for all $\|\mathcal{V}\|_s < \beta$

$$\begin{aligned}
&2\pi \|\mathcal{F}(\mathcal{U}_0) - \mathcal{F}(\mathcal{U}_0 + \mathcal{V}) - \Theta(\mathcal{U}_0)\mathcal{V}\|_{L^2(\mathbb{S},Y)}^2 = \\
&\int_{\mathbb{S}} \|a(p, u_0(p)) - a(p, u_0(p) + v(p)) - D_2 a(p, u_0(p))(v(p))\|_Y^2 \, dp \leq \\
&\int_{\mathbb{S}} \varepsilon^2 \|v(p)\|_X^2 \, dp < \varepsilon^2 \|\mathcal{V}\|_s^2 \,.
\end{aligned}$$

This shows, that $\mathcal{F}$ is Fréchet differentiable at $\mathcal{U}_0$, and that $\mathcal{F}'(\mathcal{U}_0)(\mathcal{V}) = \Theta(\mathcal{U}_0)(\mathcal{V})$. □

Analogous to Theorem 3.5 we can prove the following result.

Theorem 3.13. *Assume that $u^\dagger$ is an u^*-minimum norm solution of (1) in the interior of $\mathcal{D}^s$ and let $\mu \in [1/2, 1]$. Assume that there exists $\gamma > 0$ and $\rho > 2\|u^\dagger - u^*\|$ such that*

$$\|\mathcal{F}'(u^\dagger) - \mathcal{F}'(\mathcal{U})\|_{L^2(\mathbb{S},Y)}^2 + \lambda |u^\dagger - \mathcal{U}|_s^2 \leq \gamma^2 \|u^\dagger - \mathcal{U}\|_s^2 \,, \quad \mathcal{U} \in B_\rho^s(u^\dagger) \subseteq H^s(\mathbb{S}, X) \,.$$

If there exists $\mathcal{W} \in H^s(\mathbb{S}, X)$ satisfying $\gamma \|\mathcal{W}\|_s < 1$ and

$$u^\dagger - u^* = \left(\mathcal{F}'(u^\dagger)^* \mathcal{F}'(u^\dagger) + \lambda \Lambda^{2s}\right)^\mu \mathcal{W} \,, \tag{34}$$

where Λ is the square root of (minus) Laplace-Beltrami operator. Then, for a parameter choice $\alpha = \alpha(\delta) \sim \delta^{2/(2\mu+1)}$,

$$\|\mathcal{U}_{\alpha(\delta)}^\delta - u^\dagger\|_s = O(\delta^{2\mu/(2\mu+1)}) \,.$$

Remark 3.14. For $s = 1$ and $\mu = 1$ condition (34) reads as

$$u^\dagger - u^* = \left(\mathcal{F}'(u^\dagger)^* \mathcal{F}'(u^\dagger) - \lambda \frac{\partial^2}{\partial p^2}\right) \mathcal{W} \,.$$

From this equation we conclude that $u^\dagger - u^*$ satisfies the *averaged source condition*

$$u^\dagger - u^* = \frac{1}{2\pi} \int_0^{2\pi} D_2 a(p, u^\dagger)^* D_2 a(p, u^\dagger) w(p) \, dp \,,$$

where $\mathcal{W} = \{p \mapsto w(p)\}$. For $\mathcal{W}$ constant in p this condition is the same as in Theorem 3.5, and is therefore weaker. However, it should be taken into account that the closeness condition $\gamma \|\mathcal{W}\|_s$ is more restrictive for the embedded Tikhonov regularization.

4 Embedded Landweber-Kaczmarz Iteration

In this Section we review and addend the convergence analysis of the Landweber-Kaczmarz method, and apply it to the embedding approach discussed in the introduction.

4.1 A Survey and Addendum on the Convergence Analysis of the Landweber-Kaczmarz Method

For given iteration index n, let us denote by $i = i(n) \in \{0, \ldots, N-1\}$ the integer value satisfying

$$n = \left\lfloor \frac{n}{N} \right\rfloor N + i\,.$$

With this notation we define the Landweber-Kaczmarz iteration for approximating the solution $u \in X$ of a system

$$a_i(u) = g(p_i) \quad i = 0, 1, \ldots, N-1\,, \tag{35}$$

using available noisy data $g^\delta(p_i)$ satisfying

$$\|g^\delta(p_i) - g(p_i)\| \leq \delta_i\,,$$

by

$$\begin{aligned} u^{\delta,n+1} &= u^{\delta,n} - \omega_n a_i'(u^{\delta,n})^* (a_i(u^{\delta,n}) - g^\delta(p_i))\,, \\ &\text{with} \\ \omega_n &= \begin{cases} 1 & \text{if} \quad \|a_i(u^{\delta,n}) - g^\delta(p_i)\| > \tau\delta_i\,, \\ 0 & \text{else}\,. \end{cases} \end{aligned} \tag{36}$$

The positive number τ is chosen in dependence of properties of a_i (cf. condition (40) below) such that

$$\tau > 2\frac{1+\eta}{1-2\eta} > 2\,. \tag{37}$$

Note, that for noise free data we have $\omega_n \equiv 1$, and hence the iteration (36) reduces to the familiar Landweber-Kaczmarz iteration (10). From (37) it follows that

$$2(1+\eta)\delta_i - (1-2\eta)\|g^\delta(p_i) - a_i(u_n^\delta)\| \geq 0 \Longrightarrow w_n = 0\,. \tag{38}$$

The following assumptions are standard in the convergence analysis of iterative regularization methods (cf., e.g., [11]):

1. We assume that for fixed $i \in \{0, 1, \ldots, N-1\}$

   $$\|a_i(u)\| \leq 1\,, \qquad \forall u \in \mathcal{B}_{2\rho}(u^*) \subset D\,, \tag{39}$$

   where $\mathcal{B}_{2\rho}(u^*)$ denotes a closed ball of radius 2ρ around the starting value u^*.

2. Moreover, we assume that the (local) tangential cone condition hold. This is a central assumption in the analysis of iterative methods for the solution on non-linear ill–posed problems (cf., e.g., [6, 11]):

$$\|a_i(u) - a_i(\tilde{u}) - a_i'(u)(u - \tilde{u})\| \leq \eta \|a_i(u) - a_i(\tilde{u})\|\,, \qquad \eta < \frac{1}{2}\,, \quad u, \tilde{u} \in \mathcal{B}_{2\rho}(u^*) \subset D\,. \tag{40}$$

3. In the case of noisy data, iterative regularization methods require early termination, which is enforced by an appropriate stopping criterion: The termnination index is denoted by $n_\star := n_\star(\delta)$ and is the smallest integer multiple of N such that

$$u^{\delta,n_\star} = u^{\delta,n_\star+1} = \cdots = u^{\delta,n_\star+N-1}\,. \tag{41}$$

Here we adopt the notation $\delta := (\delta_0, \delta_1, \ldots, \delta_{N-1})$.

Before stating the main result of this Section we prove an auxiliary lemma which guarantees the existence of a finite $n_\star$.

Lemma 4.1. *For any solution u of (9) the following estimate holds true*

$$\|u^{\delta,n+1} - u\| - \|u^{\delta,n} - u\| \leq \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\| \left(2(1+\eta)\delta_i - (1-2\eta)\|g^\delta(p_i) - a_i(u^{\delta,n})\| \right)\,, \tag{42}$$

for all n. Moreover, the stoping rule (41) and (42) imply $\omega_{n_\star+i} = 0$ for all $i \in \{0, \ldots, N-1\}$, i.e.

$$\|a_i(u^{\delta,(n_\star)}) - g^\delta(p_i)\| \leq \tau\delta_i\,,\ i = 0, \ldots, N-1\,. \tag{43}$$

Furthermore,

$$\frac{n_\star}{N}(\tau \min(\delta))^2 \leq \sum_{n=0}^{n_\star-1} \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\|^2 \leq \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)} \|u - u^*\|^2\,. \tag{44}$$

Proof. Let us prove (42). For $n \leq n_\star$ we shall consider two cases: If $\omega_n = 1$, then this inequality is analog to [6, Inequality (11.11)]; for $\omega_n = 0$ the inequality is trivial. For $n > n_\star$ (42) follows from $\omega_n = 0$.

To prove the second assertion note that for $n = n_\star$ we have

$$0 \leq \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\| \left(2(1+\eta)\delta_i - (1-2\eta)\|g^\delta(p_i) - a_i(u^{\delta,n})\| \right)\,.$$

If $\omega_n \neq 0$, we would have $2(1+\eta)\delta_i - (1-2\eta)\|g^\delta(p_i) - a_i(u^{\delta,n})\| \geq 0$, contradicting (38).

To prove (44) we add up (42) from 0 through $n_\star$ and obtain

$$\|u - u^*\| - \|u - u^{\delta,n^*}\| \geq \frac{(1-2\eta)\tau - 2(1+\eta)}{\tau} \sum_{n=0}^{n_\star} \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\|^2\,.$$

This implies the second inequality in (44). The first inequality follows from

$$\begin{aligned}\sum_{n=0}^{n_\star} \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\|^2 &= \sum_{l=0}^{\frac{n_\star}{N}} \sum_{i=0}^{N-1} \omega_n \|g^\delta(p_i) - a_i(u^{\delta,n})\|^2 \\ &\geq \sum_{l=0}^{\frac{n_\star}{N}} \|g^\delta(p_i) - a_i(u^{\delta,n})\|^2 \geq \frac{n_\star}{N}(\tau \min(\delta))^2 .\end{aligned}$$

□

For the Landweber-Kaczmarz algorithm we have the following results:

Theorem 4.2. *Assume that the operators a_i are Fréchet-differentiable in the open ball $\mathcal{B}_{2\rho}(u^*) \subseteq \bigcap_{i=0}^{N-1} D(a_i)$ and satisfy conditions (39) and (40). Moreover, we assume that the system $a_i(u) = g(p_i)$, $i = 0, 1, \ldots, N-1$ has a solution in $\mathcal{B}_\rho(u^*)$. Then*

1. *If $\delta = 0$ (exact data), the Landweber-Kaczmarz iteration converges to a solution of (35). Additionally, if for all $i = 0, \ldots, N-1$ we have*

   (45) $$\mathcal{N}(a_i'(u^\dagger)) \subseteq \mathcal{N}(a_i'(u)) \text{ for all } u \in \mathcal{B}_\rho(u^\dagger) ,$$

   then $u^n \to u^\dagger$, the solution of (35) with minimal distance to u^*.
2. *For noisy data, we assume that $\delta_i > 0$. The Landweber-Kaczmarz iterates $u^{\delta,(n_\star)}$ converge to a solution of (35) as $\delta \to 0$. If in addition (45) holds, then $u^{\delta,(n_\star)}$ converges to $u^\dagger$ as $\delta \to 0$.*

Proof. The proof of the first item is analogous to the proof in [12, Proposition 4.3] (see also [11]). We emphasize that, for exact data, the variants (10) and (36) are identical, which allows to apply the results of the above papers.

The proof of the second item is analogous to the proof of the corresponding result for the Landweber iteration as in [10, Theorem 2.9]. For the first case within this proof, (43) is required. For the second case we need the monotony result from Lemma 4.1. □

In the case of noisy data (i.e. the second item of Theorem 4.2), it has been shown in [12] that the Landweber-Kaczmarz iteration (36) with $\omega_n \equiv 1$ is convergent if the iteration is terminated after the $n_\star$-th step, where $n_\star$ is the smallest iteration index such that one component of the residual vector $\left(a_i(u^{\delta,(n_\star)}) - g^\delta(p_i)\right)_{i=1,\ldots,N-1}$ satisfies

(46) $$\|a_i(u^{\delta,(n_\star)}) - g^\delta(p_i)\| \leq \tau\delta_i .$$

4.2 Application to the Embedded Landweber-Kaczmarz Algorithm

Here, we apply the general results of the previous subsection to the embedded Landweber-Kaczmarz method (11) and (12) (in the case of exact data) and to a variant taking into account appropriate termination in the presence of noisy data. For the sake of simplicity we present the case $s = 1$.

We use the following notation: For $u_i \in X$, $i = 0, 1, \ldots, N-1$ let

$$U := (u_i)_{i=0,\ldots,N-1} .$$

We define the diagonal operator

$$F := \begin{pmatrix} F_0 \\ \vdots \\ F_{N-1} \end{pmatrix} := \begin{pmatrix} a_0 & & \\ & \ddots & \\ & & a_{N-1} \end{pmatrix} : X^N \to Y^N$$

and introduce the functional

$$B(U) := \sum_{i=0}^{N-1} \|u_i - u_{i-1}\|_X^2$$

over X^N such that $\lambda B(U)$ replaces the functional (13) in the case $s = 1$.

We consider the problem of approximating a solution of the following system consisting of $N+1$ equations

$$\begin{aligned} F_i(U) &= g(p_i)\,, \qquad i = 0, \ldots, N-1\,, && (47) \\ \sqrt{\lambda B(U)} &= 0\,, && (48) \end{aligned}$$

from noisy data $G^\delta = (g^\delta(p_i))_{i=0,\ldots,N-1}$ satisfying

$$\|g^\delta(p_i) - g(p_i)\| \leq \delta_i\,. \qquad (49)$$

Note that (47), (48) and (35) are equivalent for exact data in the sense that each solution u of (35) provides a solution $U = (u, \ldots, u)$ of (47), (48) and vice-versa.

Since F_i acts on the i-th component of $U \in X^N$ only, one cycle of the Landweber-Kaczmarz iteration applied to the system (47), (48) with initial value $U^* = (u^*, \ldots, u^*)$ reads as follows:

$$U^{\delta,(n+1/2)} = U^{\delta,(n)} - \Omega_n F'(U^{\delta,(n)})^* \left(F(U^{\delta,(n)}) - g^\delta \right)\,, \qquad (50)$$

$$U^{\delta,(n+1)} = U^{\delta,(n+1/2)} - \tilde{\omega}\lambda B^* B \left(U^{\delta,(n+1/2)} \right)\,, \qquad (51)$$

were $\Omega_n = \mathrm{diag}(\omega_n^0, \ldots, \omega_n^{N-1})$ is the diagonal matrix with

$$\omega_n^i = \begin{cases} 1 & \text{if} \quad \|a_i(u^{\delta,n}) - g^\delta(p_i)\| > \tau\delta_i\,, \\ 0 & \text{else}\,, \end{cases} \qquad \text{for } i = 1, \ldots, N-1\,.$$

The additional parameter $\tilde{\omega}_n$ ensures a finite stopping index $n_\star$ for noisy data and is defined by

$$\tilde{\omega} = \begin{cases} 1 & \text{if} \quad \|B(U^{\delta,(n+1/2)})\| > \tau\epsilon\,, \\ 0 & \text{else}\,, \end{cases}$$

where $\epsilon = \epsilon(\delta) \to 0$ as $\delta := (\delta_0, \ldots, \delta_{N-1}) \to 0$. According to (41) the iteration is terminated if

$$U^{\delta,(n+1)} = U^{\delta,(n+1/2)} = U^{\delta,(n)} \tag{52}$$

for the first time. We call (50), (51) the *embedded Landweber-Kaczmarz* iteration for the solution of (1)

Note, that in order to determine $U^{\delta,(n+1/2)}$ each component can be updated independently. Indeed,

$$u_i^{\delta,(n+1/2)} = u_i^{\delta,(n)} - \omega_n^i a_i'(u_i^{\delta,(n)})^* \left(a_i(u_i^{\delta,(n)}) - g^\delta(p_i)\right), \; i = 0, \ldots, N-1 \,.$$

In the second half-step the $U^{\delta,(n+1)}$ is determined from $U^{\delta,(n+1/2)}$ by a matrix vector multiplication with the sparse matrix

$$\begin{pmatrix} 2-\tilde{\omega}_n\lambda & -1 & 0 & & -1 \\ -1 & 2-\tilde{\omega}_n\lambda & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ & \ddots & \ddots & 2-\tilde{\omega}_n\lambda & -1 \\ -1 & & 0 & -1 & 2-\tilde{\omega}_n\lambda \end{pmatrix} \otimes \mathrm{Id}_X \,.$$

What concerns the computational effort, we consider the case where X and Y are approximated by finite dimensional spaces X_h and Y_h of dimensions N_X and 1. Note, that in contrast to Kaczmarz and ART (*algebraic reconstruction technique*, cf. [8, 17, 7]), for the embedded Landweber-Kaczmarz algorithm $4N \cdot N_X$ additional operations are required for each cycle.

In order to prove convergence and stability of the Landweber-Kaczmarz iteration we follow the lines of the proof of Theorem 4.2.

Theorem 4.3. *Assume that the operators a_i are Fréchet-differentiable in $\mathcal{B}_{2\rho}(u^*) \subseteq \bigcap_{i=0}^{N-1} D(a_i)$ and satisfy the conditions (40) and (39). Moreover, assume that system (1) has a solution in $\mathcal{B}_\rho(u^*)$, and*

$$\|\sqrt{\lambda} B\| < 1 \,. \tag{53}$$

Then we have:

1. *If $\delta = 0$ (exact data), the embedded Landweber-Kaczmarz iteration (50), (51) converges to a solution of (47), (48). Additionaly, if for all $i = 0, \ldots, N-1$ we have*

   $$\mathcal{N}(a_i'(u^\dagger)) \subseteq \mathcal{N}(a_i'(u)) \text{ for all } u \in \mathcal{B}_\rho(u^\dagger)\,, \tag{54}$$

   then $U^{(n)} \to U^\dagger$, the solution of (47), (48) with minimal distance to U^*. Furthermore, $U^\dagger = (u^\dagger, \ldots, u^\dagger)$, where $u^\dagger$ is a solution of (35) with minimal distance to u^*.

2. For noisy data, let $\epsilon = \epsilon(\delta) \to 0$ as $\delta \to 0$, and $n_\star = n_\star(\delta)$ be defined by (52). Then the embedded Landweber-Kaczmarz iterates $U^{\delta,(n_\star)}$ converge to a solution of (47), (48) as $\delta \to 0$. If in addition (54) holds, then each component of $U^{\delta,(n_\star)}$ converges to the minimum distance solution $u^\dagger$ as $\delta \to 0$.

Proof. For the first assertion, the proof is analogous to the proof of Theorem 4.2. One has to take into account that (40) and (39) are satisfied with F_i instead of a_i, since $F_i(U) = a_i(u_i)$. Moreover, since B is linear we have

$$0 = \|B(U) - B(\tilde{U}) - B'(U)(U - \tilde{U})\| \leq \eta \|B(U) - B(\tilde{U})\|$$

and, therefore, it satisfies (40). The second assertion follows immediately from the second item in Theorem 4.2. □

Notice that the termination criterium requires that $\sqrt{\lambda}\|BU^{\delta,(n_\star)}\| \leq \tau\epsilon$. This ensures that the embedded Landweber-Kaczmarz iteration terminates after a finite number of iterations.

5 Conclusion

We have suggested two novel regularization techniques for solving a system of ill-posed operator equations of the form (1). Such equations appear in a variety of applications such as tomography and impedance tomography, inverse doping profile, to name but a few. We developed a convergence analysis for both the variational regularization method as well as the iterative regularization technique. The later turned out to be of Landweber-Kaczmarz type. However, this modification reveals certain advantages against previously analyzed methods in terms of appropriate stopping.

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References

[1] A.B. Bakushinskii. The problem of the convergence of the iteratively regularized Gauß-Newton method. *Comput. Maths. Math. Phys.*, 32:1353–1359, 1992.

[2] A. B. Bakushinsky and M. Y. Kokurin. *Iterative Methods for Approximate Solution of Inverse Problems*, volume 577 of *Mathematics and Its Applications.* Springer, Dordrecht, 2004.

[3] L. Borcea. Addendum to: “Electrical impedance tomography” [Inverse Problems **18** (2002), no. 6, R99–R136]. *Inverse Problems*, 19(4):997–998, 2003.

[4] L. Borcera. Electrical impedance tomography. *Inverse Problems*, 18(6):R99–R136, 2002.

[5] P. Deuflhard, H.W. Engl, and O. Scherzer. A convergence analysis of iterative methods for the solution of nonlinear ill–posed problems under affinely invariant conditions. *Inverse Probl.*, 14:1081–1106, 1998.

[6] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems.* Kluwer Academic Publishers, Dordrecht, 1996.

[7] C. L. Epstein. *Introduction to the Mathematics of Medical Imaging.* Pearson Prentice Hall, Upper Saddle River, NJ, 2003.

[8] C. Hamaker and D. C. Solmon. The angles between the null spaces of Xrays. *J. Math. Anal. Appl.*, 62(1):1–23, 1978.

[9] M. Hanke and M. Brühl. Recent progress in electrical impedance tomography. *Inverse Problems*, 19(6):S65–S90, 2003. Special section on imaging.

[10] M. Hanke, A. Neubauer, and O. Scherzer. A convergence analysis of Landweber iteration for nonlinear ill-posed problems. *Numer. Math.*, 72:21–37, 1995.

[11] B. Kaltenbacher, A. Neubauer, and O. Scherzer. *Iterative Regularization Methods for Nonlinear Ill–Posed Problems.* 2005. in preparation.

[12] R. Kowar and O. Scherzer. Convergence analysis of a Landweber-Kaczmarz method for solving nonlinear ill-posed problems. *Ill posed and inverse problems (book series)*, 23:69–90, 2002.

[13] L. Landweber. An iteration formula for Fredholm integral equations of the first kind. *Amer. J. Math.*, 73:615–624, 1951.

[14] A. Leitão M. Burger, H. W. Engl and P. A. Markowich. On inverse problems for semiconductor equations. *Milan J. Math.*, 72:273–313, 2004.

[15] P. A. Markowich M. Burger, H. W. Engl and P. Pietra. Identification of doping profiles in semiconductor devices. *Inverse Problems*, 17(6):1765–1795, 2001.

[16] D. Isaacson M. Cheney and J. C. Newell. Electrical impedance tomography. *SIAM Rev.*, 41(1):85–101 (electronic), 1999.

[17] F. Natterer. *The Mathematics of Computerized Tomography.* SIAM, Philadelphia, 2001.

[18] T.I. Seidman and C.R. Vogel. Well posedness and convergence of some regularisation methods for non–linear ill posed problems. *Inverse Probl.*, 5:227–238, 1989.

[19] A. N. Tikhonov and V. Y. Arsenin. *Solutions of Ill-Posed Problems*. John Wiley & Sons, Washington, D.C., 1977. Translation editor: Fritz John.

[20] A.N. Tikhonov. Regularization of incorrectly posed problems. *Soviet Math. Dokl.*, 4:1624–1627, 1963.

[21] J. Wloka. *Partial differential equations*. Cambridge University Press, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.

[22] K. Yosida. *Functional analysis*. Springer-Verlag, Berlin, Heidelberg, New York, 1995. 5th edition.