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Abstract

The taut string method is classically used in statistical applications to obtain a sparse estimation for a density given by point measurements. Mostly, a discrete formulation is employed that interpretes the data and the output as piecewise constant splines.

This paper deals with the continuous formulation of this algorithm. We show that it is able to deal with continuous data as well as with discrete data interpreted as Dirac measures. In fact, any onedimensional finite signed Radon measures is suited as input for the method.

Moreover, we study the usage of tubes of non-constant diameter. Examples indicate that such tubes can be useful in various applications. An existence and uniqueness theorem is given for the continuous formulation of the taut string algorithm with arbitrary tubes of nonnegative diameter.

1 Introduction

The taut string algorithm (cf. [2, 7, 9]) is a method for denoising onedimensional data f on an interval (a, b). The first step is to pass from the data f to its antiderivative

$$F(x) := \int_a^x f(y)\,dy\ .$$

In order to denoise f consider a tube $\mathcal{G}$ of radius $\alpha > 0$ around F. Imagine a string that is contained in the tube $\mathcal{G}$ and fixed in the points $(a, 0)$ and $\big(b, F(b)\big)$. Now pull on both sides until the string, still contained in $\mathcal{G}$, is taut. The derivative of this taut string then gives a smoothed approximation to the data f.

More formally, the denoised function u_α is defined as derivative of the minimizer U_α of the constrained minimal surface functional

$$\begin{aligned} J_F(V) := \int_a^b \sqrt{1 + V'(x)^2}\,dx + |V(a) - F(a)| + |V(b) - F(b)| \to \min\,, \\ \max_{a \le x \le b} |U(x) - F(x)| \le \alpha\ . \end{aligned} \tag{1}$$

This paper intends to present a twofold generalization of the taut string algorithm. First, it has to be noted that the regularization works on the integrated instead of the original data. As a consequence, the algorithm still makes sense, if we require less regularity of f. We only need that the data can be integrated. This is the case if we assume the data to be a finite Radon measure, e.g. a sum of weighted Dirac measures. In particular, this provides a theoretical justification for the usage of the taut string algorithm for density estimation in statistical applications. Note, however, that in this setting the derivative in (1) has to be regarded in a weak sense, and, consequently, J_F as convex functional depending on a Radon measure (cf. [3]).

The second generalization concerns the shape of the tube used for regularization. In (1) we simply use a tube that extends in vertical direction by a parameter α starting from the integrated data. Although there is a good motivation for using exactly this kind of tube, namely equivalence with total variation regularization (see [7, 9, 5]), one can easily find applications where different forms yield better results. In fact, already in [2] tubes of varying size have been used under the heading of local squeezing.

The structure of this article is as follows: In Section 2 we recall the basic properties of Radon measures and functions of bounded variation that are needed for the main results. We present the mathematical definition of the generalized taut string method in Section 3 and state the main existence and uniqueness theorem. Additionally, we give a characterization of the solution and some simple consequences. In Section 4 we show in different examples how this generalization can be applied. The proofs of the main theorems are given in Section 5.

2 Preliminaries

We require the following results stated in [1]. Many results concerning functions of bounded variation can also be found in [6].

A positive Radon measure on an interval $(a, b) \subset \mathbb{R}$ is Borel regular measure that is finite on each closed interval $[c, d] \subset (a, b)$. If ν is a positive Radon measure, and u a ν-summable function, we define the measure $u\,\nu$ setting

$$u\,\nu(E) = \int_E u\,d\nu$$

for every Borel set $E \subset (a, b)$.

We say that ν is a finite Radon measure, if there exist a positive Radon measure $|\nu|$ with $|\nu|(a, b) < \infty$, and a $|\nu|$-measurable function $\sigma : (a, b) \to \{\pm 1\}$ such that $\nu = \sigma\,|\nu|$. The positive Radon measure $|\nu|$ is uniquely determined by ν, and is called the total variation of ν. Similarly, σ is uniquely determined $|\nu|$-almost everywhere and coincides with the Radon Nikodým derivative $d\nu/d|\nu|$.

Let ν be a finite Radon measure on (a, b). Then there exists a unique decomposition

$$\nu = u\,\mathcal{L}^1 + \nu^s$$

such that $u \in L^1(a,b)$ and there exists a Borel set $E \subset (a,b)$ with $\mathcal{L}^1\big((a,b)\backslash E\big) = 0$ and $|\nu^s|(E) = 0$. The function u is called the absolutely continuous part of the measure ν. Here, $\mathcal{L}^1$ denotes the Lebesgue measure on the real numbers.

The Dirac measure δ_x centered at $x \in (a,b)$ is defined by $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ else.

For $U \in L^1_{\text{loc}}$ and $(c,d) \subset (a,b)$ the variation of U on (c,d) is defined as

$$|DU|(c,d) := \sup\Big\{\int_c^d U(x)\phi'(x)\,dx : \phi \in \mathcal{C}_c^\infty(c,d),\ \|\phi\|_\infty \le 1\Big\} = \infty\ .$$

Here, $\mathcal{C}_c^\infty(c,d)$ denotes the space of all arbitrarily differentiable functions compactly supported in (c,d). An integrable function $U \in L^1(a,b)$ is said to be of bounded variation, if $|DU|(a,b) < \infty$. If U is of bounded variation, then there exists a Radon measure DU with total variation $|DU|$ satisfying

$$\int_a^b U(x)\phi'(x)\,dx = -\int_a^b \phi(x)\,dDU \qquad \text{for all } \phi \in \mathcal{C}_c^\infty(a,b)\ .$$

Since DU is a finite Radon measure, it can be decomposed as $DU = V\,\mathcal{L}^1 + (DU)^s$. The function V coincides almost everywhere with the classical derivative U' of U. The singular part $(DU)^s$ is often denoted as D^sU instead. The function U is called absolutely continuous, if the singular part D^sU of its derivative vanishes.

Let $U : (a,b) \to \mathbb{R}$. We define

$$\begin{aligned} U^{(l)}(x) &:= \lim_{y\to x^-} U(y), \quad & U^{(+)}(x) &:= \limsup_{y\to x} U(y)\,, \\ U^{(r)}(x) &:= \lim_{y\to x^+} U(y), \quad & U^{(-)}(x) &:= \liminf_{y\to x} U(y) \end{aligned}$$

whenever the above limits are defined.

The space $\mathrm{BV}(a,b)$ is by definition a subspace of $L^1(a,b)$ and thus consists of equivalence classes of functions rather than of functions. There exists, however, for every class $U \in \mathrm{BV}(a,b)$ a unique 'good representative' $\tilde{U}$ satisfying

$$\tilde{U}(x) = \frac{1}{2}\left(\tilde{U}^{(l)}(x) + \tilde{U}^{(r)}(x)\right)$$

for all $x \in (a,b)$. In the following we will always identify U with its good representative $\tilde{U}$. It is easy to see that either $U^{(+)}(x) = U^{(l)}(x)$ and $U^{(-)}(x) = U^{(r)}(x)$, or $U^{(-)}(x) = U^{(l)}(x)$ and $U^{(+)}(x) = U^{(r)}(x)$.

3 Generalization of the Taut String Method

Let μ be a finite Radon measure on (a,b), and let F be the antiderivative of μ, i.e.,

$$F(x) := \mu(a,x).$$

Then the function F is of bounded variation, and $DF = \mu$.

We define for $U \in \mathrm{BV}(a,b)$

$$J_F(U) := \int_a^b \sqrt{1+U'(x)^2}\,dx + |D^s U|(a,b) + |U(a)-F(a)| + |U(b)-F(b)|\ .$$

The functional J_F is the natural extension to $\mathrm{BV}(a,b)$ of the (unconstrained) taut string functional defined in (1).

Let now $T : [a,b] \to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous, and $B : [a,b] \to \mathbb{R}\cup\{+\infty\}$ be upper semicontinuous. Assume moreover that

$$\begin{aligned} T(a) \ge F(a) \ge B(a)\,, \quad & T(b) \ge F(b) \ge B(b)\,,\\ T(x) \ge F(x) \ge B(x) \quad & \text{for a.e. } x \in (a,b)\ . \end{aligned}$$

Define

$$\mathcal{G} := \left\{ U \in \mathrm{BV}(a,b) : B(x) \le U(x) \le T(x) \text{ for a.e. } x \in (a,b) \right\}\ .$$

We consider the minimization problem

$$J_F(U) \to \min\,, \qquad U \in \mathcal{G}\,, \tag{2}$$

and define the regularization of μ with tube $\mathcal{G}$ as $\mu_{\mathcal{G}} := DU_{\mathcal{G}}$, where $U_{\mathcal{G}}$ solves (2).

In the following we prove existence and uniqueness of a solution $U_{\mathcal{G}}$ of (2). Moreover, we provide a characterization of $U_{\mathcal{G}}$. To that end we need the following subsets of (a,b):

We denote

$$\begin{aligned} \Sigma^- &:= \left\{ x \in (a,b) : \limsup_{y\to x^-} B(y) \ge \liminf_{y\to x^+} T(y) \right\}\,,\\ \Sigma^+ &:= \left\{ x \in (a,b) : \limsup_{y\to x^+} B(y) \ge \liminf_{y\to x^-} T(y) \right\}\,, \end{aligned} \tag{3}$$

and

$$\Sigma := \Sigma^+ \cup \Sigma^- = \left\{ x \in (a,b) : B(x) \ge T(x) \right\}\ . \tag{4}$$

Since T is lower semicontinuous and B is upper semicontinuous, it follows that the set Σ is closed.

We define moreover for a function $V \in \mathcal{G}$

$$\begin{aligned} S^{(+)}(V) &:= \left\{ x \in (a,b)\setminus\Sigma : V^{(-)}(x) > B(x) \right\}\,,\\ S^{(-)}(V) &:= \left\{ x \in (a,b)\setminus\Sigma : V^{(+)}(x) < T(x) \right\}\,, \end{aligned} \tag{5}$$

i.e., $S^{(+)}(V)$ is the part of (a,b) where V does not touch the lower boundary of the tube, and $S^{(-)}(V)$ is the part of (a,b) where V does not touch the upper boundary. Note that the set $S^{(+)}(V)$ is open, since the set Σ is closed, the function $V^{(-)}$ is lower semicontinuous, and B is upper semicontinuous. Similarly we obtain that the set $S^{(-)}(V)$ is open.

Theorem 3.1. *The minimization problem* (2) *attains a unique solution* $U_{\mathcal{G}}$ *characterized by the conditions*

1. $U_{\mathcal{G}}(a) = F(a)$, $U_{\mathcal{G}}(b) = F(b)$,
2. *for every* $x \in \Sigma^-$ *we have*

$$U_{\mathcal{G}}^{(l)}(x) = \limsup_{y \to x^-} B(y), \qquad U_{\mathcal{G}}^{(r)}(x) = \liminf_{y \to x^+} T(y)\,,$$

for all $x \in \Sigma^+$,

$$U_{\mathcal{G}}^{(l)}(x) = \liminf_{y \to x^-} T(y), \qquad U_{\mathcal{G}}^{(r)}(x) = \limsup_{y \to x^+} U(y)\;.$$

3. $U_{\mathcal{G}}$ *is convex on each connected component of* $S^{(+)}(U_{\mathcal{G}})$ *and concave on each connected component of* $S^{(-)}(U_{\mathcal{G}})$.

Proof. The existence of a minimizer follows from Lemma 5.1. From Lemma 5.2 it follows that $U_{\mathcal{G}}$ satisfies Item 2. Item 1 follows from Lemma 5.3, and Item 3 from Lemma 5.4. Using Lemma 5.5 it follows that $U_{\mathcal{G}}$ is uniquely characterized by Items 1–3. This shows the assertion. □

Remark 3.2. Note that by construction $\big((a,b) \setminus \Sigma\big) \subset S^{(+)}(U_{\mathcal{G}}) \cup S^{(-)}(U_{\mathcal{G}})$. In particular, it follows from Item 3 in Theorem 3.1 that in every point $x \in (a,b) \setminus \Sigma$ the function $U_{\mathcal{G}}$ is either locally convex or locally concave. In particular, $U_{\mathcal{G}}$ is continuous outside of Σ. Since every convex or concave function is absolutely continuous (see e.g. [8, Thm. 11.A]), we additionally obtain that the singular part of the measure $\mu_{\mathcal{G}}$ is concentrated on Σ. We therefore have a decomposition

$$\mu_{\mathcal{G}} = u_{\mathcal{G}}\, \mathcal{L}^1 + (\mu_{\mathcal{G}})^s$$

with $|(\mu_{\mathcal{G}})^s|\big((a,b) \setminus \Sigma\big) = 0$.

Moreover, if $U_{\mathcal{G}}$ is not locally convex in a point $x \in (a,b) \setminus \Sigma$, then necessarily $U_{\mathcal{G}}(x) = B(x)$, and if $U_{\mathcal{G}}$ is not locally concave in $x \in (a,b) \setminus \Sigma$, then $U_{\mathcal{G}}(x) = T(x)$.

Now we discuss the influence of the regularity of the functions B and T on the outcome of the algorithm.

Example 3.3 (Continuous Tubes)**.** Assume that $T(x) \geq B(x)$ for all $x \in (a,b)$. Then Σ consists of all points $x \in (a,b)$ satisfying $B(x) = T(x)$. Thus, it follows from Item 2 that $U_{\mathcal{G}}^{(l)}(x) = U_{\mathcal{G}}^{(r)}(x)$ for every $x \in \Sigma$, which implies that $U_{\mathcal{G}}$ is continuous in Σ. Since by Remark 3.2 the function $U_{\mathcal{G}}$ is continuous outside Σ, we obtain that $U_{\mathcal{G}}$ is continuous on the whole interval (a,b). Thus $\mu_{\mathcal{G}}(\{x\}) = 0$ for every $x \in (a,b)$, which shows that $\mu_{\mathcal{G}}$ contains no Dirac measures. In particular, this applies if T and B are continuous.

Example 3.4 (Absolutely Continuous Tubes)**.** Assume that B and T are absolutely continuous. From Example 3.3 it follows that $U_{\mathcal{G}}$ is continuous. Now denote

$$S_B := \{x \in (a,b) : U_{\mathcal{G}}(x) = B(x)\}\,, \qquad S_T := \{x \in (a,b) : U_{\mathcal{G}}(x) = T(x)\}\ .$$

From Item 3 in Theorem 3.1 it follows that $U_{\mathcal{G}}$ is locally affine outside $S_B \cup S_T$ and thus its derivative a piecewise constant function. Using [1, Rem. 3.93] it follows that $DU_{\mathcal{G}}$ and DB coincide on S_B. Similarly, $DU_{\mathcal{G}}$ and DT coincide on S_T. Since by assumption B and T are absolutely continuous, this implies that $U_{\mathcal{G}}$ is absolutely contiuous too, that is, $\mu_{\mathcal{G}} = u_{\mathcal{G}}\,\mathcal{L}^1$ for some $u_{\mathcal{G}} \in L^1(a,b)$.

Example 3.5 (Standard Taut String for Radon Measures)**.** Assume that $T := F^{(-)} + \alpha$ and $B = F^{(+)} - \alpha$ for some $\alpha > 0$. Then

$$\Sigma^- = \left\{x : F^{(l)} - \alpha \geq F^{(r)} + \alpha\right\}, \qquad \Sigma^+ = \left\{x : F^{(l)} + \alpha \leq F^{(r)} - \alpha\right\}\ .$$

From the definition of F it follows that

$$\Sigma^- = \left\{x : \mu(\{x\}) \leq -2\alpha\right\}, \qquad \Sigma^+ = \left\{x : \mu(\{x\}) \geq +2\alpha\right\}\ .$$

Consequently, we obtain by applying Item 2 in Thm. 3.1 that

$$\begin{aligned} \mu_{\mathcal{G}}(\{x\}) &= \mu(\{x\}) + 2\alpha, \quad \text{for } x \in \Sigma^-\ ,\\ \mu_{\mathcal{G}}(\{x\}) &= \mu(\{x\}) - 2\alpha, \quad \text{for } x \in \Sigma^+\ . \end{aligned}$$

Moreover, it follows from Remark 3.2 that $U_{\mathcal{G}}$ is absolutely continuous outside of Σ. We therefore have the decomposition

$$\mu_{\mathcal{G}} = u_{\mathcal{G}}\,\mathcal{L}^1 + \sum_{\mu(\{x\}) \leq -2\alpha} \big(\mu(\{x\}) + 2\alpha\big)\,\delta_x + \sum_{\mu(\{x\}) \geq 2\alpha} \big(\mu(\{x\}) - 2\alpha\big)\,\delta_x\ .$$

In particular we obtain that $\mu_{\mathcal{G}}$ is absolutely continuous, if $|\mu(\{x\})| \leq 2\alpha$ for all $x \in (a,b)$.

4 Examples

Piecewise constant tubes are already used for the smoothing of density estimators in statistical applications (see [2]). Here, we will concentrate on non-constant tubes depending on the data f. By designing customized tubes we can obtain problem-specific smoothing-features of the solution and therefore a higher quality in the postprocessing of the results.

4.1 Tubes depending on f

For multimeters (also known as multitesters) like ammeters, voltmeters or ohmmeters the accuracy of measurement depends on the absolute value of the measured quantity, i.e., the error in the measurement is not equally distributed over the effective range. Thus it is natural to allow for a stronger smoothing in areas where the data f is large and stick to the measured values where f is small.

This goal can be obtained by defining

$$\begin{aligned} T(x) &:= F(x) + \alpha|f(x)| \,, \\ B(x) &:= F(x) - \alpha|f(x)| \,. \end{aligned} \tag{6}$$

Figure 1 shows the results for the original taut string method as well as the results for the tube depending on f. The parameters were chosen such that the results are comparable in the mid-range of the measurement. For small measurement values the original taut string smoothing is too far away from the data (and even fails to differ between two levels, see figure 2, right) whereas for large values the result sticks too strong to the measurements. Using the tubes defined in (6), both weaknesses of a constant tube are overcome.

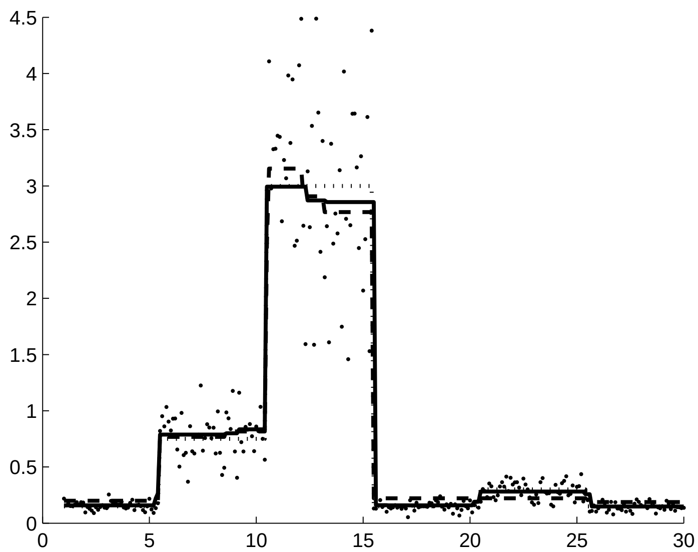

Figure 1: Original (dotted) and noisy (dots) data, original taut string (dashed) and new tube (solid) result

4.2 The min-max tube

For fixed $\delta > 0$ let

$$\begin{aligned} T(x) &:= \sup\left\{F^{(-)}(y) : y \in (x-\delta, x+\delta) \cap (a,b)\right\} \,, \\ B(x) &:= \inf\left\{F^{(+)}(y) : y \in (x-\delta, x+\delta) \cap (a,b)\right\} \,. \end{aligned}$$

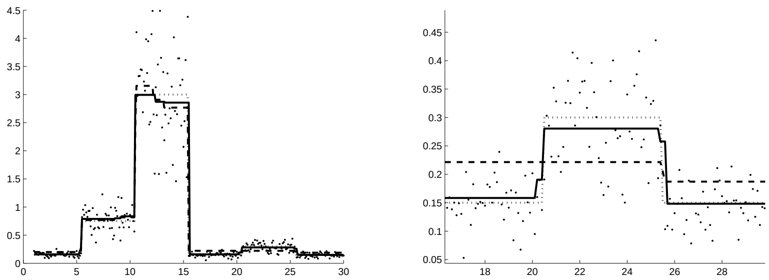

Figure 2: Left: Original (dotted) and noisy (dots) data, original taut string (dashed) and new tube (solid) result; Right: Zoom of Region

The main goals of applying this tube are the separation of regions and the enhancement of plateaus. In Figure 3 one can see that the solution $u_{\mathcal{G}}$ has a jump where the input function f is zero, thus separating the region where f is positive from the region where f is negative.

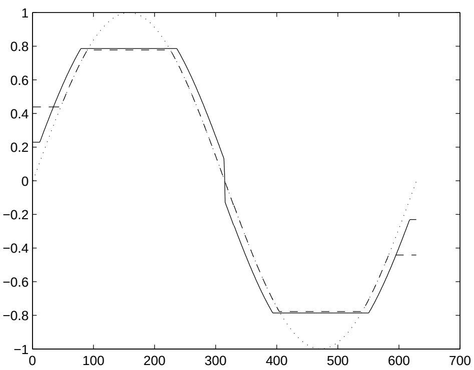

Figure 3: data (dotted), min-max tube (solid) and constant tube (dashed)

An interesting application for the min-max tube is line recognition within pattern recognition. Large amounts of (especially older) publications are about to be digitized, which requires optical text recognition (OCR) systems. To ease precise character recognition, the lines in scanned pages should be positioned horizontally.

Let $g(x, y)$ be the gray values of a scanned page, which is in general not positioned exactly horizontally, but rotated by a small angle β, and define the row sum $S(y) = \int_0^{x_{\max}} g(x, y)\, dx$. Note that $S(y)$ has a jump at the end and beginning of each line, if the page is positioned almost horizontally. Moreover, the BV semi-norm of $S(y)$ should become maximal if the position of the page is exactly horizontal. This effect is even more pronounced if one uses the H^1-semi-norm, or rather the sum of squares of the jumps of S, as decision criterion for the evenness of the scan.

For the application of the method it is necessary to smooth the row sum S. Since the min-max tube defined above yields a good separation of the jumps and preserves the plateaus in S, it seems to be a good choice for a presmoother. Figure 4 shows a part of a scanned page with the row sums as well as the smoothed row sums. One can see that the row sum has high variations within the lines which explains the need for smoothing $S(y)$ before maximizing its norm. Figure 5 shows one of the advantages of the min-max tube compared to the method using a constant tube. Even small peaks in the row sums (like page numbers or delimiters) are preserved whereas single peaks or noise in the signal are removed or at least damped.

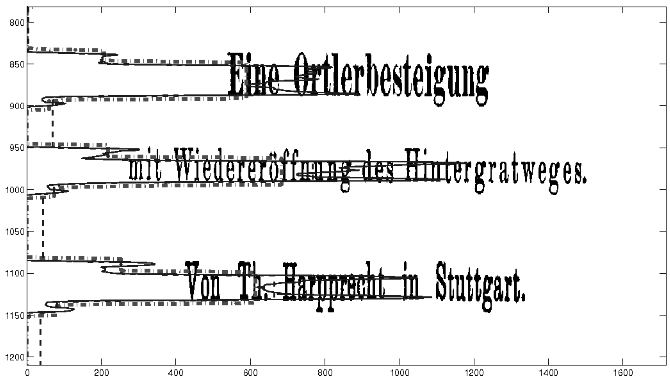

Figure 4: Row sum (solid) with peaks, smoothed row sums (dash-dot line) and constant tube (dashed line)

5 Proofs

Lemma 5.1. *The minimization problem* (2) *attains a solution.*

Proof. By assumption we have $F \in \mathcal{G}$, which proves that

$$\inf_{U \in \mathcal{G}} J_F(U) \leq J_F(F) \leq (b-a) + |DF|(a,b) < \infty.$$

From [4, Chp. 14] it follows that the functional J_F is weak* lower semicontinuous in $\mathrm{BV}(a,b)$. Let now $\{V^{(k)}\}_{k \in \mathbb{N}}$ be a minimizing sequence for J_F in $\mathcal{G}$, i.e.,

$$\lim_{k \to \infty} J_F(V^{(k)}) = \inf\{J_F(V) : V \in \mathcal{G}\}.$$

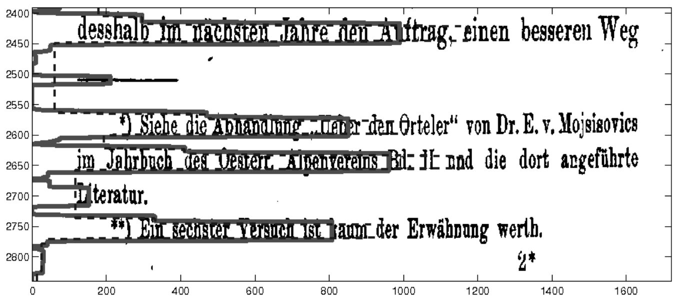

Figure 5: Result of min-max tube (solid) compared to constant tube (dashed)

Then $\sup_{k\in\mathbb{N}}|DV^{(k)}|(a,b) < \infty$. Moreover,

$$\sup_{k\in\mathbb{N}} \operatorname*{ess\,sup}_{x\in(a,b)} V^{(k)}(x) \le \sup_{k\in\mathbb{N}}\bigl\{F(a) + |V^{(k)}(a) - F(a)| + |DV^{(k)}|(a,b)\bigr\} \le$$
$$\le F(a) + \sup_{k\in\mathbb{N}} J_F(V^{(k)}) < \infty.$$

Similarly,

$$\inf_{k\in\mathbb{N}} \operatorname*{ess\,inf}_{x\in(a,b)} V^{(k)}(x) > -\infty,$$

which shows that $\sup_k\|V^{(k)}\|_\infty < \infty$. Thus, after possibly passing to a subsequence, we may assume without loss of generality that the sequence $\{V^{(k)}\}_{k\in\mathbb{N}}$ weak* converges to some $V \in \mathcal{G}$. From the weak* lower semicontinuity of J_F it follows that $J_F(V) \le \lim_{k\to\infty} J(V^{(k)})$. Thus, V is a minimizer of the restriction of J_F to $\mathcal{G}$. $\square$

In the following we denote by $U_{\mathcal{G}}$ any minimizer of J_F restricted to $\mathcal{G}$. For the definitions of $\Sigma^\pm$ we refer to (3).

Lemma 5.2. *For every $x \in \Sigma^-$ we have*

$$U_{\mathcal{G}}^{(l)}(x) = \limsup_{y\to x^-} B(y)\,, \qquad U_{\mathcal{G}}^{(r)}(x) = \liminf_{y\to x^+} T(y)\,,$$

for all $x \in \Sigma^+$,

$$U_{\mathcal{G}}^{(l)}(x) = \liminf_{y\to x^-} T(y)\,, \qquad U_{\mathcal{G}}^{(r)}(x) = \limsup_{y\to x^+} B(y)\,.$$

Proof. Let $x \in \Sigma^+$. Since $U_{\mathcal{G}} \in \mathcal{G}$, it follows that

$$U_{\mathcal{G}}^{(l)}(x) \le \liminf_{y\to x^-} T(y) \le \limsup_{y\to x^+} B(y) \le U_{\mathcal{G}}^{(r)}(x).$$

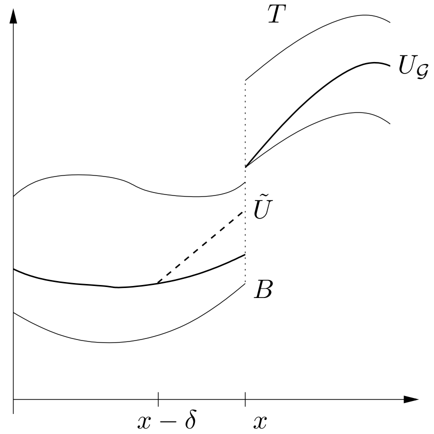

Figure 6: Sketch of $\tilde{U}$ in the proof of Lemma 5.2.

Suppose that $U_{\mathcal{G}}^{(l)}(x) < \liminf_{y\to x^-} T(y)$. Then there exists $\varepsilon > 0$ such that

$$U_{\mathcal{G}}^{(l)}(x) + 4\varepsilon < \liminf_{y\to x^-} T(y)\ .$$

Since $U_{\mathcal{G}}^{(l)}$ is continuous from the left, it follows that there exists $\delta > 0$ such that

$$U_{\mathcal{G}}^{(l)}(x) - \varepsilon \le U_{\mathcal{G}}^{(l)}(y) \le U_{\mathcal{G}}^{(l)}(x) + \varepsilon \text{ for all } y \in [x-\delta, x).$$

Since T is lower semicontinuous we may assume, after possibly choosing a smaller $\delta > 0$, that $T(y) > U_{\mathcal{G}}^{(l)}(x) + 4\varepsilon$ for all $y \in (x-\delta, x)$. Since B is upper semicontinuous, and $B(y) \le U_{\mathcal{G}}^{(l)}(y)$ for almost every y, it follows, again after choosing a smaller $\delta > 0$, that $B(y) < U_{\mathcal{G}}^{(l)}(x) + \varepsilon$ for every $y \in (x-\delta, x)$. Since $U_{\mathcal{G}}$ is of bounded variation, it has at most countably many discontinuities. Thus we may additionally assume by slightly varying δ that $U_{\mathcal{G}}$ is continuous in the point $x - \delta$.

Define

$$\tilde{U}(y) := \begin{cases} U_{\mathcal{G}}(y), & \text{if } y \notin (x-\delta, x)\,, \\ U_{\mathcal{G}}^{(l)}(x) + \varepsilon + 3(y-x)\varepsilon/\delta, & \text{if } y \in (x-\delta, x)\,, \end{cases}$$

i.e., $\tilde{U}$ linearly interpolates $U_{\mathcal{G}}^{(l)}(x)+\varepsilon$ and $U_{\mathcal{G}}^{(l)}(x)+4\varepsilon$ on the interval $(x-\delta, x)$. Since $B(y) < U_{\mathcal{G}}^{(l)}(x)+\varepsilon < U_{\mathcal{G}}^{(l)}(x)+4\varepsilon < T(y)$ for every $y \in (x-\delta, x)$, it follows that $\tilde{U} \in \mathcal{G}$.

Now note that

$$
\begin{aligned}
J_F(U_{\mathcal{G}}) - J_F(\tilde{U}) &= \int_{x-\delta}^{x} \sqrt{1 + U_{\mathcal{G}}'(t)^2}\, dt + |D^s U_{\mathcal{G}}|(x-\delta, x) \\
&\quad + U_{\mathcal{G}}^{(l)}(x) + \varepsilon - U_{\mathcal{G}}(x-\delta) + \sqrt{9\varepsilon^2 + \delta^2} - 4\varepsilon \\
&\le \sqrt{\big(U_{\mathcal{G}}^{(l)}(x) - U_{\mathcal{G}}(x-\delta)\big)^2 + \delta^2} \\
&\quad + U_{\mathcal{G}}^{(l)}(x) - U_{\mathcal{G}}(x-\delta) + \sqrt{9\varepsilon^2 + \delta^2} - 3\varepsilon\,.
\end{aligned}
$$

Since by assumption $|U_{\mathcal{G}}^{(l)}(x) - U_{\mathcal{G}}(x-\delta)| \le \varepsilon$ this shows that $J_F(U_{\mathcal{G}}) < J_F(\tilde{U})$, which is a contradiction to the minimality of $J_F(U_{\mathcal{G}})$.

Thus we obtain that $U_{\mathcal{G}}^{(l)}(x) = \limsup_{y \to x^+} T(y)$. All other equalities can be shown in a similar manner. □

Lemma 5.3. *We have $U_{\mathcal{G}}(a) = F(a)$ and $U_{\mathcal{G}}(b) = F(b)$.*

Proof. This is similar to the proof of Lemma 5.2. □

For the definition of the sets $S^{(\pm)}(U_{\mathcal{G}})$ used in the following Lemma we refer to (5).

Lemma 5.4. *The function $U_{\mathcal{G}}$ is convex on each connected component of the set $S^{(+)}(U_{\mathcal{G}})$ and concave on each connected component of $S^{(-)}(U_{\mathcal{G}})$. In particular, $U_{\mathcal{G}}$ is piecewise affine on $S^{(+)}(U_{\mathcal{G}}) \cap S^{(-)}(U_{\mathcal{G}})$.*

Proof. Let $x \in S^{(+)}(U_{\mathcal{G}})$. Since $U_{\mathcal{G}}^{(-)}$ is lower semicontinuous, and B is upper semicontinuous, there exist $\varepsilon > 0$ and $c \in \mathbb{R}$ such that

$$
B(y) < c < U_{\mathcal{G}}^{(-)}(y) \qquad \text{for all } y \in (x-\varepsilon, x+\varepsilon). \tag{7}
$$

Let $x_1 < x_2 \in (x-\varepsilon, x+\varepsilon)$. We have to show that

$$
U_{\mathcal{G}}\big(\lambda x_1 + (1-\lambda)x_2\big) \le \lambda U_{\mathcal{G}}(x_1) + (1-\lambda) U_{\mathcal{G}}(x_2) \tag{8}
$$

for all $0 < \lambda < 1$. Define

$$
\tilde{U}(y) := \begin{cases} \min\big\{U_{\mathcal{G}}(y),\ \lambda U_{\mathcal{G}}(x_1) + (1-\lambda) U_{\mathcal{G}}(x_2)\big\}, & \begin{array}{l}\text{if } y = \lambda x_1 + (1-\lambda)x_2 \\ \text{for some } \lambda \in (0,1)\,.\end{array} \\ U_{\mathcal{G}}(y), & \text{if } y \notin (x_1, x_2)\,, \end{cases}
$$

From (7) it follows that $\tilde{U}(y) \ge c \ge B(y)$ for every $y \in (x_1, x_2)$. On the other hand, $\tilde{U}(y) \le U_{\mathcal{G}}(y) \le T(y)$ for almost every y. Consequently $\tilde{U} \in \mathcal{G}$.

Now note that (8) is equivalent to the inequality $U_{\mathcal{G}}(y) \le \tilde{U}(y)$ on (x_1, x_2). Suppose to the contrary that $\tilde{U}(y) < U_{\mathcal{G}}(y)$ for some $y \in (x_1, x_2)$. Denote

$$
\begin{aligned}
y_1 &:= \sup\big\{z < y : \tilde{U}(z) < U_{\mathcal{G}}^{(-)}(z)\big\}, \\
y_2 &:= \inf\big\{z > y : \tilde{U}(z) < U_{\mathcal{G}}^{(-)}(z)\big\}.
\end{aligned}
$$

From the lower semicontinuity of $U_{\mathcal{G}}^{(-)}$ it follows that $x_1 \leq y_1 < y < y_2 \leq x_2$. Since by construction $\tilde{U}$ is affine on (y_1, y_2), it follows that $J_F(\tilde{U}) < J_F(U_{\mathcal{G}})$, which is a contradiction to the minimality of $J_F(U_{\mathcal{G}})$.

This shows that $U_{\mathcal{G}}(y) \leq \tilde{U}(y)$ for all y. Thus (8) holds for every $x_1 < x_2 \in (x - \varepsilon, x + \varepsilon)$ and $0 < \lambda < 1$, which shows that $U_{\mathcal{G}}$ is convex on $(x - \varepsilon, x + \varepsilon)$. Since this holds for every $x \in S^{(+)}(U_{\mathcal{G}})$, it follows that $U_{\mathcal{G}}$ is convex on each connected component of $S^{(+)}(U_{\mathcal{G}})$.

The proof that $U_{\mathcal{G}}$ is concave on each connected component of $S^{(-)}(U_{\mathcal{G}})$ is similar. □

Lemma 5.5. *There exists at most one function in $\mathcal{G}$ satisfying Items 1–3 in Theorem 3.1.*

Proof. Assume that V and $W \in \mathcal{G}$ are two functions satisfying Items 1–3 in Theorem 3.1. From Items 1 and 2 it follows that V and W coincide on $\Sigma \cup \{a, b\}$. Now suppose that there exists $x \in (a, b) \setminus \Sigma$ such that $V(x) > W(x)$.

Denote

$$x_1 := \sup\{y < x : V^{(r)}(y) > W^{(r)}(y)\},$$
$$x_2 := \inf\{y > x : V^{(l)}(y) > W^{(l)}(y)\}.$$

By Remark 3.2 the functions V and W are continuous on $(a, b) \setminus \Sigma$. Since $x \notin \Sigma$, it therefore follows that $x_1 < x < x_2$. From Item 1 it follows that $x_1 \geq a$, $x_2 \leq b$, and by Item 2 we have x_1, $x_2 \notin \Sigma$. Thus the continuity of V and W outside Σ implies that $V^{(r)}(x_1) = W^{(r)}(x_1)$ and $V^{(l)}(x_2) = W^{(l)}(x_2)$. Since $T(y) \geq V(y) > W(y) \geq B(y)$ for every $y \in (x_1, x_2)$, it follows that $(x_1, x_2) \subset S^{(+)}(V) \cap S^{(-)}(W)$. Thus, V is convex on (x_1, x_2) and W is concave on (x_1, x_2). From this it easily follows that $V(x) \leq W(x)$, a contradiction to the definition of x. Consequently $V(x) \leq W(x)$ for every $x \in (a, b) \setminus \Sigma$. The converse inequality follows in a similar manner. □

6 Conclusions

In this paper we have presented a generalization of the taut string algorithm both theoretically and through examples. The new framework allows for the smoothing of arbitrary one-dimensional Radon measures with non-constant tubes. In contrast, the classical taut string method only works with L^1-data and tubes of constant radius.

Existence and uniqueness are shown, and a unique characterization of the solution is given. The theoretical examples presented in Section 3 show that the algorithm yields regular results provided the employed tube is of sufficient regularity. The examples in Section 4 indicate that non-constant tubes can be superior to classical tubes, since knowledge about varying noise levels can directly be incorporated in the tube. Section 4.2 additionally shows that non-standard tubes can be made very efficient for the solution of specialized problems.

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