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# The Residual Method for Regularizing Ill-Posed Problems

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**Abstract**

Although the *residual method*, or *constrained regularization*, is frequently used in applications, a detailed study of its properties is still missing. In particular, the questions of stability and convergence rates have hardly been treated in the literature. This sharply contrasts the progress of the theory of Tikhonov regularization, where for instance the notion of the Bregman distance has recently led to a series of new results for regularization in Banach spaces. The present paper intends to bridge the gap between the existing theories as far as possible. We develop a stability and convergence theory for the residual method in general topological spaces. In addition, we prove convergence rates in terms of (generalized) Bregman distances.

Exemplarily, we apply our theory to compressed sensing. Here, we show the well-posedness of the method and derive convergence rates both for convex and non-convex regularization under rather weak conditions. It is for instance shown that in the non-convex setting the linear convergence of the regularized solutions already follows from the sparsity of the true solution and the injectivity of the (linear) operator in the equation to be solved.

**Key words.** Ill-posed problems, Regularization, Residual Method, Sparsity, Stability, Convergence Rates.

**AMS subject classifications.** 65J20; 47J06; 49J27.

## 1 Introduction

We study the solution of ill-posed operator equations

$$F(x) = y\,, \tag{1}$$

where $F\colon X \to Y$ is an operator between the topological spaces $X$ and $Y$, and $y \in Y$ are given, noisy data, which are assumed to be close to some unknown, noise-free data $y^\dagger = F(x^\dagger)$. If the operator $F$ is not continuously invertible, then (1) may not have a solution and, if a solution exists, arbitrarily small perturbations of the data may lead to unacceptable results.

If it is known that the given data satisfy an estimate $\|y^\dagger - y\| \leq \beta$, one strategy for defining an approximate solution of (1) is to solve the *constrained minimization problem*

$$\mathcal{R}(x) \to \min \qquad \text{subject to} \qquad \|F(x) - y\| \leq \beta \,. \tag{2}$$

Here, the *regularization term* $\mathcal{R}\colon X \to [0, +\infty]$ is intended to enforce certain regularity properties of the approximate solution and to stabilize the process of solving (1). In [29, 45], this strategy is called the *residual method*. It is closely related to *Tikhonov regularization*, which consists in minimizing the regularization functional

$$\mathcal{T}(x, y) := \|F(x) - y\|^2 + \alpha \mathcal{R}(x)$$

for some regularization parameter $\alpha > 0$.

While the theory of Tikhonov regularization has received much attention in the literature (see [1, 2, 12, 18, 19, 24, 27, 35, 41, 43, 46, 48]), the same cannot be said about the residual method. Nevertheless, several results are available. The existence theory of (2) and also the question of convergence, which asks whether solutions of (2) converge to a solution of (1) as $\|y - y^\dagger\| \leq \beta \to 0$, have been treated in a quite general setting in [28] (see also [44, 45]). Also, convergence rates have for instance been derived in [4] in a Hilbert space setting for a linear operator $F$ and in [5, 7] for the reconstruction of sparse sequences. Still, no attempts have been made to carry over these results to more general spaces and functionals, as opposed to the recent developments in Tikhonov regularization (see [26, 36, 38, 39, 42]).

Even more, it seems that the problem of stability, that is, continuous dependence of the solution of (2) on the input data $y$ and the presumed noise level $\beta$, has hardly been considered at all. One reason is that, in contrast to Tikhonov regularization, stability simply does not hold for general non-linear operator equations. But even for the linear case, where we indeed prove stability, so far results are non-existent in the literature.

The present paper intends to carry out the above indicated generalizations of the existent theory as far as possible. We assume that $X$ and $Y$ are mere topological spaces and consider the minimization of $\mathcal{R}(x)$ subject to the constraint $\mathcal{S}\big(F(x), y\big) \leq \beta$. Here $\mathcal{S}$ is some distance like functional taking over the role of the norm in (2). In addition, we discuss the case where the operator $F$ is not known exactly. This subsumes errors due to the modeling process as well as discretizations of the problem necessary for its numerical solution.

We provide different criteria that ensure stability (Theorem 3.6 and Propositions 3.10, 4.3) and convergence (Propositions 3.9, 4.3) of the residual method. In particular, our conditions also include certain non-linear operators (see Example 4.6). Section 5 is concerned with the derivation of convergence rates. We define a generalized Bregman distance that allows us to state and prove rates on arbitrary topological spaces (see Theorem 5.5). In Section 6 we apply our general results to the case of sparse $\ell^p$-regularization with $p \in (0, 2)$. We prove the well-posedness of the method and derive convergence rates with respect to the norm in a fairly general setting. In the case of convex regularization, that is, $p \geq 1$, we derive a convergence rate of order $\mathcal{O}(\delta^{1/p})$. In the non-convex case $0 < p < 1$, we show that the rate $\mathcal{O}(\delta)$ holds.

# 2 Definitions and Mathematical Preliminaries

Let $X$ and $Y$ be sets and $F\colon X \to Y$. Assume moreover that $\mathcal{R}\colon X \to [0, +\infty]$ and $\mathcal{S}\colon Y \times Y \to [0, +\infty]$ is such that $\mathcal{S}(y, z) = 0$ if and only if $y = z$. We consider for given $y \in Y$ and $\beta \geq 0$ the constrained minimization problem

$$\mathcal{R}(x) \to \min \qquad \text{subject to} \qquad \mathcal{S}(F(x), y) \leq \beta\,. \tag{3}$$

For the study of the properties of the solutions of (3), it is convenient to introduce the following notation. Let $\beta \geq 0$, $t \geq 0$, $y \in Y$, and $F\colon X \to Y$. We define the *feasible set* for the solution of (3) as

$$\Phi(\beta, y, F) := \bigl\{x \in X : \mathcal{S}(F(x), y) \leq \beta\bigr\}\,.$$

In addition, we denote

$$\Phi_{\mathcal{R}}(\beta, y, F, t) := \Phi(\beta, y, F) \cap \bigl\{x \in X : \mathcal{R}(x) \leq t\bigr\}\,.$$

The *value* of (3) is defined as

$$v(\beta, y, F) := \inf\bigl\{\mathcal{R}(x) : x \in \Phi(\beta, y, F)\bigr\}\,.$$

The set of solutions of (3) is denoted by

$$\Sigma(\beta, y, F) := \bigl\{x \in \Phi(\beta, y, F) : \mathcal{R}(x) = v(\beta, y, F)\bigr\}\,.$$

**Remark 2.1.** An immediate consequence of the definition of $\Sigma(\beta, y, F)$ is the identity

$$\Sigma(\beta, y, F) = \Phi_{\mathcal{R}}\bigl(\beta, y, F, v(\beta, y, F)\bigr)\,.$$

The elements of $\Sigma(0, y, F)$ satisfy $F(x) = y$ and are referred to as $\mathcal{R}$*-minimizing solutions* of the equation $F(x) = y$. ■

**Lemma 2.2.** *The sets $\Phi_{\mathcal{R}}$ satisfy*

$$\Phi_{\mathcal{R}}(\beta, y, F, t) \subset \Phi_{\mathcal{R}}(\beta + \delta, y, F, t + \varepsilon) \tag{4}$$

*for every $\varepsilon \geq 0$ and $\delta \geq 0$, and*

$$\Phi_{\mathcal{R}}(\beta, y, F, t) = \bigcap_{\delta,\varepsilon>0} \Phi_{\mathcal{R}}(\beta + \delta, y, F, t + \varepsilon)\,. \tag{5}$$

*In particular,*

$$\Sigma(\beta, y, F) = \bigcap_{\varepsilon>0} \Phi_{\mathcal{R}}\bigl(\beta, y, F, v(\beta, y, F) + \varepsilon\bigr)\,. \tag{6}$$

*Proof.* The inclusion (4) is a trivial consequence of the definition of $\Phi_{\mathcal{R}}$. For the proof of (5) note that $x \in \bigcap_{\delta,\varepsilon>0} \Phi_{\mathcal{R}}(\beta+\delta, y, F, t+\varepsilon)$ if and only if $\mathcal{S}(F(x), y) \leq \beta+\delta$ for all $\delta > 0$ and $\mathcal{R}(x) \leq t+\varepsilon$ for all $\varepsilon > 0$. This, however, amounts to saying that $\mathcal{S}(F(x), y) \leq \beta$ and $\mathcal{R}(x) \leq t$, which means that $x \in \Phi_{\mathcal{R}}(\beta, y, F, t)$. This proves one inclusion in (5), and the other inclusion is an obvious consequence of (4). Finally, equation (6) follows from Remark 2.1 and (5). $\square$

In the next section we study convergence and stability of the residual method, that is, the behavior of the set of solutions $\Sigma(\beta_k, y_k, F)$ for $\beta_k \to \beta$ and $y_k \to y$. In [18, 26, 42], where convergence and stability of Tikhonov regularization have been investigated, the results are of the form: every sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \arg\min\{\|F(x) - y_k\|^2 + \alpha_k \mathcal{R}(x)\}$ has a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ converging to some element $x \in \arg\min\{\|F(x) - y\|^2 + \alpha\mathcal{R}(x)\}$. We prove similar results for the residual method but with a different notation involving a type of convergence of sets (see [31, §29]). In addition, it is necessary to define a notion of convergence of $(y_k)_{k\in\mathbb{N}}$ in a way compatible with the distance measure $\mathcal{S}$ on $Y$.

**Definition 2.3.** The sequence $(y_k)_{k\in\mathbb{N}} \subset Y$ converges $\mathcal{S}$*-uniformly* to $y \in Y$, if

$$\sup\bigl\{\bigl|\mathcal{S}(z, y_k) - \mathcal{S}(z, y)\bigr| : z \in Y\bigr\} \to 0\,.$$

The sequence of mappings $F_k \colon X \to Y$ converges *locally* $\mathcal{S}$*-uniformly* to $F \colon X \to Y$, if

$$\sup\bigl\{\bigl|\mathcal{S}(F_k(x), y) - \mathcal{S}(F(x), y)\bigr| : y \in Y,\, x \in X,\, \mathcal{R}(x) \leq t\bigr\} \to 0$$

for $t \geq 0$. ■

**Remark 2.4.** If the distance measure $\mathcal{S} = d$ equals a metric on $Y$, then the $\mathcal{S}$-uniform convergence of a sequence $(y_k)_{k\in\mathbb{N}}$ to $y$ coincides with its convergence with respect to the metric. This result easily follows from the triangle inequality, as

$$\bigl|d(z, y_k) - d(z, y)\bigr| \leq d(y_k, y)$$

for all $z \in Y$. ■

**Lemma 2.5.** *Assume that $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$ and the mappings $F_k \colon X \to Y$ converge to $F \colon X \to Y$ locally $\mathcal{S}$-uniformly. Then there exists for every $\beta > 0$, $t > 0$, and $\varepsilon > 0$ some $k_0 \in \mathbb{N}$ such that*

$$\begin{aligned}\Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t') &\subset \Phi_{\mathcal{R}}(\beta, y_k, F_k, t')\\ &\subset \Phi_{\mathcal{R}}(\beta + \varepsilon, y, F, t') \quad \textit{for every } t' \leq t \textit{ and } k \geq k_0\,. \end{aligned} \tag{7}$$

*Proof.* Since $y_k \to y$ $\mathcal{S}$-uniformly and $F_k \to F$ locally $\mathcal{S}$-uniformly, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}\bigl|\mathcal{S}(F_k(x), y_k) - \mathcal{S}(F_k(x), y)\bigr| &\leq \varepsilon/2\,,\\ \bigl|\mathcal{S}(F_k(x), y) - \mathcal{S}(F(x), y)\bigr| &\leq \varepsilon/2\,,\end{aligned} \tag{8}$$

for all $x \in X$ with $\mathcal{R}(x) \leq t$ and $k \geq k_0$.

Now let $x \in \Phi_{\mathcal{R}}(\beta-\varepsilon, y, F, t)$. Then (8) implies that

$$\begin{aligned}
&\left|\mathcal{S}(F_k(x), y_k) - \mathcal{S}(F(x), y)\right| \\
&\qquad \le \left|\mathcal{S}(F_k(x), y_k) - \mathcal{S}(F_k(x), y)\right| + \left|\mathcal{S}(F_k(x), y) - \mathcal{S}(F(x), y)\right| \le \varepsilon\,,
\end{aligned}$$

and thus

$$\mathcal{S}(F_k(x), y_k) \le \mathcal{S}(F(x), y) + \varepsilon \le \beta\,,$$

that is, $x \in \Phi_{\mathcal{R}}(\beta, y_k, F_k, t)$, which proves the first inclusion in (7). The second inclusion can be shown in a similar manner. □

**Definition 2.6.** Let $\tau$ be a topology on the set $X$, and let $U_k \subset X$, $k \in \mathbb{N}$, be a sequence of subsets of $X$. We define the upper limit of $(U_k)_{k\in\mathbb{N}}$ as

$$\tau\text{-}\operatorname{Lim\,sup}_k U_k := \bigcap_{k\in\mathbb{N}} \Bigl(\tau\text{-cl} \bigcup_{k'\ge k} U_{k'}\Bigr)\,.$$

Here $\tau$-cl denotes the closure with respect to $\tau$. ■

**Lemma 2.7.** *Let $U_k \subset X$, $k \in \mathbb{N}$, be a sequence of subsets of $X$. Then $x \in \tau\text{-}\operatorname{Lim\,sup}_k U_k$, if and only if for every neighborhood $N$ of $x$ and every $k \in \mathbb{N}$ there exists $k' \ge k$ such that $N \cap U_{k'} \ne \emptyset$.*

*Proof.* This is a direct consequence of the definition of $\tau\text{-}\operatorname{Lim\,sup}_k U_k$. □

Now assume that $X$ satisfies the first axiom of countability, that is, every point $x \in X$ has a countable basis of neighborhoods, an assumption that is for instance satisfied for the weak topology on separable Banach spaces. Then one can characterize the upper limit of sets in terms of subsequences.

**Lemma 2.8.** *Assume that $X$ satisfies the first axiom of countability. Then $x \in \tau\text{-}\operatorname{Lim\,sup}_k U_k$ if and only if there exists a subsequence $(U_{k_j})_{j\in\mathbb{N}}$ and elements $x_j \in U_{k_j}$ such that $x_j \to_\tau x$.*

*Proof.* See [31, §29.IV]. □

**Definition 2.9.** Let $\tau$ be a topology on the set $X$, and let $U_k \subset X$, $k \in \mathbb{N}$, be a sequence of subsets of $X$. An element $x \in X$ is contained in the *lower limit* of the sequence $(U_k)_{k\in\mathbb{N}}$, in short, $x \in \tau\text{-}\operatorname{Lim\,inf}_k U_k$, if for every neighborhood $N$ of $x$ there exists $k \in \mathbb{N}$ such that $N \cap U_{k'} \ne \emptyset$ for every $k \ge k'$.

If the lower limit and the upper limit of the sequence $(U_k)_{k\in\mathbb{N}}$ coincide, we define $\tau\text{-}\operatorname{Lim}_k U_k := \tau\text{-}\operatorname{Lim\,inf}_k U_k = \tau\text{-}\operatorname{Lim\,sup}_k U_k$. ■

**Lemma 2.10.** *We have the characterization $U = \tau\text{-}\operatorname{Lim}_k U_k$, if and only if every subsequence $(U_{k_j})_{j\in\mathbb{N}}$ satisfies $U = \tau\text{-}\operatorname{Lim\,sup}_j U_{k_j}$.*

*Proof.* See [31, §29.V]. □

**Lemma 2.11.** *Assume that the topology $\tau$ on $X$ is defined by a metric $d$. Then $x \in \tau\text{-}\operatorname{Lim\,sup}_k U_k$, if and only if*

$$\liminf_k \operatorname{dist}(x, U_k) = \liminf_k \inf\bigl\{d(x,u) : u \in U_k\bigr\} = 0 \,.$$

*Similarly, $x \in \tau\text{-}\operatorname{Lim\,inf}_k U_k$, if and only if*

$$\limsup_k \operatorname{dist}(x, U_k) = \limsup_k \inf\bigl\{d(x,u) : u \in U_k\bigr\} = 0 \,.$$

*Proof.* See [31, §29.I,§29.III]. □

The following lemma clarifies the relation between the stability and convergence results in [18, 26, 42] and the results in the present paper.

**Lemma 2.12.** *Let $U_k \subset X$, $k \in \mathbb{N}$, be non-empty and assume that there exists a compact set $K$ such that $U_k \subset K$ for all $K$. Then $\tau\text{-}\operatorname{Lim\,sup}_k U_k$ is non-empty.*

*If, in addition, $X$ satisfies the first axiom of countability, then every sequence $x_k \in U_k$ has a subsequence converging to some $x \in \tau\text{-}\operatorname{Lim\,sup}_k U_k$.*

*Proof.* By assumption, the sets $S_k := \tau\text{-}\operatorname{cl} \bigcup_{k' \geq k} U_k$ form a decreasing family of non-empty, compact sets. Thus also their intersection $\bigcap_{k\in\mathbb{N}} S_k = \tau\text{-}\operatorname{Lim\,sup}_k U_k$ is non-empty (see [30, Thm. 5.1]).

Now assume that $X$ satisfies the first axiom of countability. Then in particular every compact set is sequentially compact (see [30, Thm. 5.5]). Let now $x_k \in U_k$, $k \in \mathbb{N}$. Then the sequence $(x_k)_{k\in\mathbb{N}}$ has a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ converging to some $x \in K$. From Lemma 2.8 we obtain that $x \in \tau\text{-}\operatorname{Lim\,sup}_k U_k$, which shows the assertion. □

# 3 Well-posedness

In the following we investigate the existence of minimizers, and the stability and the convergence of the residual method. Throughout the whole section, we assume that $(X, \tau)$ is a topological space, $Y$ a set, $F\colon X \to Y$ some operator, $y \in Y$, and $\beta \geq 0$.

**Theorem 3.1 (Existence).** *Assume that $\Phi_{\mathcal{R}}(\beta, y, F, t)$ is $\tau$-compact for every $t$ and non-empty for some $t$. Then Problem* (3) *has a solution.*

*Proof.* Remark 2.1 and (6) show that

$$\Sigma(\beta, y, F) = \bigcap_{\varepsilon > 0} \Phi_{\mathcal{R}}\bigl(\beta, y, F, v(\beta, y, F) + \varepsilon\bigr)$$

is the intersection of a decreasing family of non-empty $\tau$-compact sets and thus non-empty (see [30, Thm. 5.1]). □

**Remark 3.2.** Recall that a mapping $\mathcal{T}\colon X \to [0, +\infty]$ is called *lower semi-continuous*, if its lower level sets $\{x \in X : \mathcal{T}(x) \le t\}$ are closed for every $t \ge 0$. Moreover, it is *coercive*, if the lower level sets are pre-compact. Thus, $\mathcal{T}$ is lower semi-continuous and coercive, if and only if its lower level sets are compact. Since the intersection of a closed set and a compact set is itself compact, the sets $\Phi_{\mathcal{R}}(\beta, y, F, t)$ are $\tau$-compact, if both mappings $\mathcal{R}$ and $x \mapsto \mathcal{S}(F(x), y)$ are lower semi-continuous and at least one of them (or their sum) is coercive.

The lower semi-continuity of $x \mapsto \mathcal{S}(F(x), y)$ certainly holds if $F$ is continuous and $\mathcal{S}$ lower semi-continuous with respect to the first component. It is, however, also possible to obtain lower semi-continuity, if $F$ is not continuous but the functional $\mathcal{S}$ satisfies a stronger condition. Assume therefore that the mapping $z \mapsto \mathcal{S}(z, y)$ is lower semi-continuous and coercive, and $F\colon X \to Y$ has a closed graph. Then the set $\{z \in Y : \mathcal{S}(z, y) \le \beta\}$ is compact for every $\beta \ge 0$. Because $F$ has a closed graph, the pre-image under $F$ of every compact set is closed (see [28, Thm. 4]). This shows that $\{x \in X : \mathcal{S}(F(x), y) \le \beta\}$ is closed for every $\beta$, that is, the composition $x \mapsto \mathcal{S}(F(x), y)$ is lower semi-continuous. ∎

Stability is concerned with the continuous dependence of the solutions of (3) of the input data, that is, the element $y$, the parameter $\beta$, and, possibly, the operator $F$. Given sequences $\beta_k \to \beta$, $y_k \to y$, and $F_k \to F$, we ask whether the sequence of sets $\Sigma(\beta_k, y_k, F_k)$ converges to $\Sigma(\beta, y, F)$. As already indicated in Section 2, we will make use of the upper convergence of sets introduced in Definition 2.6. The topology, however, with respect to which the results are formulated, is finer than $\tau$.

**Definition 3.3.** The topology $\tau_{\mathcal{R}}$ on $X$ is generated by all sets of the form $U \cap \{x \in X : s < \mathcal{R}(x) < t\}$ with $U \in \tau$ and $s < t \in \mathbb{R}$. A sequence $(x_k)_{k\in\mathbb{N}} \subset X$ converges to $x$ with respect to $\tau_{\mathcal{R}}$, if and only if $x_k \to_\tau x$ and $\mathcal{R}(x_k) \to \mathcal{R}(x)$. ∎

Below we provide conditions that guarantee upper semi-continuity of the set of solutions with respect to $\tau_{\mathcal{R}}$ in the sense that $\emptyset \ne \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k) \subset \Sigma(\beta, y, F)$. That is, the minimizing sets for $\beta_k$, $y_k$, and $F_k$ converge to the minimizing set for $\beta$, $y$, and $F$. If $\Sigma(\beta, y, F)$ consists of a single element $x^\dagger$, then this already implies that every sequence of approximate solutions converges to $x^\dagger$. Before proving these results, we require an additional lemma stating that the value of the minimization problem (3) behaves well as the regularization parameter $\beta$ decreases.

**Lemma 3.4.** *Assume that $\Phi_{\mathcal{R}}(\gamma, y, F, t)$ is $\tau$-compact for every $\gamma$ and every $t$. Then the value $v$ of* (3) *satisfies*

$$v(\beta, y, F) = \sup_{\varepsilon>0} v(\beta + \varepsilon, y, F) \,. \tag{9}$$

*Proof.* Since $\Phi_{\mathcal{R}}(\beta, y, F, t) \subset \Phi_{\mathcal{R}}(\beta + \varepsilon, y, F, t)$, it follows that

$$v(\beta, y, F) \ge v(\beta + \varepsilon, y, F)$$

for every $\varepsilon > 0$, and therefore $v(\beta, y, F) \geq \sup_{\varepsilon>0} v(\beta + \varepsilon, y, F)$.

In order to show the converse inequality, let $\delta > 0$. Then the definition of $v(\beta, y, F)$ implies that $\Phi_{\mathcal{R}}\big(\beta, y, F, v(\beta, y, F) - \delta\big) = \emptyset$. Since (cf. Lemma 2.2)

$$\emptyset = \Phi_{\mathcal{R}}\big(\beta, y, F, v(\beta, y, F) - \delta\big) = \bigcap_{\varepsilon>0} \Phi_{\mathcal{R}}\big(\beta + \varepsilon, y, F, v(\beta, y, F) - \delta\big) \tag{10}$$

and the right hand side of (10) is a decreasing family of compact sets, it follows that already $\Phi_{\mathcal{R}}\big(\beta + \varepsilon, y, F, v(\beta, y, F) - \delta\big) = \emptyset$ for some $\varepsilon > 0$, and thus

$$v(\beta + \varepsilon, y, F) \geq v(\beta, y, F) - \delta \, .$$

Since $\delta$ was arbitrary, this shows the assertion. □

For the main stability results we make the following assumption:

**Assumption 3.5.** *Let $\beta \geq 0$, $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of non-negative numbers, $y \in Y$, $(y_k)_{k\in\mathbb{N}} \subset Y$, and $F$, $F_k \colon X \to Y$, $k \in \mathbb{N}$. The sets $\Phi_{\mathcal{R}}(\gamma, w, F_k, t)$ and $\Phi_{\mathcal{R}}(\gamma, w, F, t)$ are compact for all $\gamma$, $w$, $t$, and $k$ and non-empty some $t$.*

**Theorem 3.6 (Stability).** *Let Assumption 3.5 hold. Assume that $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$, the mappings $F_k \colon X \to Y$ converge locally $\mathcal{S}$-uniformly to $F \colon X \to Y$, and $\beta_k \to \beta$. If*

$$\limsup_k v(\beta_k, y_k, F_k) \leq v(\beta, y, F) < \infty \, , \tag{11}$$

*then*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k) \subset \Sigma(\beta, y, F) \, . \tag{12}$$

*If the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$, then*

$$\{x_\beta\} = \tau_{\mathcal{R}}\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F_k) \, . \tag{13}$$

*Proof.* Define the set $T := \tau\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k)$. Because the topology $\tau_{\mathcal{R}}$ is finer than $\tau$, it follows that $\tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k) \subset T$. We proceed by showing that $\emptyset \neq T \subset \Sigma(\beta, y, F)$ and $T \subset \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k)$, which then gives the assertion (12).

In order to simplify the notation, we define

$$\begin{aligned}
\Phi_k(t) &:= \Phi_{\mathcal{R}}(\beta_k, y_k, F_k, t) \, , & \Phi(t) &:= \Phi_{\mathcal{R}}(\beta, y, F, t) \, , \\
v_k &:= v(\beta_k, y_k, F_k) \, , & v &:= v(\beta, y, F) \, , \\
\Sigma_k &:= \Sigma(\beta_k, y_k, F_k) \, , & \Sigma &:= \Sigma(\beta, y, F) \, .
\end{aligned}$$

The inequality (11) implies that for every $\varepsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that $v_k \leq v + \varepsilon$ for all $k \geq k_0$. Since $\beta_k \to \beta$, we may additionally assume that $\beta_k \leq \beta + \varepsilon$. Applying Lemma 2.5, we see that, after possibly enlarging $k_0$,

$$\begin{aligned}
\Phi_k(v_k) &\subset \Phi_{\mathcal{R}}(\beta + \varepsilon, y_k, F_k, v_k) \\
&\subset \Phi_{\mathcal{R}}(\beta + 2\varepsilon, y, F, v_k) \subset \Phi_{\mathcal{R}}(\beta + 2\varepsilon, y, F, v + \varepsilon)
\end{aligned} \tag{14}$$

for all $k \geq k_0$. Thus,

$$T = \tau\text{-Lim}\sup_k \Sigma_k = \bigcap_{k\in\mathbb{N}} \Big(\tau\text{-cl} \bigcup_{k'\geq k} \Sigma_{k'}\Big)$$
$$= \bigcap_{k\geq k_0} \Big(\tau\text{-cl} \bigcup_{k'\geq k} \Phi_{k'}(v_{k'})\Big) \subset \Phi_{\mathcal{R}}(\beta + 2\varepsilon, y, F, v + \varepsilon)\ . \quad (15)$$

The sets $\tau\text{-cl}\bigcup_{k'\geq k} \Sigma_{k'}$ are closed and non-empty and, by assumption, the set $\Phi_{\mathcal{R}}(\beta+2\varepsilon, y, F, v+\varepsilon)$ is compact. Thus $T$ is the intersection of a decreasing family of non-empty compact sets and therefore non-empty. Moreover, because (15) holds for every $\varepsilon > 0$, we have

$$\emptyset \neq T \subset \bigcap_{\varepsilon>0} \Phi_{\mathcal{R}}(\beta + 2\varepsilon, y, F, v + \varepsilon) = \Phi(v) = \Sigma\ . \quad (16)$$

Next we show the inclusion $T \subset \tau_{\mathcal{R}}\text{-Lim}\sup_k \Sigma_k$. To that end, we first prove that

$$v = \lim_k v_k\ . \quad (17)$$

Recall that Theorem 3.1 implies that $\Phi_k(v_k) = \Sigma_k$ is non-empty. Therefore, (14) implies that also $\Phi_{\mathcal{R}}(\beta + 2\varepsilon, y, F, v_k)$ is non-empty, which in turn shows that $v_k \geq v(\beta + 2\varepsilon, y, F)$ for all $k$ large enough. Consequently,

$$\liminf_k v_k \geq v(\beta + 2\varepsilon, y, F) \quad (18)$$

for all $\varepsilon > 0$. From Lemma 3.4 we obtain that $v = \sup_{\varepsilon>0} v(\beta + 2\varepsilon, y, F)$. Together with (18) and (11) this shows (17).

Let now $x \in N$, let $N$ be a neighborhood of $x$ with respect to $\tau$, let $\delta > 0$ and $k_0 \in \mathbb{N}$. Since $T \subset \Sigma$ (see (16)), it follows that $\mathcal{R}(x) = v$. Thus it follows from (17) that there exists $k_1 \geq k_0$ such that

$$|v_k - \mathcal{R}(x)| < \delta$$

for all $k \geq k_1$. In particular,

$$\Sigma_k \subset \big\{\tilde{x} \in X : \mathcal{R}(x) - \delta < \mathcal{R}(\tilde{x}) < \mathcal{R}(x) + \delta\big\} \quad (19)$$

for all $k \geq k_1$. Lemma 2.7 implies that there exists $k_2 \geq k_1$ such that

$$N \cap \Sigma_{k_2} \neq \emptyset\ . \quad (20)$$

Now recall that the sets $N \cap \big\{\tilde{x} \in X : \mathcal{R}(x) - \delta < \mathcal{R}(\tilde{x}) < \mathcal{R}(x) + \delta\big\}$ form a basis of neighborhoods of $x$ for the topology $\tau_{\mathcal{R}}$. Therefore (19), (20), and the characterization of the upper limit of sets given in Lemma 2.7 imply that $x \in \tau_{\mathcal{R}}\text{-Lim}\sup_k \Sigma_k$. Thus the inclusion (12) follows.

If the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$, then the first part of the assertion implies that for every subsequence $(k_j)_{j\in\mathbb{N}}$ we have

$$\tau_{\mathcal{R}}\text{-Lim}\sup_j \Sigma(\beta_{k_j}, y_{k_j}, F_{k_j}) = \{x_\beta\}\ .$$

Thus the assertion follows from Lemma 2.10. $\square$

The crucial assumption in Theorem 3.6 is the inequality (11). Indeed, one can easily construct examples, where this condition fails and the solution of Problem (3) is unstable (see Example 3.7 below). What happens in the example is that $\tau_{\mathcal{R}}$-$\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F)$ consists of local minima of $\mathcal{R}$ on $\Phi(\beta, y, F)$ that fail to be global minima of $\mathcal{R}$ restricted to $\Phi(\beta, y, F)$.

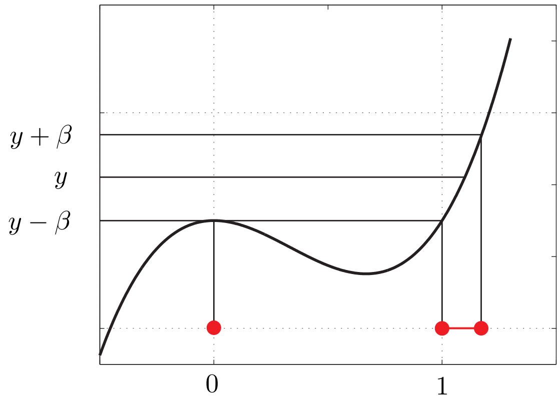


Figure 1: The nonlinear function $F$ from Example 3.7. The set $\Sigma(\beta, y, F)$ consists of an interval and the isolated point $\{0\}$.

**Example 3.7.** Consider the function $F\colon \mathbb{R} \to \mathbb{R}$, $F(x) = x^3 - x^2$, and the regularization functional $\mathcal{R}(x) = x^2$. Let $y > 0$ and choose $\beta = y$. Then

$$\arg\min\bigl\{\mathcal{R}(x) : |F(x) - y| \le \beta\bigr\} = \arg\min\bigl\{x^2 : |x^3 - x^2 - y| \le y\bigr\} = 0 \,. \quad (21)$$

Now let $y_k > y$. Then

$$\arg\min\bigl\{\mathcal{R}(x) : |F(x) - y_k| \le \beta\bigr\} = \arg\min\bigl\{x^2 : |x^3 - x^2 - y_k| \le y\bigr\} = x_k \,,$$

where $x_k$ is the unique solution of the equation $F(x) = y_k - y$. Thus, if the sequence $(y_k)_{k\in\mathbb{N}}$ converges to $y$ from above, we have $x_k > 1$ for all $k$ and $\lim_k x_k = 1$. According to (21), however, the solution of the limit problem equals zero. ∎

In the above example the solution is unstable, because the feasible set $\Phi(\beta, y, F)$ for $y$ contains elements that cannot be reached by the sets $\Phi(\beta_k, y_k, F)$. As a consequence, the limit of the sets $\Sigma(\beta_k, y_k, F)$ consists of local minima of the limit problem instead of global ones. The next result shows that by only slightly increasing the parameters $\beta_k$, the feasible sets $\Phi(\beta_k, y_k, F)$ becomes sufficiently large as to contain the solution set $\Sigma(\beta, y, F)$.

**Proposition 3.8 (Approximate Stability).** *Let Assumption 3.5 hold. Assume that $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$, the mappings $F_k\colon X \to Y$ converge locally $\mathcal{S}$-uniformly to $F\colon X \to Y$, and $\beta_k \to \beta$. Then there exists a sequence $\varepsilon_k \to 0$ such that*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k + \varepsilon_k, y_k, F_k) \subset \Sigma(\beta, y, F) \,.$$

*Proof.* Define

$$\varepsilon_k := \inf\bigl\{\varepsilon > 0 : \Phi_{\mathcal{R}}\bigl(\beta, y, F, v(\beta, y, F)\bigr) \subset \Phi_{\mathcal{R}}\bigl(\beta_k + \varepsilon, y_k, F_k, v(\beta, y, F)\bigr)\bigr\} \,.$$

Lemma 2.5 and the assumption that $\beta_k \to \beta$ imply that $\varepsilon_k \to 0$. Since by assumption

$$\emptyset \neq \Sigma(\beta, y, F) = \Phi_{\mathcal{R}}\bigl(\beta, y, F, v(\beta, y, F)\bigr) \subset \Phi_{\mathcal{R}}\bigl(\beta_k + \varepsilon_k, y_k, F_k, v(\beta, y, F)\bigr) \,,$$

we obtain that $v(\beta_k + \varepsilon_k, y_k, F_k) \leq v(\beta, y, F)$. Thus the assertion follows from Theorem 3.6. □

**Proposition 3.9 (Convergence).** *Let Assumption 3.5 hold. Assume that the sequence $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$ and $\mathcal{S}(y, y_k) \leq \beta_k \to 0$. Assume moreover that there exists $x \in X$ with $\mathcal{R}(x) < \infty$ and $F(x) = y$. Then*

$$\limsup_k v(\beta_k, y_k, F) \leq v(0, y, F) \,. \tag{22}$$

*In particular,*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F) \subset \Sigma(0, y, F) \,. \tag{23}$$

*If the set $\Sigma(0, y, F)$ consists of a single element $x^\dagger$, then*

$$\{x^\dagger\} = \tau_{\mathcal{R}}\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F) \,. \tag{24}$$

*Proof.* By assumption $S(y, y_k) \leq \beta_k$, which implies that $v(\beta_k, y_k, F) \leq \mathcal{R}(x')$ for all $x' \in \Phi(0, y, F)$. This proves (22). Now (23) and (24) follow from Theorem 3.6. □

**Proposition 3.10 (Stability).** *Let Assumption 3.5 hold. Assume that the sequence $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$, the mappings $F_k \colon X \to Y$ converge locally $\mathcal{S}$-uniformly to $F \colon X \to Y$, and $\beta_k \to \beta > 0$. Assume moreover that*

$$\Phi_{\mathcal{R}}(\beta, y, F, t) \subset \bigcap_{\delta>0}\Bigl(\tau\text{-cl}\bigcup_{\varepsilon>0}\Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t + \delta)\Bigr) \tag{25}$$

*for every $t \geq 0$. Then*

$$\limsup_k v(\beta_k, y_k, F_k) \leq v(\beta, y, F) \,. \tag{26}$$

*In particular,*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k) \subset \Sigma(\beta, y, F) \,. \tag{27}$$

*If the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$, then*

$$\{x_\beta\} = \tau_{\mathcal{R}}\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F_k) \,.$$

*Proof.* The convergence of $(\beta_k)_{k\in\mathbb{N}}$ to $\beta$ and Lemma 2.5 imply that for every $\varepsilon > 0$ and $t \in \mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that

$$\Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t) \subset \Phi_{\mathcal{R}}(\beta_k, y_k, F_k, t)$$

for all $k \geq k_0$. Consequently,

$$\begin{aligned}\limsup_k v(\beta_k, y_k, F_k) &= \limsup_k \ \inf\bigl\{t : \Phi_{\mathcal{R}}(\beta_k, y_k, F_k, t) \neq \emptyset\bigr\} \\ &\leq \inf_{\varepsilon>0} \inf\bigl\{t : \Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t) \neq \emptyset\bigr\} \ .\end{aligned}$$

From (25) we obtain that

$$\inf_{\varepsilon>0} \inf\bigl\{t : \Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t) \neq \emptyset\bigr\} \leq \inf\bigl\{t : \Phi_{\mathcal{R}}(\beta, y, F, t) \neq \emptyset\bigr\} = v(\beta, y, F) \ .$$

This shows (26). Now (27) follows from Theorem 3.6. □

# 4 Linear Spaces

Now we assume that $X$ and $Y$ are topological vector spaces. Then their linear structure allows us to introduce more tangible conditions implying stability of the residual method.

**Assumption 4.1.** *Assume that the following hold:*

1. *The sets $X$ and $Y$ are topological vector spaces.*
2. *For all $x_0$, $x_1 \in X$ with $\mathcal{S}\bigl(F(x_0), y\bigr)$, $\mathcal{S}\bigl(F(x_1), y\bigr) < \infty$, and all $0 < \lambda < 1$ we have*

   $$\mathcal{S}\bigl(F(\lambda x_0 + (1-\lambda)x_1), y\bigr) \leq \max\bigl\{\mathcal{S}\bigl(F(x_0), y\bigr), \mathcal{S}\bigl(F(x_1), y\bigr)\bigr\} \ .$$

   *Moreover, the inequality is strict for all $0 < \lambda < 1$ whenever $\mathcal{S}\bigl(F(x_0), y\bigr) \neq \mathcal{S}\bigl(F(x_1), y\bigr)$.*
3. *For every $\beta > 0$ there exists $x \in X$ with $\mathcal{S}\bigl(F(x), y\bigr) \leq \beta$ and $\mathcal{R}(x) < \infty$.*
4. *The domain $\operatorname{dom}\mathcal{R} = \bigl\{x \in X : \mathcal{R}(x) < +\infty\bigr\}$ of $\mathcal{R}$ is convex and for every $x_0$, $x_1 \in \operatorname{dom}\mathcal{R}$, the restriction of $\mathcal{R}$ to*

   $$L = \bigl\{\lambda x_0 + (1-\lambda)x_1 : 0 \leq \lambda \leq 1\bigr\}$$

   *is continuous.*

We now show that the Assumption 4.1 implies the main condition, the inclusion (25), of the stability result Proposition 3.10.

**Lemma 4.2.** *Assume that Assumption 4.1 holds. Then* (25) *is satisfied.*

*Proof.* Let $x_0 \in \Phi_{\mathcal{R}}(\beta, y, F, t)$ for some $\beta > 0$. We have to show that for every neighborhood $N \subset X$ of $x_0$ and every $\delta > 0$ there exist $\varepsilon > 0$ and $x' \in N$ such that $x' \in \Phi_{\mathcal{R}}(\beta - \varepsilon, y, F, t + \delta)$.

Item 3 in Assumption 4.1 implies the existence of some $x_1 \in X$ satisfying $\mathcal{S}\big(F(x_1), y\big) < \beta$ and $\mathcal{R}(x_1) < \infty$. Since $\mathcal{S}\big(F(x_1), y\big) < \beta$ and $\mathcal{S}\big(F(x_0), y\big) \leq \beta$, we obtain from Item 2 that $\mathcal{S}(F(x), y) < \beta$ for every $x \in L := \big\{\lambda x_0 + (1-\lambda)x_1 : 0 \leq \lambda \leq 1\big\}$. Since $x_0$, $x_1 \in \operatorname{dom} \mathcal{R}$, it follows from Item 4 that $\mathcal{R}$ is continuous on $L$. Consequently $\lim_{\lambda \to 1} \mathcal{R}(\lambda x_0 + (1-\lambda)x_1) = \mathcal{R}(x_0) \leq t$. In particular, there exists $\lambda_0 < 1$ such that $\mathcal{R}(\lambda x_0 + (1-\lambda)x_1) \leq t + \delta$ for all $1 > \lambda > \lambda_0$. Since $X$ is a topological vector space (Item 1), it follows that $x' := \lambda x_0 + (1-\lambda)x_1 \in N$ for some $1 > \lambda > \lambda_0$. This shows the assertion with $\varepsilon := \beta - \mathcal{S}\big(F(x'), y\big) > 0$.□

Lemma 4.2 allows us to apply the stability result Proposition 3.10, which shows that Assumption 4.1 implies the continuous dependence of the solutions of (3) on the data $y$ and the regularization parameter $\beta$.

**Proposition 4.3 (Stability & Convergence).** *Let Assumption 4.1 hold and assume that $\Phi_{\mathcal{R}}(\gamma, w, F, t)$ is compact for every $\gamma \geq 0$, $t \in \mathbb{R}$, and $w \in Y$. Assume moreover that $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$, and $\beta_k \to \beta$. If $\beta = 0$, assume in addition that $\mathcal{S}(y, y_k) \leq \beta_k$. Then*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F) \subset \Sigma(\beta, y, F) .$$

*If the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$, then*

$$\{x_\beta\} = \tau_{\mathcal{R}}\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F) .$$

*Proof.* If $\beta = 0$, the assertion follows from Proposition 3.9. In the case $\beta > 0$, Lemma 4.2 implies that (25) holds. Thus, the assertion follows from Proposition 3.10. □

**Proposition 4.4 (Stability).** *Let Assumption 4.1 hold. Assume that $(y_k)_{k\in\mathbb{N}}$ converges $\mathcal{S}$-uniformly to $y \in Y$, the mappings $F_k \colon X \to Y$ converge locally $\mathcal{S}$-uniformly to $F \colon X \to Y$ (see Definition 2.3), and $\beta_k \to \beta > 0$. Assume that the sets $\big\{x \in X : \mathcal{R}(x) \leq t\big\}$, $\Phi_{\mathcal{R}}(\gamma, w, F_k, t)$ and $\Phi_{\mathcal{R}}(\gamma, w, F, t)$ are compact for every $\gamma \geq 0$, $t \in \mathbb{R}$, and $w \in Y$. Then*

$$\emptyset \neq \tau_{\mathcal{R}}\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F_k) \subset \Sigma(\beta, y, F) .$$

*If the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$, then*

$$\{x_\beta\} = \tau_{\mathcal{R}}\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F_k) .$$

*Proof.* Again, Lemma 4.2 shows that (25) holds. Thus the assertion follows from Proposition 3.10. □

Item 2 in Assumption 4.1 is concerned with the interplay of the functional $F$ and the distance measure $\mathcal{S}$. The next two examples consider two situations, where this part of the assumption holds. Example 4.5 considers linear operators $F$ and convex distance measures $\mathcal{S}$. Example 4.6 introduces a class of *non-linear* operators on Hilbert spaces, where Item 2 is satisfied if the distance measure equals the squared Hilbert space norm.

**Example 4.5.** Assume that $F\colon X \to Y$ is linear and $\mathcal{S}$ is convex in its first component. Then Item 2 in Assumption 4.1 is satisfied. Indeed, in this case

$$\begin{aligned}&\mathcal{S}\big(F(\lambda x_0 + (1-\lambda)x_1), y\big) = \mathcal{S}\big(\lambda F(x_0) + (1-\lambda)F(x_1), y\big)\\ &\quad\leq \lambda\mathcal{S}\big(F(x_0), y\big) + (1-\lambda)\mathcal{S}\big(F(x_1), y\big) \leq \max\big\{\mathcal{S}\big(F(x_0), y\big), \mathcal{S}\big(F(x_1), y\big)\big\}\ .\end{aligned}$$

If moreover, $\mathcal{S}\big(F(x_0), y\big) \neq \mathcal{S}\big(F(x_1), y\big)$ and $0 < \lambda < 1$, then the last inequality is strict. ∎

**Example 4.6.** Assume that $Y$ is a Hilbert space, $\mathcal{S}(y,z) = \|y-z\|$, and $F\colon X \to Y$ is two times Gâteaux differentiable. Then Item 2 in Assumption 4.1 is equivalent to the assumption that for all $x_0$, $x_1 \in X$ the mapping

$$t \mapsto T(t; x_0, x_1) := \|F(x_0 + tx_1) - y\|^2$$

has no local maxima. This condition holds, if $\partial_t^2 T(0; x_0, x_1) > 0$ whenever $\partial_t T(0; x_0, x_1) = 0$. The computation of the derivatives of $T(\,\cdot\,; x_0, x_1)$ at zero yields that

$$\partial_t T(0; x_0, x_1) = 2\big\langle F'(x_0)(x_1), F(x_0)\big\rangle$$

and

$$\partial_t^2 T(0; x_0, x_1) = 2\big\langle F''(x_0)(x_1; x_1), F(x_0)\big\rangle + 2\big\|F'(x_0)x_1\big\|^2\ .$$

Consequently, Item 2 in Assumption 4.1 is satisfied if, for every $x_0$, $x_1 \in X$ with $x_1 \neq 0$, the equality $\big\langle F'(x_0)(x_1), F(x_0)\big\rangle = 0$ implies that

$$\big\langle F''(x_0)(x_1; x_1), F(x_0)\big\rangle + \big\|F'(x_0)(x_1)\big\|^2 > 0\ .$$ ∎

**Example 4.7.** Let $p > 1$ and $X = L^p(\Omega, \mu)$ for some $\sigma$-finite measure space $(\Omega, \mu)$. Assume that $Y$ is a Banach space and $F\colon X \to Y$ is a bounded linear operator with dense range. Let $\mathcal{R}(x) = \|x\|_p^p$ and $\mathcal{S}(w, y) = \|w - y\|$. We thus consider the minimization problem

$$\|x\|_p^p \to \min \qquad \text{subject to} \qquad \|Fx - y\| \leq \beta\ .$$

We now show that in this situation the assumptions of Proposition 4.3 are satisfied. To that end, let $\tau$ be the weak topology on $L^p(\Omega, \mu)$. As $L^p(\Omega, \mu)$ is reflexive, the level sets $\big\{x \in X : \mathcal{R}(x) \leq t\big\}$ are weakly compact. Moreover, the mapping $x \mapsto \|Fx - y\|$ is weakly lower semi-continuous. Thus all the sets $\Phi_{\mathcal{R}}(\gamma, w, F, t)$ are weakly compact. Example 4.5 shows that Item 2 in Assumption 4.1 holds. Item 3 follows from the density of the range of $F$. Finally, Item 4 holds, because $\mathcal{R}$ is norm continuous and convex.

Now assume that $y_k \to y$ and $\beta_k \to \beta$. If $\beta = 0$ assume in addition that $\|y_k - y\| \leq \beta_k$. The strict convexity of $\mathcal{R}$ and convexity of the mappings $x \mapsto \|Fx - y_k\|$ imply that each set $\Sigma(\beta_k, y_k, F)$ consists of a single element $x_k$. Similarly, $\Sigma(\beta, y, F)$ consists of a single element $x^\dagger$. From Proposition 4.3 we now obtain that $(x_k)_{k\in\mathbb{N}}$ weakly converges to $x^\dagger$ and $\|x_k\|_p^p \to \|x^\dagger\|_p^p$. Thus, in fact, the sequence $(x_k)_{k\in\mathbb{N}}$ strongly converges to $x^\dagger$ (see [34, Cor. 5.2.19]).

Let $\beta > 0$ and assume that $F_k \colon X \to Y$ is a sequence of bounded linear operators converging to $F$ with respect to the strong topology on $L(X, Y)$, that is, $\sup\bigl\{\|F_k x - Fx\| : \|x\| \leq 1\bigr\} \to 0$. Let again $\beta_k \to \beta$ and $y_k \to y$, and denote by $x_k$ the single element in $\Sigma(\beta_k, y_k, F_k)$. Applying Proposition 4.4, we again obtain that $x_k \to x^\dagger$. ■

**Remark 4.8.** Example 4.7 relies heavily on the assumption that $p > 1$, which implies that the space $L^p(\Omega, \mu)$ is reflexive. In the case $X = L^1(\Omega, \mu)$, the level sets $\bigl\{x \in X : \|x\|_1 \leq t\bigr\}$ fail to be weakly compact, and thus even the existence of a solution of Problem (3) need not hold. ■

**Remark 4.9.** The assertions of Example 4.7 concerning stability and convergence with respect to the norm topology remain valid, if $X$ is any uniformly convex Banach space and $\mathcal{R}$ the norm on $X$ to some power $p > 1$. Also in this case, weak convergence and convergence of norms imply the strong convergence of a sequence [34, Thm. 5.2.18]. More generally, this property is called the *Radon–Riesz property* [34, p. 453]. Spaces satisfying this property are also called *Efimov–Stechkin* spaces in [45]. ■

# 5 Convergence Rates

In this section we derive quantitative estimates (convergence rates) for the difference between regularized solutions $x_\beta \in \Sigma(\beta, y, F)$ and the exact solution of the equation $F(x^\dagger) = y^\dagger$.

For Tikhonov regularization, convergence rates have been derived in [4, 26, 39] in terms of the *Bregman distance*. However, its classical definition,

$$D_\xi(x, x^\dagger) = \mathcal{R}(x) - \bigl(\mathcal{R}(x^\dagger) + \langle \xi, x - x^\dagger \rangle_{X^*,X}\bigr)\,,$$

where $\xi \in \partial\mathcal{R}(x^\dagger)$, requires the space $X$ to be linear and the functional $\mathcal{R}$ to be convex, as the subdifferential $\partial\mathcal{R}(x^\dagger)$ is only defined for convex functionals. In the sequel we will extend the notion of Bregman distances to work for arbitrary functionals $\mathcal{R}$ on arbitrary sets $X$.

**Definition 5.1 (Generalized Bregman Distance).** Let $\mathcal{R} \colon X \to \mathbb{R} \cup \{+\infty\}$ and let

$$x^\dagger \in \operatorname{dom}(\mathcal{R}) := \bigl\{x \in X : \mathcal{R}(x) < \infty\bigr\}$$

be an element in its domain.

A functional $\mathcal{T}\colon X \to \mathbb{R} \cup \{+\infty\}$ is called a *Bregman tangent* for $\mathcal{R}$ at $x^\dagger$ if $\mathcal{T}(x^\dagger) = 0$, $\mathrm{dom}(\mathcal{R}) \subset \mathrm{dom}(\mathcal{T})$, and the mapping

$$x \mapsto D_{\mathcal{T}}(x, x^\dagger) := \begin{cases} \mathcal{R}(x) - \big(\mathcal{R}(x^\dagger) + \mathcal{T}(x)\big), & \text{if } x \in \mathrm{dom}(\mathcal{R})\,, \\ +\infty, & \text{if } x \in X \setminus \mathrm{dom}(\mathcal{R})\,, \end{cases} \tag{28}$$

is non-negative. The mapping $D_{\mathcal{T}}(\,\cdot\,, x^\dagger)$ is called the *Bregman distance* corresponding to $\mathcal{T}$. ■

**Remark 5.2.** Let $X$ be a Banach space, let $\mathcal{R}$ be convex, and let $\xi\colon X \to \mathbb{R}$ be a bounded linear functional. Then the mapping

$$\mathcal{T}_\xi(x) := \langle \xi, x - x^\dagger \rangle \tag{29}$$

a is a Bregman tangent for $\mathcal{R}$ at $x^\dagger$, if and only if $\xi \in \partial\mathcal{R}(x^\dagger)$. Thus, the standard Bregman distance $D_\xi = D_{\mathcal{T}_\xi}$ is indeed a special case of our generalized notion. ■

Convergence rates in terms of the Bregman distance $D_{\mathcal{T}}$ will be obtained under the following assumption:

**Assumption 5.3.**

1. *There exists a monotonically increasing function* $\theta : [0,\infty) \to [0,\infty)$ *such that*
$$\mathcal{S}(w_1, w_2) \le \theta\big(\mathcal{S}(w_1, w_3) + \mathcal{S}(w_2, w_3)\big) \tag{30}$$
*for all* $w_1$, $w_2$, $w_3 \in Y$.

2. *There exists an element* $x^\dagger \in \mathrm{dom}(\mathcal{R})$*, a Bregman tangent* $\mathcal{T}$ *for* $\mathcal{R}$ *at* $x^\dagger$*, and constants* $\gamma_1 \in [0,1)$ *and* $\gamma_2 \ge 0$ *such that*
$$-\mathcal{T}(x) \le \gamma_1 D_{\mathcal{T}}(x, x^\dagger) + \gamma_2 \mathcal{S}\big(F(x), F(x^\dagger)\big) \tag{31}$$
*for every* $x \in \Phi_{\mathcal{R}}\big(\theta(2\beta), F(x^\dagger), F, \mathcal{R}(x^\dagger)\big)$.

**Remark 5.4.** In a Banach space setting (see Subsection 5.1 below), the *source inequality* (31) has already been used in [26, 42] to derive convergence rates for Tikhonov regularization with convex functionals. ■

**Theorem 5.5 (Convergence Rates).** *Let Assumption 5.3 hold. Then*

$$\sup\big\{D_{\mathcal{T}}(x_\beta, x^\dagger) : x_\beta \in \Sigma(\beta, y, F)\big\} \le \frac{\gamma_2}{1-\gamma_1}\, \theta\big(\beta + \mathcal{S}\big(F(x^\dagger), y\big)\big)\,. \tag{32}$$

*for all* $y \in Y$ *with* $\mathcal{S}\big(F(x^\dagger), y\big) \le \beta$.

*Proof.* Let $x_\beta \in \Sigma(\beta, y, F)$. This, together with (30) and the assumption that $\mathcal{S}\big(F(x^\dagger), y\big) \le \beta$, implies that

$$\mathcal{S}\big(F(x_\beta), F(x^\dagger)\big) \le \theta\big(\mathcal{S}\big(F(x_\beta), y\big) + \mathcal{S}\big(F(x^\dagger), y\big)\big) \le \theta(2\beta)\,.$$

Thus we can apply (31) and the definition of the Bregman distance, to deduce that

$$
\begin{aligned}
-\mathcal{T}(x_\beta) &\le \gamma_1 D_{\mathcal{T}}(x_\beta, x^\dagger) + \gamma_2 \mathcal{S}\big(F(x_\beta), F(x^\dagger)\big) \\
&= \gamma_1\big(\mathcal{R}(x_\beta) - \mathcal{R}(x^\dagger) - \mathcal{T}(x_\beta)\big) + \gamma_2 \mathcal{S}\big(F(x_\beta), F(x^\dagger)\big)\,.
\end{aligned}
$$

Together with the assumption $\gamma_1 \in [0,1)$ this shows the inequality

$$
-\mathcal{T}(x_\beta) \le \frac{\gamma_1}{1-\gamma_1}\big(\mathcal{R}(x_\beta) - \mathcal{R}(x^\dagger)\big) + \frac{\gamma_2}{1-\gamma_1}\,\mathcal{S}\big(F(x_\beta), F(x^\dagger)\big)\,. \tag{33}
$$

Since $\mathcal{S}\big(F(x^\dagger), y\big) \le \beta$, it follows that $\mathcal{R}(x_\beta) - \mathcal{R}(x^\dagger) \le 0$. Therefore (33) implies that

$$
D_{\mathcal{T}}(x_\beta, x^\dagger) \le \frac{\gamma_2}{1-\gamma_1}\,\mathcal{S}\big(F(x_\beta), F(x^\dagger)\big)\,. \tag{34}
$$

Consequently we obtain from (30) and the estimate $\mathcal{S}\big(F(x_\beta), y\big) \le \beta$ the required inequality

$$
D_{\mathcal{T}}(x_\beta, x^\dagger) \le \frac{\gamma_2}{1-\gamma_1}\,\theta\big(\beta + \mathcal{S}\big(F(x^\dagger), y\big)\big)\,. \qquad \square
$$

**Remark 5.6.** Typically, convergence rates are formulated in a setting which slightly differs from the one of Theorem 5.5 (see [4, 18, 26, 42]). There one assumes the existence of an $\mathcal{R}$-minimizing solution $x^\dagger \in X$ of the equation $F(x^\dagger) = y^\dagger$, for some exact data $y^\dagger \in Y$. Instead of $y^\dagger$, only noisy data $y \in Y$ and the error bound $\mathcal{S}(y^\dagger, y) \le \beta$ are given. For this setting, (32) implies the rate

$$
D_{\mathcal{T}}(x_\beta, x^\dagger) \le \frac{\gamma_2}{1-\gamma_1}\,\theta(2\beta) = \mathcal{O}\big(\theta(2\beta)\big) \qquad \text{as } \beta \to 0\,,
$$

where $x_\beta \in \Sigma(\beta, y, F)$ denotes any regularized solution. ■

## 5.1 Convergence rates in Banach spaces

In the following, assume that $X$ and $Y$ are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and set $\mathcal{S}(y,z) := \|y - z\|_Y$. Let $\mathcal{R}$ be a convex and lower semi-continuous functional on $X$, and let $D_\xi := D_{\mathcal{T}_\xi}$ with $\xi \in \partial\mathcal{R}(x^\dagger)$ denote the classical Bregman distance (see Remark 5.2).

Given data $y$ satisfying $\|F(x^\dagger) - y\| \le \beta$, Theorem 5.5 implies the convergence rate $D_\xi(x_\beta, x^\dagger) = \mathcal{O}(\beta)$, where $x_\beta \in \Sigma(\beta, y, F)$ is a regularized solution and $x^\dagger$ satisfies the source inequality

$$
-\langle \xi, x - x^\dagger\rangle \le \gamma_1 D_\xi(x, x^\dagger) + \gamma_2 \mathcal{S}\big(F(x), F(x^\dagger)\big)\,. \tag{35}
$$

Equation (35) has already been used in [26] to derive convergence rates for Tikhonov regularization.

In the special case where $X$ is a Hilbert space and $\mathcal{R}(x) = \|x\|_X^2/2$ we have $D_\xi(x, x^\dagger) = \|x - x^\dagger\|_X^2/2$, which implies the rate $\mathcal{O}(\sqrt{\beta})$ with respect to the norm. In Proposition 5.8 below we show that this rate holds on any 2-convex space. For $r$-convex Banach spaces with $r > 2$, we derive the rate $\mathcal{O}(\delta^{1/r})$.

**Remark 5.7.** The book [42, pp. 70ff] clarifies the relation between (35) and the source conditions used to derive convergence rates for convex functionals on Banach spaces (see [4, 39]). In particular, it is shown that, if $F$ and $\mathcal{R}$ are Gâteaux differentiable at $x^\dagger$ and there exist $\gamma > 0$ and $\omega \in Y^*$ such that $\gamma\|\omega\|_{Y^*} < 1$ and

$$\xi = F'(x^\dagger)^*\,\omega \in \partial\mathcal{R}(x^\dagger)\,, \tag{36}$$

$$\big\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\big\|_Y \le \gamma D_\xi(x, x^\dagger) \tag{37}$$

for every $x \in X$, then (35) holds on $X$. Here $F'(x^\dagger)^* : Y^* \to X^*$ is the adjoint of $F'(x^\dagger)$. Conversely, if $\xi \in \partial\mathcal{R}(x^\dagger)$ satisfies (35), then (36) holds for every $x \in X$.

In the particular case that $F\colon X \to Y$ is linear and bounded, the inequality (37) is trivially satisfied with $\gamma = 0$. Thus, (35) is equivalent to the sourcewise representability of the subgradient,

$$\xi \in \partial\mathcal{R}(x^\dagger) \cap \operatorname{ran}(F^*)\,. \tag{38}$$

Here $\operatorname{ran}(F^*) = \big\{F'(x^\dagger)^*\omega : \omega \in X^*\big\}$ denotes the range of $F'(x^\dagger)^*$. ■

Recall that the Banach space $X$ is called $r$-convex (or is said to have modulus of convexity of power type $r$), if there exists a constant $C > 0$ such that

$$\inf\big\{1 - \|(x+y)/2\| : \|x\| = \|y\| = 1,\ \|x - y\| \ge \epsilon\big\} \ge C\varepsilon^r$$

for all $\varepsilon \in [0, 2]$. Notice that all Hilbert spaces are 2-convex and that there is *no* Banach space (of dimension $\ge 2$) that is $r$-convex for some $r < 2$ (see [32, pp. 63ff]).

**Proposition 5.8 (Convergence rates in the norm).** *Let $X$ be an $r$-convex Banach space with $r \ge 2$ and let $\mathcal{R}(x) := \|x\|_X^r/r$. Assume that there exists $x^\dagger \in X$, a subgradient $\xi \in \partial\mathcal{R}(x^\dagger)$, and constants $\gamma_1 \in [0, 1)$, $\gamma_2 \ge 0$ such that (35) holds for every $x \in \Phi_{\mathcal{R}}\big(2\beta, F(x^\dagger), F, \mathcal{R}(x^\dagger)\big)$.*

*Then there exists a constant $c > 0$ such that*

$$\sup\big\{\|x_\beta - x^\dagger\| : x_\beta \in \Sigma(\beta, y, F)\big\} \le c\big(\beta + \|F(x^\dagger) - y\|\big)^{1/r} \tag{39}$$

*for all $\beta > 0$ and $y \in Y$ with $\|F(x^\dagger) - y\| \le \beta$.*

*Proof.* Let $J_r : X \to 2^{X^*}$ denote the duality mapping with respect to the weight function $s \mapsto s^{r-1}$. In [49, Equation (2.17)′] it is shown that there exists a constant $K > 0$ such that

$$\|x^\dagger + z\|_X^r \ge \|x^\dagger\|_X^r + r\langle j_r(x^\dagger), z\rangle_{X,X^*} + K\|z\|_X^r \tag{40}$$

for all $x^\dagger$, $z \in X$ and $j_r(x^\dagger) \in J_r(x^\dagger)$. By Asplund's theorem [11, Chap. 1, Thm. 4.4], the subgradient of $\mathcal{R}(x) = \|x\|^r/r$ equals the duality mapping $J_r$. Therefore, by taking $z = x - x^\dagger$ and $j_r(x^\dagger) = \xi$, inequality (40) implies

$$D_\xi(x, x^\dagger) \ge K/r\ \|x - x^\dagger\|_X^r \qquad \text{for all } x^\dagger, x \in X \text{ and } \xi \in \partial\mathcal{R}(x^\dagger)\,. \tag{41}$$

Consequently, (39) follows from Theorem 5.5. □

Exact values for the constant $K$ in (41) (and thus for the constant $c$ in (39)) can be derived from [49]. Bregman distances satisfying (41) are called $r$-coercive in [25]. This $r$-coercivity has already been applied in [3] for the minimization of Tikhonov functionals in Banach spaces.

**Example 5.9.** The spaces $X = L^p(\Omega, \mu)$ for $p \in (1, 2)$ and some $\sigma$-finite measure space $(\Omega, \mu)$ are examples of 2-convex Banach spaces (see [32, p. 81, Remarks following Theorem 1.f.1.]). Consequently we obtain for these spaces the convergence rate $\mathcal{O}(\sqrt{\beta})$. The spaces $X = L^p(\Omega, \mu)$ for $p \geq 2$ are only $p$-convex, leading to the rate $\mathcal{O}(\beta^{1/p})$ in those spaces. ■

# 6 An Application: Sparse Regularization

Let $\Lambda$ be an at most countable index set, define

$$\ell^2(\Lambda) := \Big\{ x = (x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R} : \sum_\lambda |x_\lambda|^2 < \infty \Big\} ,$$

and assume that $F \colon X := \ell^2(\Lambda) \to Y$ is a bounded linear operator with dense range in the Hilbert space $Y$. We consider for $p \in (0, 2)$ the minimization problem

$$\mathcal{R}_p(x) := \|x\|^p_{\ell^p(\Lambda)} := \sum_{\lambda \in \Lambda} |x_\lambda|^p \to \min \qquad \text{subject to} \qquad \|Fx - y\|^2_Y \leq \beta \,. \tag{42}$$

For $p > 1$, the subdifferential $\partial \mathcal{R}_p(x^\dagger)$ is at most single valued and is identified with its single element.

**Remark 6.1 (Compressed Sensing).** In a finite dimensional setting with $p = 1$, the minimization problem (42) has received a lot of attention during the last years under the name of *compressed sensing* (see [5, 6, 8, 14, 15, 16, 21, 47]). Under some assumptions, the solution of (42) with $y = Fx^\dagger$ and $\beta = 0$ has been shown to recover $x^\dagger$ *exactly* provided the set $\{\lambda \in \Lambda : x^\dagger_\lambda \neq 0\}$ has sufficiently small cardinality (that is named, it is sufficiently sparse). Results for $p < 1$ can be found in [9, 13, 20, 40]. ■

In this section we prove well-posedness of (42) and derive convergence rates in a possibly infinite dimensional setting. This inverse problems point of view has so far only been considered for Tikhonov regularization

$$\mathcal{R}_p(x) + \alpha \|Fx - y\|^2_Y \to \min$$

(see [10, 12, 22, 23, 33, 37, 51]).

In the following $\tau$ denotes the weak topology on $\ell^2(\Lambda)$, and $\tau_p := \tau_{\mathcal{R}_p}$ denotes the topology as in Definition 3.3. Then a sequence $(x_k)_{k \in \mathbb{N}} \subset \ell^2(\lambda)$ converges to $x \in \ell^2(\lambda)$ with respect to $\tau_p$ if and only if $x_k \rightharpoonup x$ and $\mathcal{R}_p(x_k) \to \mathcal{R}(x)$. As shown in [23, Lemma 2] this already implies $\mathcal{R}_p(x_k - x) \to 0$. In particular, the topology $\tau_p$ is stronger than the topology induced by $\|\cdot\|_{\ell^2(\Lambda)}$.

## 6.1 Convex $\ell^p$ Regularization

We first assume that $p \in [1, 2)$, in which case the functional $\mathcal{R}_p$ is convex.

**Proposition 6.2 (Well-Posedness).** *Let $1 \le p < 2$ and let $F \colon \ell^2(\Lambda) \to Y$ be a bounded linear operator with dense range. Then the following hold:*

1. Existence: *For every $\beta > 0$ and $y \in Y$, the set of regularized solutions $\Sigma(\beta, y, F)$ is non-empty.*
2. Stability: *Let $(\beta_k)$ and $(y_k)$ be sequences with $\beta_k \to \beta > 0$ and $y_k \to y \in Y$. Then $\emptyset \neq \tau_p\text{-}\operatorname{Lim\,sup}_k N(\beta_k, y_k, F) \subset N(\beta, y, F)$.*
3. Convergence: *Let $\|y_k - y\|_Y \le \beta_k \to 0$ and assume that the equation $Fx = y$ has a solution in $\ell^p(\Lambda)$. Then $\emptyset \neq \tau_p\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F) \subset \Sigma(0, y, F)$. If the equation $Fx = y$ has a unique $\mathcal{R}_p$-minimizing solution $x^\dagger$, then $\tau_p\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F) = \{x^\dagger\}$.*

**Remark 6.3.** In the case $p > 1$, the functional $\mathcal{R}_p$ is strictly convex, and therefore the $\mathcal{R}_p$-minimizing solution $x^\dagger$ of $Fx = y$ is unique. Consequently the equality $\tau_p\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F) = \{x^\dagger\}$ holds for every $y$ in the range of the operator $F$. ■

*Proof (of Proposition 6.2).* In order to prove the existence of minimizers, we apply Theorem 3.1 by showing that $\Phi_{\mathcal{R}}(\beta, y, F, t)$ is compact with respect to the weak topology on $\ell^2(\Lambda)$ for every $t > 0$ and is nonempty for some $t$. Because $F$ has dense range, the set

$$\Phi_{\mathcal{R}}(\beta, y, F, t) = \left\{x \in \ell^2(\Lambda) : \mathcal{R}_p(x) \le t, \|F(x) - y\|_Y^2 \le \beta\right\}$$

is non-empty for $t$ large enough. It remains to show that the sets $\Phi_{\mathcal{R}}(\beta, y, F, t)$ are weakly compact on $\ell^2(\lambda)$ for every positive $t$.

The functional $\mathcal{R}_p(x) = \sum_{\lambda \in \Lambda} |x_\lambda|^p$ is weakly lower semi-continuous (on $\ell^2(\lambda)$) as the sum of non-negative and weakly continuous functionals (see [17]). Moreover, the mapping $F$ is weakly continuous, and therefore $x \mapsto \|Fx - y\|_Y^2$ is weakly lower semi-continuous, too. The estimate $\mathcal{R}_p(x) \ge \|x\|_{\ell^2(\Lambda)}^p$ (see [23, Equation (5)]) shows that $\mathcal{R}_p$ is weakly coercive. Therefore the sets $\Phi_{\mathcal{R}}(\beta, y, F, t)$ are weakly compact for all $t > 0$ (cf. Remark 3.2). Thus, Theorem 3.1 shows that $\Sigma(\beta, y, F) \neq \emptyset$.

Taking into account Example 4.5, it follows that $\mathcal{R}_p$, $\mathcal{S}$, and $F$ satisfy Assumption 4.1. Consequently, Items 2 and 3 follow from Proposition 4.3. □

In the following, we derive two types of convergence rates results with respect the $\ell^2$-norm: the rate $\mathcal{O}(\sqrt{\delta})$ (for $p \in (1, 2)$) and the rate $\mathcal{O}(\delta^{1/p})$ (for every $p \in [1, 2)$) for sparse sequences—here and in the following, $x^\dagger \in \ell^2(\Lambda)$ is called sparse, if

$$\operatorname{supp}(x^\dagger) := \left\{\lambda \in \Lambda : x_\lambda^\dagger \neq 0\right\}$$

is finite. The same type of results has also been obtained for sparse Tikhonov regularization in [23, 42].

**Proposition 6.4.** *Let $1 < p < 2$, $x^\dagger = (x_\lambda^\dagger)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$, and let $F\colon \ell^2(\Lambda) \to Y$ be a bounded linear operator. Moreover, assume that there exists $\omega \in Y$ with $\partial\mathcal{R}_p(x^\dagger) = F^*\omega$.*

*Then there exists a constant $d_p > 0$ only depending on $p$ such that*

$$\begin{aligned}\sup\bigl\{\|x_\beta - x^\dagger\|^2_{\ell^2(\Lambda)} : x_\beta \in \Sigma(\beta, y, F)\bigr\} \\ \leq \frac{d_p\|\omega\|_Y}{3 + 2\mathcal{R}_p(x^\dagger)}\bigl(\beta + \|Fx^\dagger - y\|_Y\bigr)\end{aligned} \tag{43}$$

*for all $\beta > 0$ and $y \in Y$ with $\|F(x^\dagger) - y\|_Y \leq \beta$.*

*Proof.* The assumption $\partial\mathcal{R}_p(x^\dagger) = F^*\omega$ implies that

$$-\langle\partial\mathcal{R}_p(x^\dagger), x - x^\dagger\rangle \leq \bigl|\langle\partial\mathcal{R}_p(x^\dagger), x - x^\dagger\rangle\bigr| \leq \|\omega\|_Y\|F(x - x^\dagger)\|_Y\ . \tag{44}$$

Thus (31) holds with $\gamma_1 = 0$ and $\gamma_2 = \|\omega\|_Y$. Theorem 5.5 therefore implies the inequality

$$\sup\bigl\{D_{\partial\mathcal{R}_p(x^\dagger)}(x_\beta, x^\dagger) : x_\beta \in \Sigma(\beta, y, F)\bigr\} \leq \|\omega\|_Y\bigl(\beta + \|Fx^\dagger - y\|_Y\bigr)\ . \tag{45}$$

From [23, Lemma 10] we obtain the inequality

$$\|x - x^\dagger\|^2_{\ell^2(\Lambda)} \leq \frac{d_p}{3 + 2\mathcal{R}_p(x^\dagger) + \mathcal{R}_p(x)} D_{\partial\mathcal{R}_p(x^\dagger)}(x, x^\dagger) \tag{46}$$

for all $x \in \operatorname{dom}(\mathcal{R}_p)$. Now, (43) follows from (45) and (46). □

**Remark 6.5.** Since $\ell^p(\Lambda)$ is 2-convex (see [32]) and continuously embedded in $\ell^2(\Lambda)$, Proposition 5.8 provides an alternative estimate for $x_\beta - x^\dagger$ in terms of the stronger distance $\|\cdot\|_{\ell^p(\Lambda)}$. The prefactor in (39), however, is constant, whereas the prefactor in (43) tends to 0 as $\mathcal{R}_p(x^\dagger)$ increases. Thus the two estimates are somehow independent from each other. ■

**Proposition 6.6 (Sparse Case, $p > 1$).** *Let $p \in (1, 2)$, let $x^\dagger = (x_\lambda^\dagger)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ be sparse, and let $F\colon \ell^2(\Lambda) \to Y$ be bounded linear. Moreover, assume that there exists $\omega \in Y$ with $\partial\mathcal{R}_p(x^\dagger) = F^*\omega$ and that $F$ is injective on*

$$V := \bigl\{x \in \ell^2(\Lambda) : \operatorname{supp}(x) \subset \operatorname{supp}(x^\dagger)\bigr\}\ .$$

*Then*

$$\sup\bigl\{\|x_\beta - x^\dagger\|_{\ell^2(\Lambda)} : x_\beta \in \Sigma(\beta, y, F),\ \|Fx^\dagger - y\|_Y \leq \beta\bigr\} = \mathcal{O}(\beta^{1/p}) \quad \text{as } \beta \to 0\ . \tag{47}$$

*Proof.* The injectivity of $F$ on the finite dimensional space $V$ implies the existence of a constant $C > 0$ such that

$$C\|F(x)\|_Y \geq \|x\|_{\ell^2(\Lambda)}\ , \qquad \text{for all } x \in V\ .$$

Denote by $\pi$, $\pi^\perp : X \to X$ the projections

$$\pi x := \sum_{\lambda \in \mathrm{supp}(x^\dagger)} x_\lambda \, e_\lambda \,, \qquad \pi^\perp x := \sum_{\lambda \notin \mathrm{supp}(x^\dagger)} x_\lambda \, e_\lambda \,.$$

By means of the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for $a$, $b \ge 0$, it follows that for every $x \in \ell^2(\Lambda)$

$$\begin{aligned} \|x - x^\dagger\|^p_{\ell^2(\Lambda)} &\le 2^{p-1} \left\|\pi(x - x^\dagger)\right\|^p_{\ell^2(\Lambda)} + 2^{p-1}\left\|\pi^\perp x\right\|^p_{\ell^2(\Lambda)} \\ &\le 2^{p-1} C^p \|F\big(\pi(x - x^\dagger)\big)\|^p_Y + 2^{p-1} \left\|\pi^\perp x\right\|^p_{\ell^2(\Lambda)} . \end{aligned} \tag{48}$$

Applying the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ a second time shows that

$$\begin{aligned} \|F\big(\pi(x - x^\dagger)\big)\|^p_Y &\le 2^{p-1}\|F(x - x^\dagger)\|^p_Y + \|F\pi^\perp(x - x^\dagger)\|^p_Y \\ &\le 2^{p-1}\|F(x - x^\dagger)\|^p_Y + \|F\|^p\|\pi^\perp x\|^p_{\ell^2(\Lambda)} . \end{aligned} \tag{49}$$

In [23] it is shown that

$$\|\pi^\perp x\|^p_{\ell^2(\Lambda)} \le \mathcal{R}_p\big(\pi^\perp x\big) \le D_{\partial\mathcal{R}(x^\dagger)}(x, x^\dagger) \,.$$

Together with inequalities (48) and (49), this implies

$$\begin{aligned} &\|x - x^\dagger\|^p_{\ell^2(\Lambda)} \\ &\quad \le 2^{2(p-1)} C^p \big\|F(x - x^\dagger)\big\|^p_Y + 2^{p-1}\Big(1 + 2^{p-1} C^p \|F\|^p\Big)\|\pi^\perp x\|^p_{\ell^2(\Lambda)} \\ &\quad \le 2^{2(p-1)} C^p \big\|F(x - x^\dagger)\big\|^p_Y + 2^{p-1}\Big(1 + 2^{p-1} C^p \|F\|^p\Big) D_{\partial\mathcal{R}(x^\dagger)}(x, x^\dagger) \,. \end{aligned} \tag{50}$$

As in the proof of Proposition 6.4 one verifies that the inequality (46) holds for all $y \in Y$ with $\|F(x^\dagger) - y\| \le \beta$. Therefore (47) follows (50). ☐

Next we derive the rate $\mathcal{O}(\delta)$ for $p = 1$.

**Proposition 6.7 (Sparse case, $p = 1$).** *Let $x^\dagger = (x^\dagger_\lambda)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ be sparse. Assume that there exist $\xi = (\xi_\lambda)_{\lambda\in\Lambda} \in \partial\mathcal{R}_1$ and $\omega \in Y$ with $\xi = F^*\omega$, and that $F$ is injective on*

$$V = \big\{x \in \ell^2(\Lambda) : \mathrm{supp}(x) \subset \{\lambda \in \Lambda : |\xi_\lambda| = 1\}\big\} \,.$$

*Then*

$$\sup\big\{\|x_\beta - x^\dagger\|_{\ell^2(\Lambda)} : x_\beta \in \Sigma(\beta, y, F),\ \|Fx^\dagger - y\|_Y \le \beta\big\} = \mathcal{O}(\beta) \quad \text{as } \beta \to 0 \,. \tag{51}$$

*Proof.* Analogously to the proof of (50) one shows that

$$\|x - x^\dagger\|_{\ell^2(\Lambda)} \le C\big\|F(x - x^\dagger)\big\|_Y + \Big(1 + C\|F\|\Big)\|\pi^\perp x\|_{\ell^2(\Lambda)} \tag{52}$$

holds for every $x \in \ell^2(\Lambda)$. Further, in [23] it is shown that

$$\|\pi^\perp x\|_{\ell^2(\Lambda)} \leq \frac{D_{\partial\mathcal{R}_p(x^\dagger)}(x, x^\dagger)}{1 - \max\bigl\{|\xi_\lambda| : \lambda \in \Lambda, |\xi_\lambda| < 1\bigr\}} \,. \tag{53}$$

Finally one verifies as in the proof of Proposition 6.4 that

$$D_{\partial\mathcal{R}_p(x^\dagger)}(x_\beta, x^\dagger) \leq \|\omega\|_Y \bigl(\beta + \|F(x^\dagger) - y\|_Y\bigr) \tag{54}$$

holds for every $x_\beta \in \Sigma(\beta, y, F)$. Combining the estimates (54), (52), and (53) shows (51) and concludes the proof. □

**Remark 6.8.** Let $p \in [1, 2)$. If $V$ is any finite dimensional subspace of $\ell^2(\Lambda)$ and $F$ is injective on $V$, then there exists a constant $C_p > 0$ such that

$$C_p \|F(x)\|_Y \geq \|x\|_{\ell^p(\Lambda)} \,, \qquad \text{for all } x \in V \,.$$

Arguing as in the proofs of Propositions 6.6 and 6.7, one therefore derives the convergence rate

$$\sup\bigl\{\|x_\beta - x^\dagger\|_{\ell^p(\lambda)} : x_\beta \in \Sigma(\beta, y, F),\ \|Fx^\dagger - y\|_Y \leq \beta\bigr\} = \mathcal{O}(\beta) \quad \text{as } \beta \to 0$$

for the reconstruction of sparse sequences $x^\dagger$. ■

The convergence rates results for constrained $\ell^p$ regularization are summarized in Table 1.

| Rate | Norm | Premises (besides $\operatorname{ran}(F^*) \cap \partial\mathcal{R}_p \neq \emptyset$) | Result |
|---|---|---|---|
| $\beta^{1/2}$ | $\|\cdot\|_{\ell^2}$ | $p \in (1, 2)$ | Prop. 6.4 |
| $\beta^{1/2}$ | $\|\cdot\|_{\ell^p}$ | $p \in (1, 2)$ | Rem. 6.5 |
| $\beta^{1/p}$ | $\|\cdot\|_{\ell^2}$ | $p \in [1, 2)$, sparsity, injectivity on $V$ | Props. 6.6, 6.7 |
| $\beta^{1/p}$ | $\|\cdot\|_{\ell^p}$ | $p \in [1, 2)$, sparsity, injectivity on $V$ | Rem. 6.8 |
| $\beta$ | $\|\cdot\|_{\ell^2}$ | $p \in (0, 1)$, uniqueness of $x^\dagger$, sparsity, injectivity on $V$ | Prop. 6.11 |

Table 1: Convergence rates for constrained $\ell^p$ regularization.

## 6.2 Non-convex Regularization

We now drop the requirement that $p \geq 1$ and instead choose $0 < p < 1$. That is, we consider the minimization problem (42) with $0 < p < 1$. In the following we show that most results of the superlinear case $p \geq 1$ carry over to the sublinear case. Note, however, that the functional $\mathcal{R}_p$ is non-convex for $p < 1$ and thus its restriction to a set of the form $\bigl\{x \in \ell^p : \|Fx - y\| \leq \beta\bigr\}$ may have local minima. Also, the $\mathcal{R}_p$-minimizing solution of $Fx = y$ need not be unique.

**Proposition 6.9 (Well-Posedness).** *Let $F\colon \ell^2(\Lambda) \to Y$ be a bounded linear operator with dense range. Then constrained $\ell^p$ regularization with $0 < p < 1$ is well-posed:*

1. *For every $\beta > 0$ and $y \in Y$ the set of solutions $\Sigma(\beta, y, F)$ is non-empty.*
2. *Let $\beta_k \to \beta > 0$ and $y_k \to y \in Y$. Then $\emptyset \neq \tau_p\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F) \subset \Sigma(\beta, y, F)$.*
3. *Let $\|y_k - y\| \leq \beta_k \to 0$ and assume that the equation $Fx = y$ has a solution in $\ell^p(\Lambda)$. Then $\emptyset \neq \tau_p\text{-}\operatorname{Lim\,sup}_k \Sigma(\beta_k, y_k, F) \subset \Sigma(0, y, F)$. If the equation $Fx = y$ has a unique $\mathcal{R}_p$-minimizing solution $x^\dagger$, then $\tau_p\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F) = \{x^\dagger\}$.*

*Proof.* This is similar as Proposition 6.2. □

**Remark 6.10.** In case the $\mathcal{R}_p$-minimizing solution $x^\dagger$ of $Fx = y$ is unique, Item 3 in Proposition 6.9 implies that

$$\sup\bigl\{\|x_\beta - x^\dagger\|_{\ell^p(\Lambda)} : x_\beta \in \Sigma(\beta, y_\beta, F),\ \|y - y_\beta\| \leq \beta\bigr\} \to 0 \text{ as } \beta \to 0\ ;$$

otherwise there would exist a sequence $\beta_k \to 0$, $y_k \in Y$ with $\|y_k - y\| \leq \beta_k$ and $\varepsilon > 0$ such that $\|x_k - x^\dagger\|_{\ell^p(\Lambda)} > \varepsilon$ for some $x_k \in \Sigma(\beta_k, y_k, F)$, $k \in \mathbb{N}$. This, however, contradicts the assertion of Proposition 6.9 that $\{x^\dagger\} = \tau_p\text{-}\operatorname{Lim}_k \Sigma(\beta_k, y_k, F)$. ■

Now we prove convergence rates for non-convex $\ell^p$ regularization. Similar, but weaker results have been derived in [22, 51] in the context of Tikhonov regularization. In [51], the conditions for the convergence rates result are basically the same as in Proposition 6.11, but only a rate of order $\mathcal{O}(\sqrt{\delta})$ has been obtained. In [22], a linear convergence rate $\mathcal{O}(\delta)$ is proven, but with a considerably stronger range condition: each standard basis vector $e_\lambda$, $\lambda \in \Lambda$, has to satisfy $e_\lambda \in \operatorname{ran} F^*$.

**Proposition 6.11.** *Let $F\colon \ell^2(\Lambda) \to Y$ be a bounded linear operator with dense range and let $x^\dagger = (x^\dagger_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ be sparse. Assume moreover that there exists a unique $\mathcal{R}_p$-minimizing solution $x^\dagger$ of $Fx = Fx^\dagger$, and $F$ is injective on*

$$V := \bigl\{x \in \ell^2(\Lambda) : \operatorname{supp}(x) \subset \operatorname{supp}(x^\dagger)\bigr\}\ .$$

*Then*

$$\sup\bigl\{\|x_\beta - x^\dagger\|_{\ell^2(\Lambda)} : x_\beta \in \Sigma(\beta, y, F),\ \|Fx^\dagger - y\|_Y \leq \beta\bigr\} = \mathcal{O}(\beta) \quad \text{as } \beta \to 0\ .$$

*Proof.* Denote $\Omega := \operatorname{supp}(x^\dagger)$, which by assumption is a finite set, and let $\pi$, $\pi^\perp\colon \ell^2(\Lambda) \to \ell^2(\Lambda)$ be the projections

$$\pi x = \sum_{\lambda \in \Omega} x_\lambda e_\lambda\,, \qquad \pi^\perp x = \sum_{\lambda \notin \Omega} x_\lambda e_\lambda\ .$$

As in the proof of Proposition 6.7 (see (52)), we obtain the existence of $C_1 > 0$ such that

$$\|x - x^\dagger\|_{\ell^2(\Lambda)} \le C_1 \big\|F(x - x^\dagger)\big\|_Y + \big(1 + C_1\|F\|\big)\|\pi^\perp x\|_{\ell^2(\Lambda)} \tag{55}$$

for every $x \in \ell^2(\Lambda)$.

We now derive a bound for $\|\pi^\perp x\|_{\ell^2(\Lambda)}$. For every $\sigma \in \{\pm 1\}^\Omega$, we define $\zeta(\sigma) \in \ell^2(\Omega)$ by

$$\zeta(\sigma)_\lambda := \sigma_\lambda |x^\dagger_\lambda|^{p-1}, \qquad \lambda \in \Omega .$$

Note that $\zeta(\sigma)_\lambda$ is well-defined, because by assumption $x^\dagger_\lambda \neq 0$ for every $\lambda \in \Omega$. Now let $x \in \ell^2(\Lambda)$ and let $\sigma_\lambda = \operatorname{sgn}(x_\lambda - x^\dagger_\lambda)$ for every $\lambda \in \Omega$ with $x_\lambda \neq x^\dagger_\lambda$ and $\sigma_\lambda = 1$ if $x_\lambda = x^\dagger_\lambda$. Then

$$\begin{aligned}\mathcal{R}_p(x^\dagger) - \mathcal{R}_p(\pi x) &= \sum_{\lambda\in\Omega}\Big(|x^\dagger_\lambda|^p - |x_\lambda|^p\Big) \le \sum_{|x^\dagger_\lambda| \ge |x_\lambda|}\Big(|x^\dagger_\lambda|^p - |x_\lambda|^p\Big)\\ &\le \sum_{|x^\dagger_\lambda| \ge |x_\lambda|} |x^\dagger_\lambda|^{p-1}|x^\dagger_\lambda - x_\lambda| \le \sum_{\lambda\in\Omega}|x^\dagger_\lambda|^{p-1}|x^\dagger_\lambda - x_\lambda| = \langle \zeta(\sigma), \pi(x - x^\dagger)\rangle .\end{aligned}$$

Consequently,

$$\max_{\sigma\in\{\pm1\}^\Omega}\big|\langle\zeta(\sigma),\pi(x-x^\dagger)\rangle\big| \ge \mathcal{R}_p(x^\dagger) - \mathcal{R}_p(\pi x) \qquad \text{for every } x \in \ell^2(\Lambda) . \tag{56}$$

Denote by $i\colon \ell^2(\Omega) \to \ell^2(\Lambda)$ the embedding $i(x) = x$. Since $F \circ i$ is injective and $i^* = \pi$, it follows that $(F\circ i)^* = \pi \circ F^*$ is surjective (see [50, Cor. VII.5.2]). In particular, $\zeta(\sigma) \in \operatorname{ran}(\pi\circ F^*)$ for every $\sigma \in \{\pm1\}^\Omega$. Thus, there exists for every $\sigma \in \{\pm1\}^\Omega$ some $\omega(\sigma) \in Y$ such that $\pi \circ F^*\omega(\sigma) = \zeta(\sigma)$. With the abbreviation

$$C_2 := \max_{\sigma\in\{\pm1\}^\Omega}\|\omega(\sigma)\|_Y ,$$

we therefore obtain that

$$\begin{aligned}\big|\langle\zeta(\sigma),\pi(x-x^\dagger)\rangle\big| &= \big|\langle\pi\circ F^*\omega(\sigma),\pi(x-x^\dagger)\rangle\big|\\ &\le \big|\langle F^*\omega(\sigma), x-x^\dagger\rangle\big| + \big|\langle F^*\omega(\sigma),\pi^\perp(x-x^\dagger)\rangle\big|\\ &\le \big|\langle\omega(\sigma), F(x-x^\dagger)\rangle\big| + \|F^*\omega(\sigma)\|_{\ell^2(\Lambda)}\|\pi^\perp(x-x^\dagger)\|_{\ell^2(\Lambda)}\\ &\le C_2\|F(x-x^\dagger)\|_Y + C_2\|F\|\|\pi^\perp x\|_{\ell^2(\Lambda)}\end{aligned}$$

for every $x \in \ell^2(\Lambda)$ and $\sigma \in \{\pm1\}^\Omega$. Together with (56), this implies that

$$\begin{aligned}C_2\|F(x-x^\dagger)\|_Y &\ge \max_{\sigma\in\{\pm1\}^\Omega}\big|\langle\zeta(\sigma),\pi(x-x^\dagger)\rangle\big| - C_2\|F\|\|\pi^\perp x\|_{\ell^2(\Lambda)}\\ &\ge \mathcal{R}_p(x^\dagger) - \mathcal{R}_p(\pi x) - C_2\|F\|\|\pi^\perp x\|_{\ell^2(\Lambda)}\end{aligned} \tag{57}$$

for every $x \in \ell^2(\Lambda)$.

Since $p < 1$, there exists $\varepsilon > 0$ such that

$$(C_2\|F\| + 1)|t| \le |t|^p \qquad \text{whenever } t \in \mathbb{R} \text{ with } |t| < \varepsilon \, .$$

Thus, for every $x \in \ell^2(\Lambda)$ with $\|\pi^\perp x\|_{\ell^\infty(\Lambda)} < \varepsilon$ we have

$$\begin{aligned}(C_2\|F\| + 1)\|\pi^\perp x\|_{\ell^2(\Lambda)} &\le (C_2\|F\| + 1)\|\pi^\perp x\|_{\ell^1(\Lambda)} \\ &= \sum_{\lambda\notin\Omega}(C_2\|F\| + 1)|x_\lambda| \le \sum_{\lambda\notin\Omega}|x_\lambda|^p = \mathcal{R}_p(\pi^\perp x) \, .\end{aligned}$$

Thus (57) implies that, for every $x \in \ell^2(\Lambda)$ with $\|\pi^\perp x\|_{\ell^\infty(\Lambda)} < \varepsilon$,

$$\mathcal{R}_p(x^\dagger) - \mathcal{R}_p(x) + \|\pi^\perp x\|_{\ell^2(\Lambda)} \le C_2\|F(x - x^\dagger)\|_Y \tag{58}$$

Remark 6.10 and the assumption that $x^\dagger$ is the unique $\mathcal{R}_p$-minimizing solution of $Fx = Fx^\dagger$ imply that

$$\sup\bigl\{\|x_\beta - x^\dagger\|_{\ell^2(\Lambda)} : x_\beta \in \Sigma(\beta, y, F), \ \|Fx^\dagger - y\|_Y \le \beta\bigr\} \to 0 \ \text{ as } \beta \to 0 \, .$$

Since $\|x_\beta - x^\dagger\|_{\ell^\infty(\Lambda)} \le \|x_\beta - x^\dagger\|_{\ell^2(\Lambda)}$, it follows that there exists $\beta_0$ such that $\|x_\beta - x^\dagger\|_{\ell^\infty(\Lambda)} \le \varepsilon$ for all $x_\beta \in \Sigma(\beta, y, F)$, $\|Fx^\dagger - y\|_Y \le \beta$, and $\beta \le \beta_0$. In these cases, (58) applies. The estimates (55) and (58), and the inequalities $\mathcal{R}_p(x_\beta) \le \mathcal{R}_p(x^\dagger)$ and $\|F(x_\beta - x^\dagger)\| \le 2\beta$ therefore imply that

$$\begin{aligned}\|x_\beta - x^\dagger\|_{\ell^2(\Lambda)} &\le C_1\|F(x_\beta - x^\dagger)\|_Y + (1 + C_1\|F\|)\|\pi^\perp x_\beta\|_{\ell^2(\Lambda)} \\ &\le C_1\|F(x_\beta - x^\dagger)\|_Y \\ &\qquad + (1 + C_1\|F\|)C_2\bigl(\|F(x_\beta - x^\dagger)\| + \mathcal{R}_p(x_\beta) - \mathcal{R}_p(x^\dagger)\bigr) \\ &\le \bigl(C_1 + C_2 + C_1C_2\|F\|\bigr)\|F(x_\beta - x^\dagger)\|_Y \\ &\le 2\bigl(C_1 + C_2 + C_1C_2\|F\|\bigr)\beta \, ,\end{aligned}$$

which proves the assertion. □

# 7 Conclusion

Due to modeling, computing, and measurement errors, the solution of an ill-posed equation $F(x) = y$, even if it exists, typically yields inacceptable results. The residual method replaces the exact solution by the set $\Sigma(\beta, y, F) = \arg\min\bigl\{\mathcal{R}(x) : \mathcal{S}(F(x), y) \le \beta\bigr\}$, where $\mathcal{R}$ is a stabilizing functional and $\mathcal{S}$ denotes a distance measure between $F(x)$ and $y$. This paper shows that in a very general setting $\Sigma(\beta, y, F)$ is stable with respect to perturbations of the data $y$ and the operator $F$ (Theorem 3.6 and Proposition 3.10), and the regularized solutions converge to $\mathcal{R}$-minimizing solutions of $F(x) = y$ as $\beta \to 0$ (Proposition 3.9). In particular the stability issue has hardly been considered so far in the literature.

In the case where $F$ acts between linear spaces $X$ and $Y$, stability and convergence have been shown under a list of reasonable properties (see Assumption 4.1). These assumptions are satisfied for bounded linear operators, but also for a certain class of nonlinear operators (Example 4.6). If $Y$ is reflexive, $X$ satisfies the Radon–Riesz property, $F$ is a linear closed operator, and $\mathcal{R}$ and $\mathcal{S}$ are given by the norms on $X$ and $Y$, the set $\Sigma(\beta, y, F)$ consists of a single element $x_\beta$. This element is shown to converge strongly to the minimal norm solution $x^\dagger$ as $\beta \to 0$. In this special situation, norm convergence has also been shown in [29, Theorem 3.4.1].

In Section 5 we have derived quantitative estimates (convergence rates) for the difference between $x^\dagger$ and minimizers $x_\beta \in \Sigma(\beta, y, F)$ in terms of a (generalized) Bregman distance. All these estimates hold provided $\mathcal{S}(F(x^\dagger), y) \leq \beta$ and a source inequality introduced in [26] is satisfied. For linear operators, the required source inequality follows from a source wise representation of a subgradient of $\mathcal{R}$ at $x^\dagger$. This carries on the result of [4] for constrained regularization. In the case that $X$ is an $r$-convex Banach space with $r \geq 2$ and $\mathcal{R}$ is the $r$-th power of the norm on $X$, we have obtained convergence rates $\mathcal{O}(\beta^{1/r})$ with respect to the norm. The spaces $X = L^p(\Omega)$ for $p \in (1, 2]$ are examples of 2-convex Banach spaces, leading to the rate $\mathcal{O}(\sqrt{\beta})$ in those spaces.

As an application for our rather general results we have investigated sparse $\ell^p$ regularization with $p \in (0, 2)$. We have shown well-posedness in both the convex ($p \geq 1$) and the non-convex case ($p < 1$). In addition, we have studied the reconstruction of sparse sequence. There we have derived the improved convergence rates $\mathcal{O}(\beta^{1/p})$ for the convex and $\mathcal{O}(\beta)$ for the non-convex case.

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