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SHAPE METRICS BASED ON ELASTIC DEFORMATIONS

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Abstract. Deformations of shapes and distances between shapes are an active research topic in computer vision. We propose an energy of infinitesimal deformations of continuous 1- and 2-dimensional shapes that is based on the elastic energy of deformed objects. This energy defines a shape metric which is inherently invariant with respect to Euclidean transformations and yields very natural deformations which preserve details. We compute shortest paths between planar shapes based on elastic deformations and apply our approach to the modeling of 2-dimensional shapes.

1. Introduction

This paper is concerned with the problem of quantifying the differences between shapes. This leads to the notion of a *shape space* which is an appropriate representation of shapes and to a *shape metric* on the shape space.

In general, the modeling of both, the shape space and the associated metric, is a challenging task and different approaches lead to diverse models whose usefulness is decided by the application in question. This includes e.g. the statistical analysis of shapes used to regularize the detection or tracking of objects in images and movies. Also the classification of detected outlines involves statistical methods in shape spaces. Another application is the modeling of shapes. Given two deformed states of one object, the shortest path between the two shapes morphs the first state into the second. The more natural these paths look, the less intermediate steps have to be have defined by the designer.

There exists no common shape space or shape metric which satisfies all requirements of the mentioned applications optimally. The suitability of a certain approach depends very much on the demands in a given situation. A property which classifies shape metrics is their invariance with respect to certain geometric transformations. Probably the simplest example is invariance to translations. This means that two shapes are considered to be the same (and thus have distance zero) if they are transformed translationally. In general, translational invariance is considered to be inherent to the notion of "shape".

The situation is less clear when it comes to more general transformations. E.g., if the task is to classify objects in a microscope image it makes obviously no sense to consider the rotation of the shapes as their positions

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and rotations will be randomly distributed. For the recognition of digits, however, the correct rotation is crucial to distinguish e.g. "6" and "9". On the other hand, the scaling of the digits should not affect the result if one compares image data of different sources. In the above-mentioned example of microscope images, though, the size of the objects might be important. Thus, one favors a translational and rotational invariant metric in the first case but would use a metric invariant to translation and scaling but not to rotation in the second case.

These different invariance properties are reflected in the different approaches to shape spaces and metrics in literature. In principle there exist two, not strictly separated, levels of incorporating invariances. First, the shape space can be designed in a way such that the shape representation is independent of certain properties. E.g. representing shapes via their tangent vectors removes information about their absolute position, normalizing their perimeter makes them scale invariant. Secondly, it is possible to design metrics which do not consider certain transformations. Then, a shape space might still include translated shapes but the associated shape metric measures the distance between them as zero. This is the approach we chose in this paper where the shape space is only invariant to reparametrizations (a minimal requirement for a shape space) but the shape metric maps any Euclidean transformation to zero.

1.1. **Related work.** The concept of shape spaces with an associated metric was first developed by Kendall [6]. There, shapes are characterized by labelled points in Euclidean space, so called *land-marks*, and the author investigates Riemannian structures on this space. The drawback of this and related approaches is that the shapes have to be labelled before they can be processed further. On the other hand, the associated spaces are finite-dimensional and computationally easy to handle.

More modern works are concerned with continuous shape representations of infinite dimensions. Klassen et al. [8] represent planar curves by their direction and curvature functions. I.e. the shape space is a subspace of the periodic L^2-functions on $[0, 2\pi]$. Such a function corresponds to the angle of the curve tangent in the first and the curvature at a given curve point in the second case. Because the curves are assumed to be parametrized by arclength, this uniquely determines shapes. These representations are invariant to scaling and translation and to scaling, translation and rotation, respectively. The authors impose further properties such as closedness and rotation index via defining functions on the space of curves. This results in a Riemannian manifold embedded in $L^2([0, 2\pi])$ and equipped with the induced metric, i.e. the L^2-metric.

In a similar manner, Mio et al. [14] represent planar curves by the polar coordinates of their tangent vectors. I.e. the angle and the length of the tangent at a curve are smooth functions on the unit interval. They also impose the closure condition via defining functions which results in a shape manifold of infinite dimension. This manifold is Riemannian in virtue of a weighted L^2-metric on the product space of the tangent angles and the tangent lengths. This metric measures the *bending* and *stretching* of curves subject to an infinitesimal deformation. The influence of the bending energy versus the stretching energy is controlled by a parameter.

The idea of modeling the boundary curve of a shape as an elastic object is related to Younes [20] but the technical approach there is different from ours. The author considers plane curves which are parametrized by arc-length and assumes a group acting transitively on this space. Then, the transformation of one shape into another is represented by a path in this group. Hence, the measurement of distances between shapes is transformed to the problem of defining lengths of paths in the group acting on the shape space. This approach is also presented in a more general manner in Miller and Younes [13]. Invariances of the resulting metric are incorporated via the group action.

The shape metrics presented in [20, 14] are related to the elastic energy of the boundary curves of the shapes. It is important to note that they do not consider the elastic properties of the domain confined by the shape boundary. This is a significant difference to our approach as noted in Section 1.2.

Michor and Mumford [12] choose smooth embeddings of the S^1 in the plane as shape space and propose a L^2-metric which is regularized by the curvature of the shape boundary. This work is partly motivated by their observation [11] that the standard L^2-metric vanishes on this shape space. An extensive review of metrics on the same shape space can be found in Yezzi and Mennucci [18].

A different approach is chosen by Zolésio [21]. In this work the author considers the characteristic functions of measurable sets as shapes and defines *tubes* which are paths between two shapes. Each tube is associated with a time-dependent vector field that prescribes the deformation of the tube. In this formulation lengths of tubes can be defined by imposing norms on the associated function spaces. The idea of defining shape deformations via vector fields defining deformations of the ambient space is related to our approach as explained in Section 3.

Keeling and Ring [4, 5] use the elastic energy of infinitesimal deformations to regularize the optical flow between two images. If one considers characteristic functions of shapes as images the minimal energy of their cost functional can be interpreted as a shape metric based on elasticity. This idea is similar to the approach presented in Section 3. In contrast to our work, they consider the elastic energy of infinitesimal deformations of the ambient space whereas we compute them on the domain defined by the shape.

For the modeling of 3-dimensional shapes, Kilian et al. [7] propose two shape metrics which promote rigid and isometric, respectively, deformations of a discrete boundary mesh. They propose a multi-resolution approach both in time and in space to solve for shortest paths between two shapes.

1.2. **Elastic deformations of shapes.** In this work we also consider the space of smooth embeddings of the unit circle/sphere modulo reparametrizations and define an energy of infinitesimal deformations of shapes. This allows us to the define the length of homotopies between shapes and yields a distance measure on the shape space. The energy itself is based on the elastic energy of the shape (i.e. the area inside the shape boundary) caused by an infinitesimal deformation of the shape boundary. Thus, the main characteristics of our approach are:

- The elastic deformation metric takes into account the shape of the actual *object* as opposed to metrics considering only the *boundary curve*.
- It is naturally invariant to rotations and translations.
- The metric applies equally to 1- and 2-dimensional shapes.
- It is formulated for multiply connected shapes.

The first property allows us to distinguish between deforming thin parts of an object (which requires less deformation energy) and thick parts (requiring more energy). This is not possible with any formulation based on local properties of the boundary curve, which is the case for the majority of the approaches cited above.

1.3. **Outline.** This paper is organized as follows. In the next section we define the elastic deformation energy of infinitesimal deformations of shapes and the corresponding distance measure. Section 3 is devoted to an alternative interpretation of the elastic deformation energy. It is based on vector fields deforming the ambient space of shapes such that the deformation of the metric on the shape is minimal.

The elastic deformation energy is defined via a variational formulation. In Section 4 we prove that minimizers exist and that they are essentially unique. The succeeding section is devoted to the numerical computation of shortest paths between planar shapes. Section 6 considers deformations of shapes in space from a modeling point of view. This means that we do not consider shortest paths between two objects but slowly (i.e. infinitesimally) deform parts of a given object and adapt the rest of the object such that the energy of the infinitesimal deformation is minimal. In the final section we give a short summary and conclude with an outlook.

1.4. **Notations and preliminaries.** In the following we denote the trace of a matrix by tr, i.e. $\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}$ for $A \in \mathbb{R}^{n\times n}$. The norm $|A|$ is defined as the square root of the sum of the squared components of A, i.e. $|A|^2 = \sum_{i,j=1}^{n} A_{ij}^2$ and the identity

$$|A|^2 = \operatorname{tr}(A^t A) \tag{1.1}$$

holds.

The *time derivative* of a function γ defined on $[0,1]$ is denoted by $\dot\gamma$ or $\partial\gamma/\partial t$. In particular, if γ is a homotopy, i.e. a piecewise differentiable path between two shapes, $\dot\gamma$ and $\partial\gamma/\partial t$ always denote the derivative of the homotopy with respect to the time parameter and *never* the derivative of the shape parametrization.

In the next section we need the notion of *infinitesimal rotations* in 2- and 3-dimensional space. Formally, infinitesimal rotations constitute the Lie algebra of the respective rotation groups and form linear subspaces of $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{3\times 3}$, respectively. These subspaces are the sets of all skew-symmetric matrices in the respective spaces. They are spanned by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{1.2}$$

in the plane and by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3}$$

in space. Any rotation R of $\mathbb{R}^2$ or $\mathbb{R}^3$ can be written as $R = \mathrm{e}^A$, where A is a linear combination of the matrices (1.2) or (1.3), respectively.

Finally, we denote the set of all *infinitesimal Euclidean motions* on $\mathbb{R}^n$, $n = 2, 3$, as

$$M_n := \{A : \mathbb{R}^n \to \mathbb{R}^n : A(x) = a + Rx\,, a \in \mathbb{R}^n, R \text{ infinitesimal rotation}\}\,.$$

2. Elastic Deformation Energy and Distance

In this section we propose an energy of infinitesimal deformations of shapes based on linear elasticity and derive a shape metric from this energy. Moreover, we require the shapes to be smooth and to have a well defined interior area or volume. This naturally leads to smooth embeddings of a parameter domain into its ambient space. Let $n = 1, 2$ and assume $D \subseteq \mathbb{R}^{n+1}$ (the parameter domain) to be a closed, orientable $\mathcal{C}^\infty$-manifold of dimension n with no boundary. We denote the embeddings of D in $\mathbb{R}^{n+1}$ by $\mathrm{Emb}(D, \mathbb{R}^{n+1})$, the set of all functions which are $\mathcal{C}^\infty$-diffeomorphisms onto their images. The group of all diffeomorphic maps from D onto itself is denoted by $\mathrm{Diff}(D)$. Because several embeddings can represent identical shapes we consider them as equal if a smooth reparametrization can transform one into the other. I.e. we define the n-dimensional shape space

$$\mathcal{S}_n = \mathrm{Emb}(D, \mathbb{R}^{n+1}) / \mathrm{Diff}(D), \quad n = 1, 2\,.$$

For simply connected curves or surfaces one can choose $D = S^n$, the n-dimensional unit sphere. This leads to smooth embeddings of the unit circle and the unit sphere, respectively.

Our goal is to define a distance between shapes in $\mathcal{S}_n$. For this purpose we borrow some ideas from Riemannian geometry to compute distances in this space. A Riemannian manifold is a manifold equipped with a Riemannian metric. The Riemannian metric is an inner product on the tangent bundle of the manifold and as such defines angles between tangent vectors. In particular, the length of tangent vectors can be measured with respect to a Riemannian metric.

On manifolds, the distance between two arbitrary points is defined as the infimum of the lengths of all differentiable paths connecting the points. The length of a path is computed by integrating the absolute value of its first derivative along the path, i.e. by integrating its velocity. The first derivative of a differentiable path on a manifold at a given position is contained in the tangent space at this position. On Riemannian manifolds its length is measured in terms of the Riemannian metric. Note that the notion of the derivative of a path on a Riemannian manifold does not depend on the Riemannian metric. These considerations lead to the conclusion that distances can be computed as above in a more general than the Riemannian setting as long as we are able to compute the length of the velocity vectors of differentiable paths connecting two elements. In the following, we derive such a norm of velocities in $\mathcal{S}_n$ which leads to a metric on this space. As stated in Remark 4.4 later on, this norm is induced by a Riemannian metric on $\mathcal{S}_n$. In a first step we define the elastic energy of an infinitesimal deformation of a domain Ω, which plays the role of the shape later on:

2.1. Definition. Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain with smooth, i.e. $\mathcal{C}^\infty$, boundary and assume parameters $\lambda \geq 0$ and $\mu > 0$. Moreover let $\mathbf{u} \in H^1(\Omega)^{n+1}$, i.e. $\mathbf{u}$ is a vector field defined on Ω. Then the *elastic deformation energy* of the deformation $\mathbf{u}$ on Ω is given by

$$E(\mathbf{u}) = \int_\Omega \big(\lambda(\operatorname{tr}\mathbf{e}(\mathbf{u}))^2 + 2\mu\operatorname{tr}(\mathbf{e}(\mathbf{u})^t\mathbf{e}(\mathbf{u}))\big)\,dx\,, \tag{2.1}$$

where the components of the *linearized strain tensor* $\mathbf{e}(\mathbf{u})$ are given by

$$e^i_j(\mathbf{u}) = \frac{1}{2}(\partial_j u^i + \partial_i u^j)\,, \quad 1 \leq i \leq j \leq n\,. \tag{2.2}$$

This defines a map

$$E : H^1(\Omega)^{n+1} \;\to\; [0,\infty[\;.$$

In physics, expression (2.1) is the linear elastic energy of a homogeneous, isotropic material Ω which is displaced by the infinitesimal deformation $\mathbf{u}$. The parameters λ, μ are called the *Lamé parameters* and characterize the elastic properties of the object. It is easy to see that $E(\mathbf{u}) = 0$ if and only if $\mathbf{u}$ is an infinitesimal Euclidean motion, i.e. $\mathbf{u} \in M_{n+1}$.

Using (2.1) we are able to define the elastic energy of an infinitesimal deformation of shapes in $\mathcal{S}_n$. Such a deformation is given by the displacements of the shape boundary into the directions normal to the boundary. By the definition of $\mathcal{S}_n$ we do not consider displacements tangential to the shape boundary as they correspond to a reparametrization of the shape. Thus, the space of infinitesimal deformations of shapes in $\mathcal{S}_n$ is the space of smooth functions on D, i.e. $\mathcal{C}^\infty(D)^{n+1}$.

Moreover, for shape modeling it turns out to be useful to also consider deformations of parts of the boundary of a domain. Let $\Omega \subseteq \mathbb{R}^{n+1}$ be as in Definition 2.1. We assume a non-empty part $\Gamma \subseteq \partial\Omega$ of the boundary of Ω. Then we define the linear map

$$\operatorname{Tr}_{\mathbf{n}} : H^1(\Omega)^{n+1} \to H^{1/2}(\Gamma)\,, \quad \mathbf{u} \mapsto \langle \operatorname{Tr}\,\mathbf{u}, \mathbf{n}\rangle\,, \tag{2.3}$$

where $\mathbf{n}$ denotes the outer unit normal at Ω and $\operatorname{Tr} : H^1(\Omega)^{n+1} \to H^{1/2}(\Gamma)^{n+1}$ is the trace operator restricted to Γ.

2.2. Lemma. *The operator* $\operatorname{Tr}_{\mathbf{n}} : H^1(\Omega)^{n+1} \to H^{1/2}(\Gamma)$ *is continuous and onto, i.e. for every* $f \in H^{1/2}(\Gamma)$ *there exists* $\mathbf{u} \in H^1(\Omega)^{n+1}$ *such that*

$$\operatorname{Tr}_{\mathbf{n}}(\mathbf{u}) = f\,.$$

Proof. Since Tr is continuous $H^1(\Omega)^{n+1} \to H^{1/2}(\Gamma)^{n+1}$ [1, Theorem 7.57] the map $\operatorname{Tr}_{\mathbf{n}}$ is continuous.

Let $f \in H^{1/2}(\Gamma)$. The boundary normal $\mathbf{n} : D \to \mathbb{R}^{n+1}$ is of class $\mathcal{C}^\infty$. As a consequence each component of

$$\mathbf{f} := f\mathbf{n} : \Gamma \to \mathbb{R}^{n+1}$$

lies in $H^{1/2}(\Gamma)$. Moreover, the trace operator $H^1(\Omega)^{n+1} \to H^{1/2}(\Gamma)^{n+1}$ is onto [1, Theorem 7.57]. Thus, there exists $\mathbf{u} \in H^1(\Omega)^{n+1}$ with $\mathrm{Tr}\,(\mathbf{u}) = \mathbf{f}$. $\square$

The above considerations are concerned with a domain Ω and a subset Γ of its boundary. In the shape space setting Ω is the interior of a shape. This is reflected in the definition of the elastic deformation energy of a shape:

2.3. Definition. Assume a shape $a \in \mathcal{S}_n$ and denote

$$\Omega = \text{the open set bounded by the image of } a. \tag{2.4}$$

Let $\Gamma \subseteq \partial\Omega$ and $\mathrm{Tr}_{\mathbf{n}}$ be as in (2.3). Then the *elastic deformation energy* $|f|^2_{\mathrm{e},a}$ of an infinitesimal boundary deformation $f \in H^{1/2}(\Gamma)$ is defined by

$$|f|^2_{\mathrm{e},a} = \inf_{\substack{\mathbf{u} \in H^1(\Omega)^{n+1} \\ \mathrm{Tr}_{\mathbf{n}}\,\mathbf{u} = f}} E(\mathbf{u})\,. \tag{2.5}$$

In other words, we consider the energies of all infinitesimal deformations of Ω which deform the subset Γ of the boundary in the normal direction as prescribed by f and define $|f|^2_{\mathrm{e},a}$ as the infimum of these energies. We chose the notation $|\cdot|^2_{\mathrm{e}}$ for the elastic energy because its square root plays the role of the norm induced by the Riemannian metric in Definition 2.5.

2.4. Remark. The variational expression (2.5) is very similar to the *pure displacement problem* in linear elasticity [2, Section 5.1], where the energy caused by an infinitesimal vector valued boundary deformation $\mathbf{f} \in H^{1/2}(\Gamma)^{n+1}$ is given by

$$\inf_{\substack{\mathbf{u} \in H^1(\Omega)^{n+1} \\ \mathrm{Tr}\,\mathbf{u} = \mathbf{f}}} E(\mathbf{u})\,. \tag{2.6}$$

The difference between (2.5) and (2.6) is that in the latter formulation the infinitesimal deformation $\mathbf{u}$ is completely prescribed on the boundary whereas we only fix its normal components.

Using Definition 2.3 to measure the magnitude of the velocity of homotopies we can finally define the elastic shape metric on $\mathcal{S}_n$.

2.5. Definition. Let $a, b \in \mathcal{S}_n$ and $\gamma : [0,1] \to \mathcal{S}_n$ piecewise continuously differentiable such that $\gamma(0) = a$ and $\gamma(1) = b$. The *length* and the *energy* of γ are given by

$$L(\gamma) := \int_0^1 |\dot\gamma(t)|_{\mathrm{e},\gamma(t)}\,dt \quad \text{and} \quad E(\gamma) := \int_0^1 |\dot\gamma(t)|^2_{\mathrm{e},\gamma(t)}\,dt\,,$$

respectively. Here $\dot\gamma(t) \in \mathbb{R}$ is the velocity of $\gamma(t)$ normal to $\gamma(t)$, i.e. normal to the shape boundary at the time t, $0 < t < 1$. We use the notation $E(\gamma)$ for paths γ exclusively to avoid any confusion with the elastic deformation energy $E(\mathbf{u})$ of an infinitesimal deformation $\mathbf{u}$ as defined in (2.1).

We define the *elastic deformation metric* $d : \mathcal{S}_n \times \mathcal{S}_n \to [0, \infty[$ by

$$d(a,b) = \inf_{\substack{\gamma(0)=a \\ \gamma(1)=b}} L(\gamma)\,, \tag{2.7}$$

where $\gamma : [0,1] \to \mathcal{S}_n$ is as above. Note that the elastic energy of $\dot\gamma$ is given by applying the case $\Gamma = \partial\Omega$ in Definition 2.3.

2.6. Remark. Let d be as in Definition 2.5 and $a, b, c \in \mathcal{S}_n$. Then the following relations hold:

- $d(a, b) = d(b, a)$ and
- $d(a, c) \leq d(a, b) + d(b, c)$.

In Remark 4.4 we state that the elastic deformation energy $|\cdot|_{\mathrm{e}}^2$ is induced by a Riemannian metric on $\mathcal{S}_n$. For a path $\gamma : [0, 1] \to \mathcal{S}_n$, which connects two shapes $a, b \in \mathcal{S}_n$, this has following two consequences:

- $L(\gamma)$ is invariant to reparameterization of γ.
- Assume that γ is such that its energy $E(\gamma)$ is minimal among all curves connecting a and b. Then $L(\gamma)$ also minimizes the lengths of all paths between a and b and $|\dot{\gamma}|_{\mathrm{e},\gamma}$ is constant along γ.

For the actual computation of distances between shapes and geodesics connecting them several issues remain. The definitions in (2.5) and (2.7) are formulated via infima of non-negative sets. This results in well-defined values but does not necessarily imply the existence of minimizing elements. In Section 4 we prove that a unique minimizer of (2.5) exists in $H^1(\Omega)^{n+1}$. This enables us to compute $|\cdot|_{\mathrm{e}}^2$ using a finite element approach to solve the weak formulation of the optimality condition of the variational problem.

The existence of minimal geodesics, i.e. the existence of minimizers of (2.7), is a much harder question. In the finite dimensional setting, one approach to this problem is the Hopf-Rinow Theorem which states the existence of minimal geodesics on a finite dimensional manifold which is complete as a metric space. However in the infinite dimensional case this theorem fails [9]. Due to these difficulties, we computed discrete geodesics as explained in Section 5 but can not provide analytical results concerning their existence and uniqueness.

3. Metric Perturbation

In this section we give an alternative interpretation of the elastic deformation energy. We derive the perturbation of the metric on a shape which is deformed by an infinitesimal deformation and show that this perturbation is a special case of the elastic deformation energy (2.5). The basic idea is to define a Riemannian metric on $\mathbb{R}^{n+1}$ which reflects the deformation of this space according to a time-dependent flow field. We then compute the perturbation of this metric as the L^2-norm of the time derivative of the metric tensor at the time zero. In the following we give a detailed description of this approach.

Assume a time-dependent vector field $\mathbf{x} : [0, \varepsilon[\to H^1(\mathbb{R}^{n+1})^{n+1}$, $\varepsilon > 0$, which is differentiable at time zero and satisfies $\mathbf{x}(0) = \mathrm{Id}$. This vector field can be interpreted as the trajectories of points in $\mathbb{R}^{n+1}$ at a given time. If one considers the space $\mathbb{R}^{n+1}$ as a whole, then $\mathbf{x}(t)$ corresponds to a deformed state of this space at $t > 0$. At the time zero every point is mapped to its initial position, i.e. the space is not deformed, whereas for times $t > 0$ angles and distances between points get distorted. In the following we quantify these distortions in an infinitesimal setting, i.e. we consider the distortion between points at arbitrarily small distances and short times.

First note that we write $D\mathbf{x}(t,p)$ for the *spatial derivative* of $\mathbf{x}(t)$ at a fixed time $0 < t < 1$ in $p \in \mathbb{R}^{n+1}$. I.e. the directional derivative of $\mathbf{x}(t)$ into the direction $v \in \mathbb{R}^{n+1}$ is given by $D\mathbf{x}(t,p)(v)$. We choose an orthonormal basis $(v_i)_{1\leq i\leq n+1}$ of $\mathbb{R}^{n+1}$ and define the *metric tensor* $\mathbf{G}_\mathbf{x} = (g_{ij})_{1\leq i,j\leq n+1}$ in a point $p \in \mathbb{R}^{n+1}$ at the time $t \geq 0$ by

$$g_{ij}(t,p) = \big\langle D\mathbf{x}(t,p)(v_i), D\mathbf{x}(t,p)(v_j)\big\rangle \quad \text{for} \quad 1 \leq i,j \leq n\,. \tag{3.1}$$

Note that $\mathbf{G}_\mathbf{x}(0,p) = \mathrm{Id}$ for all $p \in \mathbb{R}^{n+1}$. In the next step we define the perturbation of $\mathbf{G}_\mathbf{x}$ caused by an infinitesimal deformation of a shape domain.

3.1. Definition. Assume $a \in \mathcal{S}_n$ and Ω as in (2.4). Let $\Gamma \subseteq \Omega$ and $\mathrm{Tr}_\mathbf{n}$ as in (2.3). Then the metric perturbation $|f|^2_{\mathrm{m},a}$ induced by an infinitesimal deformation $f \in H^{1/2}(\Gamma)^{n+1}$ is defined by

$$|f|^2_{\mathrm{m},a} = \inf_{\mathrm{Tr}_\mathbf{n}\,(\dot{\mathbf{x}}(0)|_\Omega)=f} \int_\Omega \Big|\frac{\partial}{\partial t}\mathbf{G}_\mathbf{x}(t,p)\Big|^2_{t=0}\,dp\,, \tag{3.2}$$

where $\mathbf{x} : [0,\varepsilon[\, \to H^1(\mathbb{R}^{n+1})^{n+1}$, $\varepsilon > 0$. According to Theorem 3.2, $|f|^2_{\mathrm{m},a}$ is independent of the choice of $(v_i)_{1\leq i\leq n+1}$.

In other words, we consider time dependent deformations of the ambient space of the shape a which coincide with the infinitesimal normal deformation f at the time zero. The time derivatives of each such deformation define a metric tensor $\mathbf{G}_\mathbf{x}$ and we minimize the L^2-norm of the time derivative of the tensor on Ω. I.e. we penalize temporal changes of the metric inside the shape at the time zero. The next theorem proves that the metric perturbation energy (3.2) coincides with the elastic deformation energy (2.1) in case of Lamé parameters $\lambda = 0$ and $\mu = 2$.

3.2. Theorem. *Assume a, Γ and f as in Definition 2.1. Let further be $\lambda = 0$ and $\mu = 2$ in* (2.2). *Then*

$$|f|^2_{\mathrm{m},a} = |f|^2_{\mathrm{e},a}\,.$$

In particular, $|f|^2_{\mathrm{m},a}$ does not depend on the choice of the orthonormal basis $(v_i)_{1\leq i\leq n+1}$.

Proof. Consider a vector field $\mathbf{x}$ as in Definition 3.1 and define $\mathbf{u} \in H^1(\mathbb{R}^{n+1})^{n+1}$ by $\mathbf{u} = \dot{\mathbf{x}}(0)$. Without loss of generality we assume that $(v_i)_{1\leq i\leq n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. Because $\mathbf{x}(0) = \mathrm{Id}$ the equality

$$\frac{\partial}{\partial t}\big\langle D\mathbf{x}(t,p)(e_i), D\mathbf{x}(t,p)(e_j)\big\rangle\Big|_{t=0} = \big\langle D\mathbf{u}(p)(e_i), e_j\big\rangle + \big\langle e_i, D\mathbf{u}(p)(e_j)\big\rangle = \partial_i u^j + \partial_j u^i \tag{3.3}$$

holds for $1 \leq i,j \leq n$ and $p \in \mathbb{R}^{n+1}$. Let $\mathbf{e}(\mathbf{u})$ be as in (2.2), i.e.

$$e^i_j(\mathbf{u}) = \frac{1}{2}(\partial_j u^i + \partial_i u^j)\,, \quad 1 \leq i \leq j \leq n\,.$$

Then it follows from (3.1) and (3.3) together with the above equation that

$$\Big|\frac{\partial}{\partial t}\mathbf{G}_\mathbf{x}(t,p)\Big|^2_{t=0} = \big|2\mathbf{e}(\mathbf{u})\big|^2 = 4\big|\mathbf{e}(\mathbf{u})\big|^2\,.$$

Applying (1.1) and comparing (2.1) and (3.2) concludes the proof. $\square$

4. Existence and Uniqueness of Minimizing Deformations

This section is devoted to the existence of unique minimizers of the variational problem which defines the elastic deformation energy in (2.5). As pointed out in the previous section this problem is very much related to the pure displacement problem in linear elasticity and the following results are a modification of the treatment of this problem in [2, Section 6.4].

First we look at a problem similar to (2.5) but with homogeneous boundary conditions and non-vanishing source term. We derive a weak formulation of the corresponding PDE and prove existence and uniqueness of solutions of this equation. Moreover, these solutions also solve the original variational problem. Finally, we adapt these results to (2.5).

We start with the following definitions: Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain with smooth boundary, $\emptyset \neq \Gamma \subseteq \partial\Omega$ and $\mathrm{Tr}_{\mathbf{n}}$ as in (2.3). Assume $\lambda \geq 0$, $\mu > 0$ and $\mathbf{g} \in H^1(\Omega)^{n+1}$. For $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^{n+1}$ define

$$B(\mathbf{u},\mathbf{v}) = \int_\Omega \big(\lambda \operatorname{tr} \mathbf{e}(\mathbf{u}) \operatorname{tr} \mathbf{e}(\mathbf{v}) + 2\mu \operatorname{tr}(\mathbf{e}(\mathbf{u})^t \mathbf{e}(\mathbf{v}))\big)\, dx\,, \tag{4.1}$$

where

$$e^i_j(\mathbf{u}) = \frac{1}{2}(\partial_j u^i + \partial_i u^j)\,, \quad 1 \leq i \leq j \leq n\,.$$

In the next theorem we are concerned with the task of minimizing a functional defined by (4.1) subject to homogenous boundary conditions. We state two different versions of this problem, the variational problem and the weak formulation of the corresponding Euler-Lagrange equations:

Variational formulation: Find $\mathbf{u} \in H^1(\Omega)^{n+1}$ such that

$$\begin{cases} \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0 \\ B(\mathbf{u}, \mathbf{u} + 2\mathbf{g}) \leq B(\mathbf{v}, \mathbf{v} + 2\mathbf{g}) \end{cases} \tag{V_{h}}$$

for all $\mathbf{v} \in H^1(\Omega)^{n+1}$ with $\mathrm{Tr}_{\mathbf{n}}(\mathbf{v}) = 0$.

Weak formulation: Find $\mathbf{u} \in H^1(\Omega)^{n+1}$ such that

$$\begin{cases} \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0 \\ B(\mathbf{u}, \boldsymbol{\varphi}) = -B(\mathbf{g}, \boldsymbol{\varphi}) \end{cases} \tag{W_{h}}$$

for all $\boldsymbol{\varphi} \in H^1(\Omega)^{n+1}$ with $\mathrm{Tr}_{\mathbf{n}}(\boldsymbol{\varphi}) = 0$.

4.1. Theorem. *A function $\mathbf{u} \in H^1(\Omega)^{n+1}$ solves (V_{h}) if and only if it solves (W_{h}), i.e. the two problems are equivalent. Moreover, a solution of (V_{h}) and (W_{h}) exists. If $\mathbf{u}, \mathbf{w} \in H^1(\Omega)^{n+1}$ solve (V_{h}) and (W_{h}), respectively, then*

$$\mathbf{w} = \mathbf{u} + A\,,$$

where $A \in M_{n+1}$ is an infinitesimal Euclidean motion.

Proof. We start by defining

$$V := \{\mathbf{u} \in H^1(\Omega)^{n+1} : \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0\}\,.$$

First we prove the above claim on the quotient space of V where we identify deformations which differ only by an infinitesimal Euclidean motion. This allows us to directly apply Korn's inequality on the quotient space. Define

$$\bar{V} := \{[\mathbf{u}] = \mathbf{u} + E : \mathbf{u} \in V\}\,.$$

According to Lemma 4.2, this space equipped with the norm

$$\|[\mathbf{u}]\|_{\bar{V}} = \inf_{\mathbf{u}\in[\mathbf{u}]} \|\mathbf{u}\|_{H^1(\Omega)^{n+1}} \quad \text{for} \quad [\mathbf{u}] \in \bar{V}$$

is a Banach space.

The map

$$\mathbf{e} : \bar{V} \to L^2(\Omega)^{n+1}, \quad \mathbf{u} \mapsto \frac{1}{2}(\partial_j u^i + \partial_i u^j),$$

is defined on $\bar{V}$ because it maps infinitesimal Euclidean motions to 0. As a consequence, $B : \bar{V} \times \bar{V} \to \mathbb{R}$ is well defined. Hence, by (1.1),

$$2\mu\|\mathbf{e}([\mathbf{u}])\|^2_{L^2(\Omega)^{n+1}} \leq B([\mathbf{u}],[\mathbf{u}]) \quad \text{for} \quad [\mathbf{u}] \in \bar{V}. \tag{4.2}$$

Korn's inequality [3] states that there exists $C > 0$ such that

$$C\|[\mathbf{u}]\|_{\bar{V}} \leq \|\mathbf{e}([\mathbf{u}])\|_{L^2(\Omega)^{n+1}} \quad \text{for} \quad [\mathbf{u}] \in \bar{V}. \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$C'\|[\mathbf{u}]\|_{\bar{V}} \leq B([\mathbf{u}],[\mathbf{u}]) \quad \text{for} \quad [\mathbf{u}] \in \bar{V} \tag{4.4}$$

for some $C' > 0$. We define the linear, bounded functional $L : \bar{V} \to \mathbb{R}$ by

$$L([\mathbf{u}]) = B([\mathbf{g}],[\mathbf{u}]) \quad \text{for} \quad [\mathbf{u}] \in \bar{V}.$$

Inequality (4.4) means that B is V-elliptic on $\bar{V}$ in the sense of [2, Theorem 6.3-2]. The same theorem states that there exists a unique solution $[\mathbf{u}] \in \bar{V}$ of

$$B([\mathbf{u}],[\boldsymbol{\varphi}]) = -L([\boldsymbol{\varphi}]) \quad \text{for all} \quad [\boldsymbol{\varphi}] \in \bar{V} \tag{4.5}$$

and that $[\mathbf{u}]$ is the unique solution of

$$B([\mathbf{u}],[\mathbf{u}]) + 2L([\mathbf{u}]) \leq B([\mathbf{v}],[\mathbf{v}]) + 2L([\mathbf{v}]) \quad \text{for all} \quad [\mathbf{v}] \in \bar{V}. \tag{4.6}$$

By assumption we can choose $\mathbf{u} \in [\mathbf{u}]$ such that $\mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0$. Then $\mathbf{u}$ solves (W_{h}) and (V_{h}).

To prove uniqueness consider two solutions $\mathbf{u}$ and $\mathbf{w}$ of (W_{h}) or (V_{h}). Obviously $[\mathbf{u}]$ and $[\mathbf{w}]$ solve (4.5) and (4.6), respectively. Because these problems have unique solutions $[\mathbf{u}] = [\mathbf{w}]$ and therefore $\mathbf{u} = \mathbf{w} + A$ for some $A \in E$. □

4.2. Lemma. *The space*

$$\bar{V} = \{[\mathbf{u}] = \mathbf{u} + E : \mathbf{u} \in H^1(\Omega)^{n+1}, \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0\}$$

with norm

$$\|[\mathbf{u}]\|_{\bar{V}} = \inf_{\mathbf{u}\in[\mathbf{u}]} \|\mathbf{u}\|_{H^1(\Omega)^{n+1}} \textit{ for } [\mathbf{u}] \in \bar{V}$$

is a Banach space.

Proof. We first consider the space

$$\bar{H}^1(\Omega)^{n+1} := H^1(\Omega)^{n+1}/E = \{[\mathbf{u}] = \mathbf{u} + E : \mathbf{u} \in H^1(\Omega)^{n+1}\}$$

with norm

$$\|[\mathbf{u}]\|_{\bar{H}^1(\Omega)^{n+1}} = \inf_{\mathbf{u}\in[\mathbf{u}]} \|\mathbf{u}\|_{H^1(\Omega)^{n+1}} \text{ for } \mathbf{u} \in \bar{H}^1.$$

$\bar{H}^1(\Omega)^{n+1}$ is a Banach space as the quotient space of a Banach space and a closed subspace [19, Section I.11].

Moreover $\bar{V} \subseteq \bar{H}^1(\Omega)^{n+1}$. To show the assertion it suffices to prove that $\bar{V}$ is a closed subspace of $\bar{H}^1(\Omega)^{n+1}$. Consider $[\mathbf{u}], [\mathbf{u}_k] \in \bar{V}$, $k \geq 0$, such that

$$\lim_{k\to\infty} [\mathbf{u}_k] = [\mathbf{u}] \quad \text{in} \quad \bar{H}^1(\Omega)^{n+1} .$$

Without loss of generality we can assume that

$$\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u} \quad \text{in} \quad H^1(\Omega)^{n+1} .$$

Moreover there exists a sequence of infinitesimal Euclidean motions $A_k \in E$, $k \geq 0$, such that

$$\operatorname{Tr}_{\mathbf{n}} (\mathbf{u}_k + A_k) = 0 . \tag{4.7}$$

Because $\operatorname{Tr}_{\mathbf{n}}$ is continuous, this implies that there exists $M \geq 0$ such that

$$\| \operatorname{Tr}_{\mathbf{n}} (A_k)\|_{L^2(\Gamma)} \leq M \quad \text{for} \quad k \geq 0 . \tag{4.8}$$

On the other hand, the map

$$E/ \ker \operatorname{Tr}_{\mathbf{n}} \to \operatorname{im}(\operatorname{Tr}_{\mathbf{n}}) , \quad [A] \mapsto \operatorname{Tr}_{\mathbf{n}} A$$

is a linear bijection of finite-dimensional linear spaces. Thus, inequality (4.8) implies that the sequence $([A_k])_{k\geq 0}$ is bounded in $E/ \ker \operatorname{Tr}_{\mathbf{n}}$. Without loss of generality (by selecting an appropriate subsequence) we conclude that there exists $A \in E$ such that

$$\lim_{k\to\infty} [A_k] = [A] \quad \text{in} \quad E/ \ker \operatorname{Tr}_{\mathbf{n}} .$$

Moreover, we can choose $(A'_k)_{k\geq 0}$ such that

$$A'_k \in [A_k] \quad \text{for} \quad k \geq 0 .$$

and

$$\lim_{k\to\infty} A'_k = A \quad \text{in} \quad E .$$

Note that by (4.7) this implies that

$$\operatorname{Tr}_{\mathbf{n}} (\mathbf{u}_k + A'_k) = 0 .$$

Then, by the continuity of $\operatorname{Tr}_{\mathbf{n}}$, we have

$$\operatorname{Tr}_{\mathbf{n}} (\mathbf{u} + A) = \operatorname{Tr}_{\mathbf{n}} \left(\lim_{k\to\infty} (\mathbf{u}_k + A'_k) \right) = \lim_{k\to\infty} \operatorname{Tr}_{\mathbf{n}} (\mathbf{u}_k + A'_k) = 0 .$$

□

In a next step we show that the existence of minimizers in (2.5) follows from Theorem 4.1. For this purpose we restate the problems (V_{h}) and (W_{h}) without source term but with inhomogeneous boundary conditions:

Variational formulation: Find $\mathbf{u} \in H^1(\Omega)^{n+1}$ such that

$$\begin{cases} \operatorname{Tr}_{\mathbf{n}} (\mathbf{u}) = f \\ B(\mathbf{u}, \mathbf{u}) \leq B(\mathbf{v}, \mathbf{v}) \end{cases} \tag{V_{ih}}$$

for all $\mathbf{v} \in H^1(\Omega)^{n+1}$ with $\operatorname{Tr}_{\mathbf{n}} (\mathbf{v}) = f$.

Weak formulation: Find $\mathbf{u} \in H^1(\Omega)^{n+1}$ such that

$$\begin{cases} \operatorname{Tr}_{\mathbf{n}} (\mathbf{u}) = f \\ B(\mathbf{u}, \boldsymbol{\varphi}) = 0 \end{cases} \tag{W_{ih}}$$

for all $\boldsymbol{\varphi} \in H^1(\Omega)^{n+1}$ with $\operatorname{Tr}_{\mathbf{n}} (\boldsymbol{\varphi}) = 0$.

4.3. Theorem. *A function* $\mathbf{u} \in H^1(\Omega)^{n+1}$ *solves* (V_{ih}) *if and only if it solves* (W_{ih})*, i.e. the two problems are equivalent. Moreover, a solution of* (V_{ih}) *and* (W_{ih}) *exists. If* $\mathbf{u}, \mathbf{w} \in H^1(\Omega)^{n+1}$ *solve* (V_{ih}) *and* (W_{ih})*, respectively, then*

$$\mathbf{w} = \mathbf{u} + A\,,$$

where $A \in M_{n+1}$ *is an infinitesimal Euclidean motion.*

Proof. According to Lemma 2.2 there exists $\mathbf{g} \in H^1(\Omega)^{n+1}$ such that

$$\mathrm{Tr}_{\mathbf{n}}(\mathbf{g}) = f.$$

By Theorem 4.1 there exists $\mathbf{u}' \in H^1(\Omega)^{n+1}$ which, for $\mathbf{g}$ as above, solves (W_{h}) and (V_{h}), respectively. Then $\mathbf{u} = \mathbf{u}' + \mathbf{g}$ satisfies $\mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = f$ and further solves (W_{ih}) and (V_{ih}), respectively.

Moreover, assume that $\mathbf{w} \in H^1(\Omega)^{n+1}$ solves (W_{ih}) and (V_{ih}), respectively. Then $\mathbf{w}' := \mathbf{w} - \mathbf{g}$ satisfies $\mathrm{Tr}_{\mathbf{n}}(\mathbf{w}') = 0$ and further solves (W_{h}) and (V_{h}), respectively. Thus, $\mathbf{u}'$ and $\mathbf{w}'$ differ only by an infinitesimal Euclidean motion and $\mathbf{u} - \mathbf{w} = \mathbf{u}' - \mathbf{w}' \in M_{n+1}$. □

4.4. Remark. The elastic deformation energy is induced by a Riemannian metric on $\mathcal{S}_n$. Let $a \in \mathcal{S}_n$, Ω and Γ be as in Definition 2.3 and define the inner product $\langle \cdot, \cdot \rangle_a$ by

$$\langle f, g \rangle_a = B(\mathbf{u}, \mathbf{v})\,, \quad \text{where} \quad E(\mathbf{u}) = |f|^2_{\text{e},a} \text{ and } E(\mathbf{v}) = |g|^2_{\text{e},a}\,, \tag{4.9}$$

for $f, g \in H^{1/2}(\Gamma)$. By Theorem 4.3 the inner product (4.9) is well-defined and linear in each component. Moreover it is symmetric and

$$\langle f, f \rangle_a = |f|^2_{\text{e},a}$$

holds for $f \in H^{1/2}(\Gamma)$.

4.5. Remark. Note that Theorem 4.3 not only states that the infimum in (2.5) is attained in $H^1(\Omega)^{n+1}$, i.e. a solution of (V_{ih}) exists, but also that it is uniquely (up to infinitesimal Euclidean motions) determined by the weak formulation (W_{ih}). Thus, the elastic energy of an infinitesimal deformation can directly be computed using a finite element approach.

5. Computation of Shortest Paths in the Plane

In the examples presented in Section 5.3 below we consider two B-spline curves in $\mathbb{R}^2$ and compute a numerical approximation of their distance as defined in Definition 2.5.

This is done by computing a sequence of discretized paths minimizing (2.7). The evaluation of the elastic deformation energy (2.5) requires to numerically solve for the displacement field $\mathbf{u}$ in the problems (V_{ih}) and (W_{ih}), respectively. Hence, the first part of this section is devoted to the computation of $\mathbf{u}$ using a finite element approach. The second part considers the minimization of the elastic shape metric by discretizing paths between two shapes and iteratively updating them using a Quasi-Newton method. This strategy is independent of the actual energy of the infinitesimal deformations of the shapes (or the metric on $\mathcal{S}_n$). Thus, we use the same approach to compute minimal paths with respect to other metrics which we compare to the elastic deformation metric in Section 5.3.

5.1. **Computation of the elastic deformation energy.** In this section we restrict ourselves to the case of planar shapes, i.e. $n = 1$. Assume a shape $a \in \mathcal{S}_1$, the domain $\Omega \subseteq \mathbb{R}^2$ defined by a and a subset $\Gamma \subseteq \partial\Omega$ of the shape as in Theorem 4.1. Furthermore, let $f \in H^{1/2}(\partial\Omega)$ and $\lambda \geq 0$, $\mu > 0$ be as in Theorem 4.3. We want to solve (W_{ih}) on a finite dimensional subspace of $H^1(\Omega)^{n+1}$. First assume that Ω is a polygon defined by points $\{q_1, \ldots, q_M\} \subseteq \mathbb{R}^2$, i.e. we consider discretizations of shapes.

For a fixed $h > 0$ we triangulate the discretized shape such that the maximal area of each triangle is smaller than h and that boundary nodes of the triangulation coincide with the boundary discretization $q_1, \ldots, q_M$. Denote the nodes of the triangulation as $\{p_1, \ldots, p_K\} \in \mathbb{R}^2$. Then

$$\{q_1, \ldots, q_M\} \subseteq \{p_1, \ldots, p_K\} \,.$$

Next we consider the family $\varphi_1, \ldots, \varphi_K$ of linear splines with nodes $\{p_1, \ldots, p_K\}$. For $1 \leq k \leq K$ define the vector-valued test functions

$$\boldsymbol{\varphi}_k^1 := (\varphi_k, 0)^t \quad \text{and} \quad \boldsymbol{\varphi}_k^2 := (0, \varphi_k)^t \,,$$

and let

$$V := {}_{\mathbb{R}}\langle \boldsymbol{\varphi}_k^1, \boldsymbol{\varphi}_k^2 : 1 \leq k \leq K \rangle \,,$$

be the $\mathbb{R}$-linear hull of all test functions, i.e. the set of all vector valued, piecewise linear functions on the triangulation defined by the nodes $\{p_1, \ldots, p_K\}$. We want to compute a solution $\mathbf{u}$ of (V_{ih}) in the space V for a given $f \in H^{1/2}(\Gamma)$. Then, by Definitions 2.1 and 2.3 and by comparision of (2.1) and (4.1), the equality

$$|f|^2_{\mathrm{e},a} = B(\mathbf{u}, \mathbf{u}) \tag{5.1}$$

holds.

Assume that $\mathbf{u} \in V$ is a solution of (V_{ih}). Then

$$B(\mathbf{u}, \boldsymbol{\varphi}_k^i) = 0 \quad \text{for} \quad p_k \notin \Gamma \,,\; 1 \leq k \leq K \,,\; 1 \leq i \leq 2 \,, \tag{5.2}$$

i.e. $\mathbf{u}$ satisfies the weak formulation (W_{ih}) for every test function without support on Γ. For every boundary discretization point $p_k \in \partial\Omega$, $1 \leq k \leq K$, (i.e. for every p_k such that $p_k = q_m$ for some $1 \leq m \leq M$) we define

$$\boldsymbol{\varphi}_k^{\mathbf{t}} = \varphi_k \mathbf{t} \,, \quad \text{where} \quad \mathbf{t} \perp \mathbf{n}(p_k) \,,\; |\mathbf{t}| = 1 \,.$$

Here $\mathbf{n}(p_k)$ is the outer unit normal at Ω in the boundary point p_k. In the next section, $\partial\Omega$ is represented as a continuously differentiable B-spline curve and the p_k are points on this curve. Thus, we can compute $\mathbf{n}(p_k)$ as the unit normal at the boundary spline curve. As an alternative, one could define the normal to be the mean of the normals of the polygon edges joining in p_k. The test function $\boldsymbol{\varphi}_k^{\mathbf{t}}$ then satisfies $\mathrm{Tr}_{\mathbf{n}}\,(\boldsymbol{\varphi}_k^{\mathbf{t}})(p_k) = 0$, and, under the simplifying assumption that the boundary normal is locally constant,

$$B(\mathbf{u}, \boldsymbol{\varphi}_k^{\mathbf{t}}) = 0 \quad \text{for} \quad p_k \in \Gamma \,,\; 1 \leq k \leq K \,, \tag{5.3}$$

holds. By "locally constant" we mean that the unit normals of the edges joining at the vertex p_k are the same as $\mathbf{n}(p_k)$, i.e. the boundary normal in p_k. In this case $\mathrm{Tr}_{\mathbf{n}}\,(\boldsymbol{\varphi}_k^{\mathbf{t}}) = 0$ on $\partial\Omega$. If $\partial\Omega$ is sampled densely enough (with more samples on regions of high curvature) the angle between the edges decreases and this condition is approximately met. Moreover,

$$\int_\Gamma \mathrm{Tr}_{\mathbf{n}}(\mathbf{u})\varphi^k \, d\tau = \int_\Gamma f\varphi^k \, d\tau \quad \text{for} \quad p_k \in \Gamma\,,\ 1 \le k \le K\,. \tag{5.4}$$

Now assume that

$$\mathbf{u} = \sum_{k=1}^{K} \varphi_k (u_1^k, u_2^k)^t \in V$$

for coefficients $u_1^k, u_2^k \in \mathbb{R}$, $1 \le k \le K$. If m_1 is the number of boundary vertices in Γ then equations (5.2), (5.3) and (5.4) yield $2(K-m_1)+m_1+m_1$ linear equations in the unknown coefficients $(u_1^k, u_2^k)_{1\le k\le K}$. Thus, (5.2), (5.3) and (5.4) define a system of linear equations

$$\underline{\mathrm{M}}\,\underline{\mathrm{u}} = \underline{\mathrm{f}}\,, \tag{5.5}$$

where $\underline{\mathrm{M}} \in \mathbb{R}^{2K\times 2K}$, $\underline{\mathrm{f}} \in \mathbb{R}^{2K}$, and $\underline{\mathrm{u}} \in \mathbb{R}^{2K}$ is the column vector of the coefficients $(u_1^k, u_2^k)_{1\le k\le K}$. In the implementation, the integrals over the domain Ω were computed using the barycenter integrator in each triangular element. The integration along the boundary $\partial\Omega$ is done by using the Gaussian integrator with two nodes on each edge of the boundary polygon.

Having obtained a solution $\mathbf{u}$ by solving (5.5), we compute the elastic energy $|f|^2_{\mathrm{e},\gamma}$ by evaluating the integral on the right hand side of (5.1). Again, we use barycentric integration on each triangle in T. Note that (5.1) is an integral of a quadric in $\partial_j u^i$, $1 \le i,j \le 2$ over Ω. As such it can be transformed into a boundary integral along $\partial\Omega$ by virtue of the divergence theorem.

The linear equation (5.5) is sparse but, because of the boundary equations, not strictly symmetric. Also remember that solutions of (V_{ih}) are unique only up to infinitesimal Euclidean transformations. For most domains Ω this has no consequences because adding an infinitesimal Euclidean transformation to a solution $\mathbf{u}$ of (V_{ih}) would violate the boundary conditions prescribed on Γ. In these cases (5.5) has a unique solution and we were able to solve the system of linear equations via LU decomposition of $\underline{\mathrm{M}}$.

One notable exception, though, is the disc in the plane. Any infinitesimal rotation around the center of the disc does not change the energy of an underlying deformation and leaves the boundary conditions unaffected:

5.1. Example. Assume $\Omega = B_1(0) \subseteq \mathbb{R}^2$ and $\emptyset \ne \Gamma \subseteq \partial\Omega = S^1$. For a function $f \in L^2(\Gamma, \mathbb{R})$ let $\mathbf{u}$ be a solution of (V_{ih}). Let $A \in E$ be an infinitesimal rotation around 0. Then $\mathbf{u} + A$ solves (V_{ih}). This means that we must expect (5.5) to be numerically underdetermined. Indeed, it turns out to be impossible to solve (5.5) in a stable way in this example. Hence, we further restrict possible solutions by seeking a minimum-norm solution of the coefficient vector $\underline{\mathrm{u}}$:

$$\begin{cases} \underline{\mathrm{M}}\,\underline{\mathrm{u}} = \underline{\mathrm{f}}\,, & \text{and} \\ |\underline{\mathrm{u}}| \le |\underline{\mathrm{v}}| & \text{for} \quad \underline{\mathrm{M}}\,\underline{\mathrm{v}} = \underline{\mathrm{f}} \end{cases} \tag{5.6}$$

Using Lagrangian multipliers, this leads to the problem of computing a stationary point of

$$|\underline{\mathrm{u}}|^2 + \Lambda^t(\underline{\mathrm{M}}\,\underline{\mathrm{u}} - \underline{\mathrm{f}})\,, \tag{5.7}$$

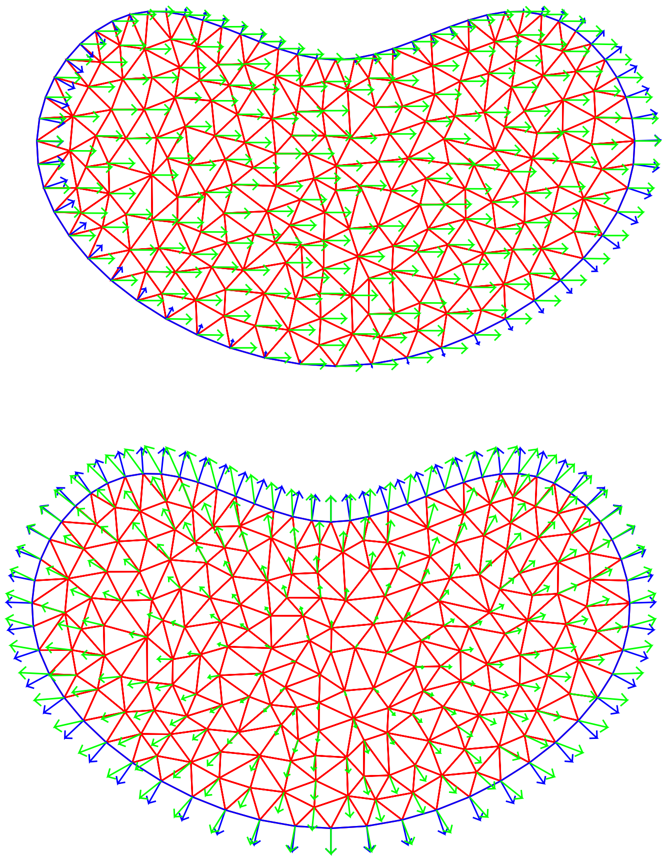

FIGURE 1. Displacement fields (*green*) minimizing the elastic energy of the infinitesimal deformation into normal direction prescribed on the shape boundary (*blue*). In the *top* image the boundary deformation corresponds to a translation of the shape, in the *bottom* image to a boundary offset.

where $\Lambda \in \mathbb{R}^{2K}$. Differentiating (5.7) yields the following system of linear equations for $\underline{\mathrm{u}}$ and Λ:

$$\begin{pmatrix} \underline{\mathrm{M}} & 0 \\ 2I_K & \underline{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \underline{\mathrm{u}} \\ \Lambda \end{pmatrix} = \begin{pmatrix} \underline{\mathrm{f}} \\ 0 \end{pmatrix} \tag{5.8}$$

In case (5.5) is underdetermined, the regularized formulation (5.7) leads to a unique solution $\underline{\mathrm{u}}$ but the system (5.8) remains underdetermined (being a simple extension of (5.7)). However, the instability of the problem now affects Λ but not $\underline{\mathrm{u}}$. In this example we were able to solve (5.8) using the *BiCgStab* solver [16].

5.2. **Computation of shortest paths.** This section is concerned with the computation of paths of minimal energy connecting two planar shapes. I.e.

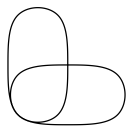

FIGURE 2. Initial configuration of two rounded rectangles rotated by $\frac{\pi}{2}$.

for shapes $a, b \in \mathcal{S}_1$ we compute a discrete path γ' which approximates

$$\inf_{\substack{\gamma(0)=a \\ \gamma(1)=b}} E(\gamma) = \inf_{\substack{\gamma(0)=a \\ \gamma(1)=b}} \int_0^1 |\dot\gamma(t)|^2_{\gamma(t)}\, dt\,, \tag{5.9}$$

for a given metric $|\cdot|_a$, $a \in \mathcal{S}_1$. Naturally we are mainly interested in the case $|\cdot|_a = |\cdot|_{\mathrm{e},a}$ but also minimize (5.9) with respect to other metrics for reasons of comparison.

To minimize (5.9) we assume $a, b \in \mathcal{S}_1$ to be cubic B-spline curves determined by $K > 4$ control points. For a discretization level $d \in \mathbb{N}$, we define the map $L_d : \mathbb{R}^{2Kd} \to \mathbb{R}$ by

$$\begin{aligned}(\mathbf{c}_1, \dots, \mathbf{c}_d) \mapsto \frac{1}{d+1} \sum_{k=0}^{d} \frac{1}{2} \Big(\int_{\gamma_k} |\langle \gamma_k - \gamma_{k+1}, \mathbf{n} \rangle|^2_{\gamma_k}\, d\tau + \\ \int_{\gamma_{k+1}} |\langle \gamma_k - \gamma_{k+1}, \mathbf{n} \rangle|^2_{\gamma_{k+1}}\, d\tau \Big)\,.\end{aligned} \tag{5.10}$$

Here $\gamma_1, \dots, \gamma_d$ are the cubic B-spline curves defined by the coefficient vectors $\mathbf{c}_1, \dots, \mathbf{c}_d$ and $\gamma_0 := a$, $\gamma_{d+1} := b$. I.e. L_d computes the length of the discrete path $a, \gamma_1, \dots, \gamma_d, b$ by approximating the derivative $\dot\gamma$ by symmetric finite differences at the discretization points. The integrals in (5.10) are computed by a simple equidistant discretization of the curve parameter τ. We minimize L_d using the L-BFGS method [10] with numerically computed gradients. As initial value we chose

$$\mathbf{c}_k = (1-t)\mathbf{c}_a + t\mathbf{c}_b\,, \quad t = \frac{k}{d+1}\,, \quad 1 \le k \le d\,, \tag{5.11}$$

where $\mathbf{c}_a, \mathbf{c}_b \in \mathbb{R}^{2K}$ are the vectors of the B-spline control points of a and b. I.e. the starting sequence of the coefficients $\mathbf{c}_1, \dots, \mathbf{c}_d$ linearly interpolates the control coefficients of a and b.

5.3. **Numerical results.** We computed shortest paths for the elastic deformation metric and compared them to the corresponding results for the L^2-energy and the regularized L^2-energy as proposed by Michor and Mumford [12]. The L^2-energy maps an infinitesimal deformation $f \in \mathcal{C}^\infty(D)$ of a curve $a \in \mathcal{S}_1$ to its L^2-norm, i.e. it is defined by

$$|f|^2_{L^2,a} := \int_D f(\tau)^2 |a'(\tau)|\, d\tau\,. \tag{5.12}$$

In [11], it is proven that the distance with respect to the above energy between two arbitrary shapes always vanishes and that shortest paths do not

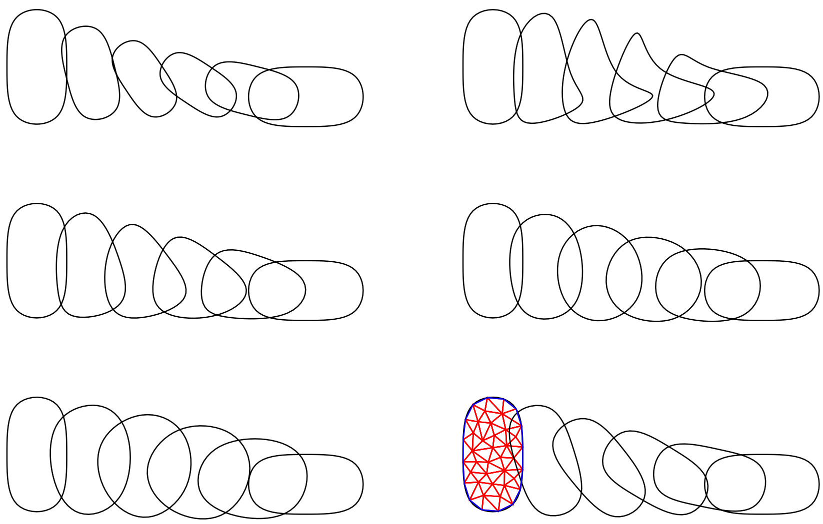

FIGURE 3. Shortest paths connecting the shapes in Figure 2 (*from left to right*). *Top row*: linear interpolation of the B-spline control points, L^2-metric. *Middle row*: regularized L^2-metric with $\alpha = 0.001$, regularized L^2-metric with $\alpha = 0.01$. *Bottom row*: regularized L^2-metric with $\alpha = 0.1$, elastic deformation metric.

exist in general. The authors propose to regularize (5.12) by the curvature of a and introduce

$$|f|^2_{L^2_\alpha,a} := \int_D (1 + \alpha\kappa_a(\tau)) f(\tau)^2 |a'(\tau)|\, d\tau\,. \tag{5.13}$$

The parameter $\alpha > 0$ controls the influence of the curvature term. Positive lower bounds for the distance between two different shapes with respect to this energy exist [12].

In our experiments we compare the elastic deformation metric to the metrics defined by (5.12) and (5.13). Note that the phenomenon of vanishing distances in case of the L^2-norm is due to the infinite dimension of $\mathcal{S}_n$. In the finite dimensional setting of discrete paths of B-spline curves, we can still compute distances with respect to (5.12). For all presented results we chose $\mu = 2$ and $\lambda = 0$ in (2.1). The triangles in the images concerning the elastic deformation metric are the triangulations and the boundary discretizations used for the finite element implementation of the elastic deformation energy.

In the first example we consider a rectangle with rounded corners which was rotated by $\frac{\pi}{2}$. The initial configuration of the shapes is illustrated in Figure 2. Obviously the elastic deformation distance between these two shapes is zero and any path which moves the start shape to the end shape by means of Euclidean motions is a minimal one. In the lower right image of Figure 3 we see one possible solution of such a path which was computed by the L-BFGS method departing from the initial value (5.11) in the top left image. It corresponds to a counter-clockwise rotation and a translation

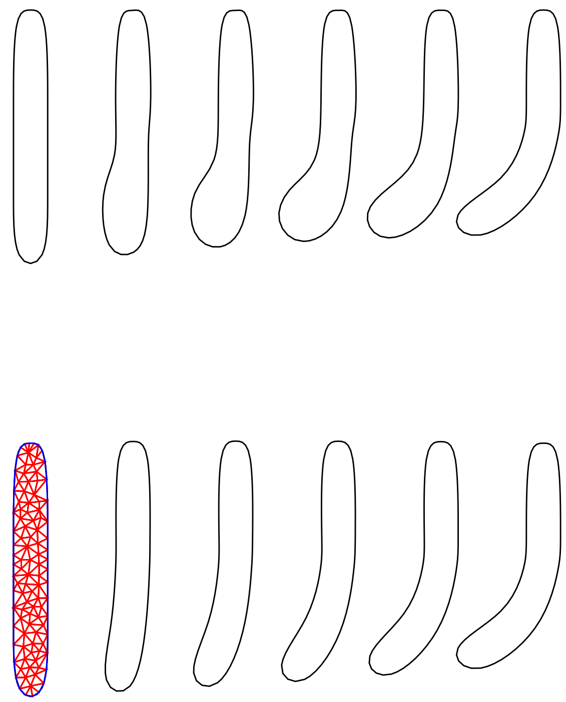

FIGURE 4. Shortest paths connecting a bent, beam-like shape with respect to the regularized L^2-metric with $\alpha = 0.0001$ (*top*) and the elastic deformation metric (*bottom*).

to the right. The L^2-metric yields a smooth "flow" of one shape into the other, whereas an increasing regularization results in intermediate shapes which resemble circles.

In the examples in Figures 3–5 we moved the computed shapes either horizontally or vertically apart. Originally the start and end shapes lie on top of each other and are not translated. The example in Figure 4 concerns a beam-like shape bent to the left. Again, the regularized L^2-metric on the left tries to minimize the boundary length weighted by the curvature of the intermediate shapes. The elastic deformation metric deforms the beam such that the required deformation energy is minimal which results in the bottom path.

In the last example we deformed a bone-like shape as illustrated in Figure 5. The shape of the two ends of the straight and the bent bone are exactly the same only their relative position varies and the connecting bar in the middle is deformed. For the (regularized) L^2-metric the details at the ends vanish during the deformation while the elastic deformation metric is able to preserve them.

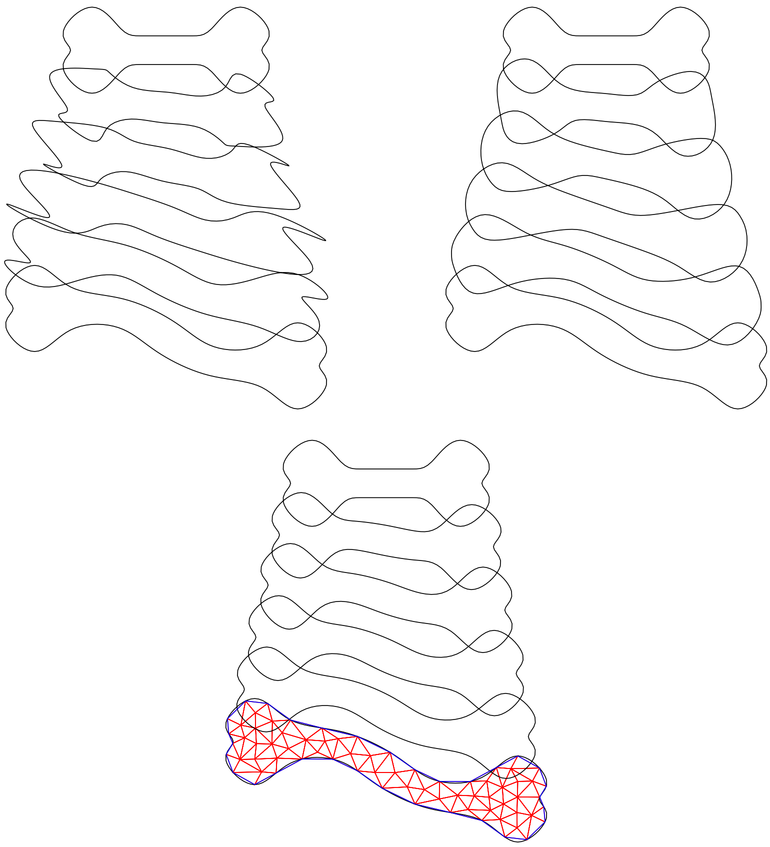

Figure 5. Shortest paths connecting a deformed bone-like shape with respect to the L^2-metric (*top left*), the regularized L^2-metric with $\alpha = 0.1$ (*top right*) and the elastic deformation metric (*bottom*).

6. Elastic Deformation Shape Modeling

In this section we describe how to use the elastic deformation energy (2.1) for elastic shape modeling in 3D. Given boundary conditions for the deformation of a 3D object represented by triangular meshes, we want to obtain a sequence of deformed objects which constitute a realistic animation of elastically deformable models. More specific, we are given a (possibly time-dependent) velocity field which prescribes the infinitesimal deformation of a shape on parts of its boundary. The task is to solve the forward problem of finding the global deformation in time of the object subject to the boundary conditions.

At a fixed time we compute the infinitesimal deformation of the boundary (which, on the respective parts, is of course determined by the boundary conditions) which minimizes the elastic deformation energy. Then we move the shape according to this deformation multiplied by a small time step. Iterating this process leads to a natural deformation of the shape which is controlled by the given boundary deformations. Note that the results turned out to be satisfactory if we prescribe the complete infinitesimal deformations (not just its normal components) on the boundary. This is reflected in the description of the implementation of one time step below.

6.1. **Computation of the deformation field in T-spline space.** For the elastic shape modeling in 3D, we compute the elastic deformation energy (2.1) in a smooth *T-spline* [15] space. We chose the approach of Section 3 and consider the metric perturbation of a globally defined deformation field. More specifically, we define the infinitesimal deformation field $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$. Then $\mathbf{u}$ corresponds to the infinitesimal displacements at the given time. In our case, we model $\mathbf{u}$ using cubic T-spline functions which means that $\mathbf{u}$ is twice continuously differentiable. This way the computation of energy (2.1) can be discretized without having to triangulate Ω, a task which is usually more complicated in 3D than in 2D.

In order to define T-spline functions for $\mathbf{u}$, a control grid of T-spline control points (also called *T-mesh*, c.f. Figure 6(b)) is constructed in the function domain. The distribution of the T-spline control points is adapted to the geometry of the deformable objects, which leads to a compact representation of $\mathbf{u}$. If $K > 0$ is the number of basis functions on the T-mesh, the infinitesimal deformation $\mathbf{u}$ of the function domain is a linear combination of the vector of the T-spline control points $c^k = (c_1^k, c_2^k, c_3^k)^t \in \mathbb{R}^3$, $1 \leq k \leq K$, and the vector of the T-spline basis functions $(b_k)_{1 \leq k \leq K}$ (refer to [15] and [17] for more details):

$$\mathbf{u} = \sum_{k=1}^{K} b_k c^k \,. \tag{6.1}$$

Once the T-mesh is fixed during the deformation, then $\mathbf{u}$ is completely determined by the coefficients $(c^k)_{1 \leq k \leq K}$.

Let $\mathbf{f} \in H^{1/2}(\Gamma)^3$ be the desired infinitesimal deformation on the boundary. Then the elastic energy of $\mathbf{f}$ is defined by (c.f. Definition 2.3)

$$|\mathbf{f}|^2_{\mathrm{e},a} = \inf_{\substack{c \in \mathbb{R}^K \\ \mathrm{Tr}\ \mathbf{u} = \mathbf{f}}} E(\mathbf{u}) \,, \tag{6.2}$$

where $\mathbf{u}$ is computed from c as in (6.1). In order to solve the above constrained optimization problem in the T-spline space, we use a penalty method to compute the T-spline control points, i.e.

$$c = \operatorname*{argmin}_{c \in \mathbb{R}^K} (E(\mathbf{u}) + F(\mathbf{u})) \,, \tag{6.3}$$

where

$$F(\mathbf{u}) = \omega \int_\Gamma |\, \mathrm{Tr}\ \mathbf{u} - \mathbf{f}|^2 \, dx \tag{6.4}$$

is the penalty function for the boundary constraint, with a large positive weight $\omega > 0$. Since $\mathbf{u}$ is a linear function of c and (6.2) is quadratic in $\mathbf{u}$, the T-spline control points c can be computed by solving a sparse linear system of equations. The displacement field $\mathbf{u}$ is then obtained by (6.1). Note that with this approach the number of degrees of freedom is determined by the T-mesh and not by the discretization of the shape. In the example in Figure 6 this leads to a considerably smaller system of linear equations than a finite

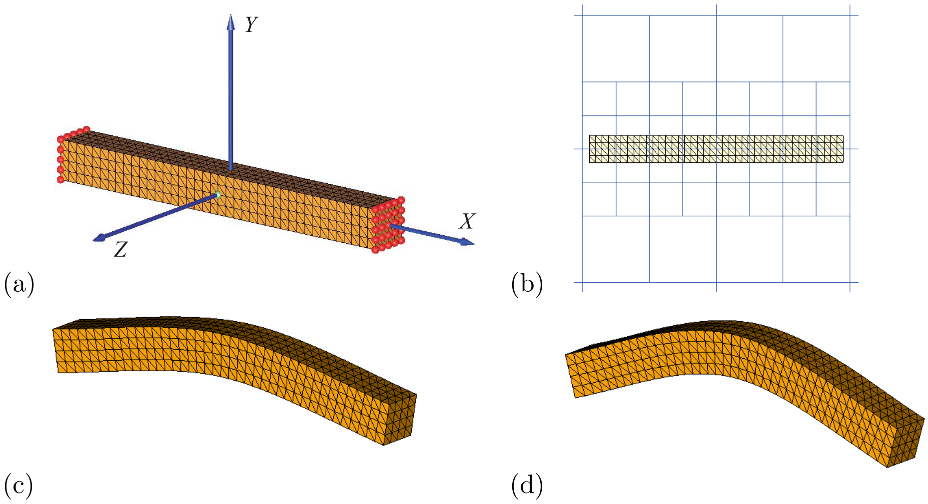

FIGURE 6. Bent beam.

element approach based on the illustrated discretization of the beam would imply.

6.2. **Numerical results.** We present two examples to illustrate the effectiveness of elastic shape modeling. For the presented results we chose $\mu = 2$ and $\lambda = 2$ when computing the elastic energy $E(\mathbf{u})$. Numerical integration is used to evaluate both $E(\mathbf{u})$ and the penalty function $F(\mathbf{u})$. The penalty weight is set to be $\omega = 10{,}000$. Both examples are concerned with a rectangular beam which is deformed according to two different velocity distributions.

In the first example we bend the beam as illustrated in Figure 6. The initial shape and the boundary constraints for deformation are illustrated in Figure 6(a). The the nodes at the ends of the beam and in its center correspond to the boundary Γ where the infinitesimal shape deformation is prescribed. The right face of the beam is rotated about the negative Z-axis and the left face about the positive Z-axis, while the intersection line segment between the beam and the Z-axis is fixed during the deformation. Figure 6(b) shows the T-spline control grid (T-mesh), which is adapted to the geometry of the beam model. The results of the elastic deformation at two times are given in Figures 6(c) and (d).

In the second example we twist the rectangular beam about the X-axis. The initial shape of the object and the boundary constraints for deformation are illustrated in Figure 7(a). The right face of the beam is now rotated about the positive X-axis and the left face about the negative X-axis. The sequence of the deformed objects is shown in Figures 7(b), (c), (d) and (e).

7. CONCLUSION AND OUTLOOK

We introduced the elastic deformation energy (2.5) of infinitesimal deformations of shapes based on the elastic energy of an isotropic material. The energy is invariant to Euclidean transformations and applies to 1- and 2-dimensional shapes. In contrast to previously proposed metrics on shape

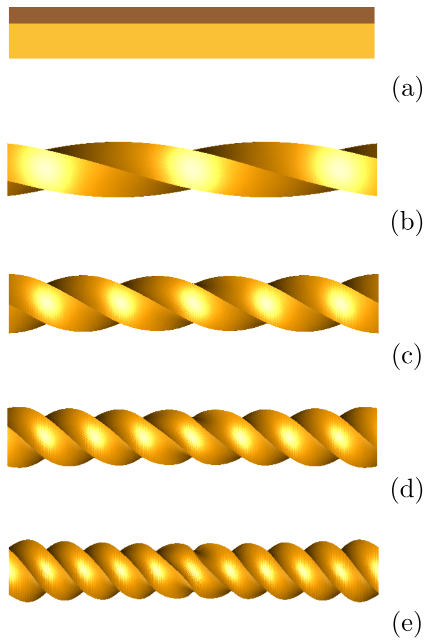

Figure 7. Twisted beam.

manifolds it considers the interior of the shape instead of only the shape boundary. Thus, it is naturally defined for multiply connected shapes. We proved existence and uniqueness of minimizers in the variational formulation (2.5) of the elastic deformation energy.

The energy induces the elastic deformation metric (2.7). In Section 5 we presented geodesics with respect to this metric which connect planar B-spline shapes. The use of the elastic deformation energy for shape modeling in space is illustrated in Section 6.

From the theoretic point of view several open questions remain. As observed in Remark 2.6, the elastic deformation metric is symmetric and satisfies the triangle inequality by definition. However, we are not able to show that $d(a,b) > 0$ if $a \neq b$ (modulo Euclidean transformations). This is due to the fact that we can not give suitable estimates for the dependence of the constant C (*Korn's constant*) in Korn's inequality (4.3) on the domain Ω. Moreover the existence of geodesics, i.e. the existence of minimizers in (2.7), is an open problem.

Our approach allows for general topologies of shapes but it requires the topology to stay the same during the evolution. However, if parts of shapes are torn apart or merged together during their deformation, the shape topology changes. To handle such cases the elastic deformation energy has to be adapted in a suitable way.

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