SOLVING CONSTRAINT ILL-POSED PROBLEMS USING GINZBURG-LANDAU REGULARIZATION FUNCTIONALS*

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Abstract. We consider constraint ill-posed operator equations such that we restrict the domain to functions, which in the first case are piecewise constant and in the second case attain values in a certain interval. We use Ginzburg-Landau regularization methods for solving this equations. In our numerical examples we consider the inverse conductivity problem which has applications in electrical impedance tomography. We present a numerical implementation along with some results and compare them with standard H^1-Tikhonov regularization.

Key words. Ill-Posed Problems, Regularization Methods, Inverse Conductivity

AMS subject classifications. 47A52, 65J20

1. Introduction. The goal of this paper is to solve the constraint ill-posed operator equation

$$F(z) = y \tag{1.1}$$

where F is a nonlinear continuous operator from $L^1(\Omega)$ into a Hilbert space Y. Here we discuss (1.1) with two different constraints:

C1: We restrict z to be in a set of admissible functions $\mathcal{P} := \{z : z = 1 + \chi_D\} \subset L^1(\Omega)$ where $D \subset\subset \Omega$ is a set of finite perimeter and χ_D denotes the characteristic function of the set D.

C2: The function z attains values in a certain interval $[a, b]$.

For solving (1.1) under C1 we work out the following methods. One possible way to enforce the restriction is by introducing a projection operator

$$P := \left\{ \begin{array}{ccc} H^1(\Omega) & \to & \mathcal{P} \subset L^1(\Omega) \\ \phi(\cdot) & \mapsto & \left\{ \begin{array}{ll} 1 & \text{for } \phi(\cdot) < 0 \\ 2 & \text{for } \phi(\cdot) \geq 0 \end{array} \right. \end{array} \right. \tag{1.2}$$

and involve P into the equation (1.1), i.e.

$$F(P(\phi)) = y\,. \tag{1.3}$$

Then we consider an ill-posed operator equation where we can decompose the operator into a continuous and a discontinuous part. Such a problem was investigated in [8, 9] by regularization, which means to minimize the functional

$$\frac{1}{2}\|F(P(\phi)) - y\|_Y^2 + \alpha\mathcal{R}(\phi) \tag{1.4}$$

over $H^1(\Omega)$, where $\mathcal{R}$ is a regularization functional.

A regularization approach involving the projection operator (1.2) is called *level set regularization* since we recover the boundary of an object which we regard as the

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zero level set of the function ϕ. Considering characteristic functions as level sets of higher dimensional data has been used before in the context of multiphase flow (see e.g. [16, 22, 6]) and segmentation (see e.g. [5]). Level set methods have been used successfully in many applications since the pioneering work of Osher and Sethian [19]. For solving inverse problems with level sets we refer to [20, 4, 8].

Another possibility to constrain z to be in $\mathcal{P}$ is by introducing an appropriate penalizer. To solve (1.1) under constraint C1 we minimize the functional

$$\frac{1}{2}\|F(z)-y\|_Y^2 + \alpha \mathcal{S}_\epsilon(z)\,. \tag{1.5}$$

We denote minimizers of (1.5) by $z_{\epsilon,\alpha}$. The functional $\mathcal{S}_\epsilon$ is taken in such a way, that for $\epsilon \to 0$ we have $z_{\epsilon,\alpha}(x) \in \{1,2\}$. We choose a real *Ginzburg-Landau* (GL) type functional as regularizer $\mathcal{S}_\epsilon$ in (1.5), which means

$$\mathcal{S}_\epsilon(z) := \int_\Omega \left(\frac{\epsilon}{2}|\nabla z|^2 + \frac{1}{\epsilon}W(z)\right)\,dx \tag{1.6}$$

where W is an appropriate potential. Such GL functionals were also used in topological optimization, see [14] or for modelling immiscible fluids, cf. [1, p.99].

Next we consider constraint C2. A standard approach for solving nonlinear problems is regularization. Therefore we minimize

$$\frac{1}{2}\|F(z)-y\|_Y^2 + \alpha \mathcal{R}(z)\,. \tag{1.7}$$

Solving such nonlinear problems is discussed in [17, 18, 7], where classical results on convergence and stability are described.

To make use of the a-priori knowledge that z has values in $[a,b]$ we take a complex GL regularizer. We achieve this by using a complex valued function v with real part $z = \Re(v)$ and minimize

$$\frac{1}{2}\|F(z)-y\|_Y^2 + \alpha \mathcal{T}(v)\,. \tag{1.8}$$

We choose as regularizer a complex GL functional

$$\mathcal{T}(v) := \int_\Omega \left(\frac{\lambda}{2}|\nabla v|^2 + \frac{1}{\lambda}W(v)\right)\,dx\,, \tag{1.9}$$

where W is taken in an appropriate way to enforce $z \in [a,b]$.

So far such functionals were applied in digital inpainting to reconstruct missing areas in pictures, cf. [10]. Equations like (1.9) have proven to be useful in several areas, for example to describe phase transitions in superconductors near critical temperatures, cf. [13], or to model some types of chemical reactions like the famous Belousov-Zhabotinsky reaction, to specify boundary layers in multi-phase systems, and to describe the development of patterns and shocks in non-equilibrium systems, cf. [11, 15, 21]. To the best of our knowledge complex regularization functionals have not been used for solving ill-posed operator equations with constraint C2.

Note that in C1 we are only interested in reconstructing the shape of D whereas in C2 we have the possibility to determine concrete values of z.

The outline of this paper is as follows. In section 2 we describe the regularization methods considered in this paper. The existence of minimizers is guaranteed. Furthermore the optimality conditions and the numerical implementation for an inverse conductivity problem is described in section 3. Finally, in section 4 we present some results and discuss them.

2. The regularization functionals. In this section we analyze properties of the regularization functionals. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with sufficiently smooth boundary. First we consider (1.1) under constraint C1:

1. **Tikhonov:** To minimize (1.4) we follow [8], where detailed analysis to this method can be found, when $\mathcal{R}$ consists of a bounded variation seminorm term and the H^1-Tikhonov regularization functional. Here we just consider the H^1-Tikhonov regularization functional, i.e., $\mathcal{R}(\phi) = \frac{1}{2}\|\phi - \phi_0\|^2_{H^1(\Omega)}$. The function ϕ_0 is an a-priori guess. We approximate the discontinuous projection P by a Lipschitz continuous operator

$$P_\epsilon(\phi)(x) := \begin{cases} 1 & \text{for } \phi(x) < -\epsilon \\ 2 + \frac{\phi(x)}{\epsilon} & \text{for } -\epsilon \le \phi(x) \le 0 \\ 2 & \text{for } \phi(x) > 0 \end{cases}, \quad x \in \Omega\,. \tag{2.1}$$

Then we minimize the functional

$$\mathcal{F}_{\epsilon,\alpha}(\phi) := \frac{1}{2}\|F(P_\epsilon(\phi)) - y\|_Y^2 + \alpha\mathcal{R}(\phi) \tag{2.2}$$

over $H^1(\Omega)$ and denote the minimizer by $\phi_{\epsilon,\alpha}$. Thereafter we understand the minimizer ϕ_α of (1.4) as

$$\phi_\alpha = \lim_{\epsilon \to 0+} \phi_{\epsilon,\alpha}$$

where the limit is understood in an appropriate sense.
For the computation of a minimum of the functional (2.2) we need the variational derivative of $\mathcal{R}(\phi) = \frac{1}{2}\|\phi - \phi_0\|^2_{H^1(\Omega)}$. It is well known to be $\mathcal{R}'(\phi) = (I - \Delta)(\phi - \phi_0)$.

2. **Real GL:** Now we observe the functional $\mathcal{S}_\epsilon$ from (1.6) with the potential W. Here W denotes a symmetric double well potential, i.e., a potential with two minima. The location of the minima is chosen at the desired values of $z_{\epsilon,\alpha}$. For our problem an obvious choice would be

$$W(s) = \frac{1}{4}\left((s-2)\cdot(s-1)\right)^2\,. \tag{2.3}$$

For fixed $\alpha > 0$, according to a result of Modica & Mortola (see [1, sections 14 and 15]), $\lim\limits_{\epsilon \to 0} \mathcal{S}_\epsilon(z) < \infty$ if and only if $z = 1 + \chi_D$.
An easy calculation shows

$$\mathcal{S}'_\epsilon(z) = -\epsilon\Delta z + \frac{1}{\epsilon}(z-1)(z-2)(z-\frac{3}{2})\,. \tag{2.4}$$

Let us denote by $z_\epsilon(\cdot, t)$ the gradient flow of $\mathcal{S}_\epsilon$, i.e., the solution of the semilinear parabolic equation

$$\frac{\partial z}{\partial t} = \epsilon\Delta z - \frac{1}{\epsilon}\,(z-1)\,(z-2)(z-\frac{3}{2}) \tag{2.5}$$

which is a real GL equation. For a bounded, open set $E \subset\subset \Omega$ with smooth boundary we have that for $\epsilon \to 0$

$$z_\epsilon(x,t) \to 2, \text{ for } (x,t) \in \bigcup_{t\in[0,T)} E_t \times \{t\} \tag{2.6}$$

$$z_\epsilon(x,t) \to 1, \text{ for } (x,t) \in \bigcup_{t\in[0,T)} (\Omega \setminus E_t) \times \{t\} \tag{2.7}$$

for solutions of (2.5) with $z_\epsilon(\cdot,0) = 1 + \chi_E^\sigma$. Here χ_E^σ denotes a slightly smoothed version of χ_E with $\mathrm{sgn}(\chi_E^\sigma) = \mathrm{sgn}(\chi_E)$, and E_t denotes the set which originates from E if its boundary ∂E evolves under mean curvature motion up to time T. In this sense (2.5) approximates mean curvature motion as ϵ tends to zero. Further details can be found in [1].

To solve (1.1) under constraint C2 we observe the following methods:

1. **Tikhonov:** To minimize the functional (1.7) we choose again the H^1-Tikhonov regularization functional $\mathcal{R}(z) = \frac{1}{2}\|z - z_0\|^2_{H^1(\Omega)}$. Here we do not introduce the constraint $z \in [a, b]$.
2. **Complex GL:** Finally we consider the complex GL functional $\mathcal{T}$ from (1.9). In the example in 4.2 we have the a-priori knowledge that $z(x) \in [1, 2]$ for $x \in \Omega$. Therefore we set

$$W(s) = \frac{1}{4}\left(|s - \frac{3}{2}|^2 - \frac{1}{4}\right)^2 \tag{2.8}$$

which attains its minimum on the complex sphere with center $\frac{3}{2}$ and radius $\frac{1}{2}$. Thus we force z to have values in $[1, 2]$.
We find the variational derivative of $\mathcal{T}$

$$\mathcal{T}'(v) = -\lambda\Delta v + \frac{1}{\lambda}\left(|v - \frac{3}{2}|^2 - \frac{1}{4}\right)\left(v - \frac{3}{2}\right) . \tag{2.9}$$

Solutions of $-\lambda\Delta v + \frac{1}{\lambda}\left(|v|^2 - 1\right) v = 0$ have been investigated in detail in [2]. Equation (2.9) is attained by an appropriate coordinate transform and therefore its solutions have similar properties.

We can guarantee the existence of a minimizer for $\mathcal{F}_{\epsilon,\alpha}$.

THEOREM 2.1. *The functional $\mathcal{F}_{\epsilon,\alpha}$ from (2.2) attains a minimizer $\phi_{\epsilon,\alpha}$, if $\mathcal{R}$ is weak lower semi continuous, if there exist $c > 0$ and $b \in \mathbb{R}$ such that $\mathcal{R}(\phi) > c\|\phi\|^2_{H^1(\Omega)} + b$ for all $\phi \in H^1(\Omega)$, and there exists at least one $\tilde{\phi} \in H^1(\Omega)$ such that $\mathcal{F}_{\epsilon,\alpha}(\tilde{\phi})$ is finite.*

The proof for the Tikhonov regularization can be found in [8]. To fulfill the required conditions of the regularization functional we need appropriate boundary conditions for the GL regularizers, for example see section 3.2. Then the theorem is also applicable for the functionals (1.5), (1.7) and (1.8).

Note that the assumptions of the theorem do not necessarily imply that $\lim_{\epsilon\to 0+} P(\phi_{\epsilon,\alpha}) \in \mathcal{P}$. In [8] this condition was enforced by introducing the BV-seminorm $|P_\epsilon(\phi)|_{BV}$ as additional penalizing term.

3. Numerics. To receive a minimizer of the functionals we calculate the formal optimality conditions. In section 2 we derived the variational derivatives of the regularizing terms. Now we consider the derivative of the data term.

3.1. The variational derivative of the data term. In our numerical examples we consider an operator F which corresponds to the inverse conductivity problem. We can formulate the problems as follows:

- **C1:** We are interested in determining inclusions of constant conductivity in a surrounding medium of a different constant conductivity.
- **C2:** We want to reconstruct conductivities which are assumed to take on values in a certain interval.

Therefore we consider the Neumann problem

$$\begin{aligned} \nabla\cdot(a\nabla u) &= 0 \quad \text{in } \Omega\,, \\ a\tfrac{\partial u}{\partial\nu} &= h \quad \text{on } \partial\Omega\,, \\ \textstyle\int_{\partial\Omega} u\,ds &= 0\,. \end{aligned} \tag{3.1}$$

It is well known that there exists a unique solution $u\in H^1(\Omega)$ if $h\in L^2_\diamond(\partial\Omega):=\left\{u\in L^2(\partial\Omega):\int_{\partial\Omega}u\,ds=0\right\}$ and $a\in L^\infty_+(\Omega)$. In C1 we restrict us to conductivities

$$a=\begin{cases} 2 & \text{in } D \\ 1 & \text{in } \Omega\setminus\overline{D}\,. \end{cases} \tag{3.2}$$

In C2 the conductivity a is piecewise $C^2(\overline{\Omega},[1,2])$. These restrictions on the conductivity guarantee that $a\in L^1(\Omega)\cap L^2(\Omega)$. Let $g=u|_{\partial\Omega}\in L^2_\diamond(\partial\Omega)$ be the trace of u and $h\not\equiv 0$ fixed. We introduce the operator

$$F_h:\begin{cases} L^1(\Omega) & \to & L^2_\diamond(\partial\Omega) \\ a & \mapsto & g \end{cases}\quad. \tag{3.3}$$

The inverse problem consist in reconstructing the unknown conductivity a from the knowledge of the Neumann data h and the corresponding Dirichlet data g. Then the following proposition applies, cf. [12, Theorem 5.7.1].

PROPOSITION 3.1. *Let D_1, D_2 be open subdomains of Ω with Lipschitz boundaries such that $\Omega\setminus\overline{D_i}$, $i=1,2$ are connected and $D_i\subset\subset\Omega$. Assume γ is a given $C^2(\overline{\Omega})$-function and*

$$a_i:=\gamma+\chi_{D_i}k_i$$

where k_i is $C^2(\overline{\Omega})$-function with $k_i\neq 0$ on ∂D_i, $i=1,2$. Then

$$F_h(a_1)=F_h(a_2) \textit{ implies } a_1=a_2.$$

The same result is applicable if $a_i\in W^{1,p}(\Omega)$ for $p>1$, cf. [3].

Let us now take $Y=L^2_\diamond(\partial\Omega)$, then we receive the variational derivatives

$$\frac{1}{2}\frac{\delta}{\delta z}\|F(z)-y\|^2_Y := F'(z)^*(F(z)-y) \tag{3.4}$$

and

$$\frac{1}{2}\frac{\delta}{\delta\phi}\|F(P_\epsilon(\phi))-y\|^2_Y := P'_\epsilon(\phi)F'(P_\epsilon(\phi))^*(F(P_\epsilon(\phi))-y) \tag{3.5}$$

where the adjoint is taken with respect to the $L^2(\Omega)$ scalar product. Following [9] the adjoint operator $F'_h(a)^*$ of the Frechet-derivative of F_h has the following form

$$F'_h(a)^*(s)=-\nabla u\cdot\nabla w_s, \quad \text{for each } s\in L^2_\diamond(\partial\Omega),$$

where u is a solution of (3.1) and w_s solves (3.1) with Neumann boundary condition $a\frac{\partial w_s}{\partial\nu}=s$. Therefore we receive the following algorithm to calculate $F'_h(P_\epsilon(\phi))^*(F_h(P_\epsilon(\phi))-y)$:

1. Calculate $F_h(P_\epsilon(\phi))$: Solve the Neumann problem

$$\begin{aligned} \nabla\cdot(P_\epsilon(\phi)\nabla u) &= 0 \quad \text{in } \Omega\,, \\ P_\epsilon(\phi)\tfrac{\partial u}{\partial\nu} &= h \quad \text{on } \partial\Omega\,, \\ \textstyle\int_{\partial\Omega} u\,ds &= 0\,. \end{aligned} \tag{3.6}$$

Then $F_h(P_\epsilon(\phi))=u|_{\partial\Omega}$.

2. Evaluate the residual $r = F_h(P_\epsilon(\phi)) - y$.
3. Calculate $F_h'(P_\epsilon(\phi))^*(r)$: Solve the Neumann problem

$$\begin{aligned} \nabla \cdot (P_\epsilon(\phi)\nabla w) &= 0 && \text{in } \Omega\,, \\ P_\epsilon(\phi)\tfrac{\partial w}{\partial \nu} &= r && \text{on } \partial\Omega\,, \\ \textstyle\int_{\partial\Omega} w\,ds &= 0\,. \end{aligned}$$

Then $F_h'(P_\epsilon(\phi))^*(r) = -\nabla u \cdot \nabla w$.

Note that the same minimization algorithm applies to $F_h'(z)^*(F_h(z) - y)$ if we replace $P_\epsilon(\phi)$ by z.

3.2. Implementation. The approach for solving C1 and C2 is essentially the same and therefore we do not distinguish between the two cases unless it is special mentioned.

1. **Tikhonov:** We use the same implementation as in [8].
2. **GL:** To minimize the functionals (1.5), resp., (1.8) we employ a gradient descent method, i.e., we solve the "time dependent" PDE

$$\frac{\partial z}{\partial t} = -F'(z)^*(F(z) - y) - \alpha \mathcal{S}_\epsilon'(z) \tag{3.7}$$

resp.,

$$\frac{\partial v}{\partial t} = -F'(z)^*(F(z) - y) - \alpha \mathcal{T}'(v) \tag{3.8}$$

up to a stationary state in time.

Since all solution methods are iterative we have to prescribe an initial guess ϕ_0, z_0 or v_0. Numerical experiments have shown that the solutions do not depend on the choice of the initial guess as long as some natural assumptions are fulfilled, see section 4.

For our numerics we want to reconstruct conductivities of the form given in proposition 3.1 with $\gamma \equiv 1$. The inclusion D is not necessarily connected. For C1 we set $k = 1$ and use the following boundary conditions. For the Tikhonov regularization method we adopt Neumann boundary conditions, cf. [8]. The boundary value $z|_{\partial\Omega} = 1$ for the real GL regularizer was chosen to be a stationary value of the potential W. For C2 we get better results if we introduce the Dirichlet boundary condition $z|_{\partial\Omega} = 1$ to the Tikhonov regularization method, since D should be completely contained in Ω. For the complex GL regularization we use the Dirichlet boundary data equal to $v = 1$ since this is a minimum of the corresponding potential.

We use a finite difference discretization, and perform explicit Euler time stepping for all methods except the Tikhonov regularization. The two elliptic PDEs related to the operator F (see section 3.1) are solved directly by using Matlab's backslash operator.

For our numerical examples we used $\Omega = (0,1)^2$ discretized with a homogeneous rectangular grid with gridsize Δx. For all regularizations in C1 the parameter ϵ corresponds roughly to the width of the inclusion boundary in the solution. Thus we set $\epsilon = \Delta x$ to achieve maximally sharp object boundaries within the limits imposed by the discretization.

The Neumann boundary data h to calculate the operator F_h (see (3.6)) were chosen as two periods of the sine function extending around the unit square.

The boundary measurement data y for solving the inverse problem are obtained by solving the elliptic boundary value problem (3.1). In order to avoid "inverse crimes" the direct problem (3.1) is solved on a finer grid than the inverse problem.

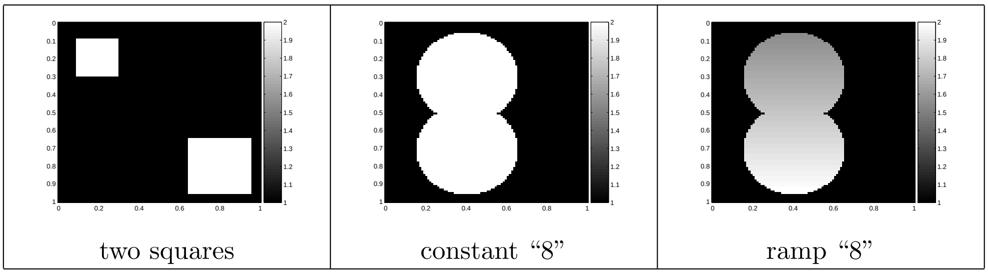

FIG. 4.1. *The three conductivities which should be reconstructed.*

4. Results and Discussion.

4.1. Examples for C1. In our experiments we take random initial data for real GL regularization, such that $1 \leq z_0(x) \leq 2$ for all $x \in \Omega$. For this choice the initial values lie between the two minima of the potential W. We used a signed distance function centered in Ω as initial condition such that $P_\epsilon(\phi_0)|_{\partial\Omega} = 1$ and that $P'_\epsilon(\phi_0) > 0$ for some $x \in \Omega$ for the Tikhonov regularization.

The conductivities which we want to reconstruct are plotted in figure 4.1 left, resp., middle. In figure 4.2, resp., 4.3 we show results obtained by using different regularizers for reconstructing. For the real GL regularizer we directly plot the solution z, whereas for the Tikhonov we plot the projection $P_\epsilon(\phi)$. Note that the resolution in figure 4.1 is finer than in the other figures.

In the first experiment we want to reconstruct two squares located at opposite corners of Ω. For both regularizers a spurious signal remains in the center of Ω for many iterations. The reason for this phenomenon is that the influence of the boundary values spreads slowly into the interior. After the spurious signal has vanished higher values remain only in the locations of the squares. The shape of the two squares is better recognized with the real GL regularization, therefor with the Tikhonov regularizer the boundaries are sharper.

In the second experiment a nonconvex inclusion should be reconstructed. We derive nearly the same effects for both regularizers as in the previous example. Again good reconstructions are achieved by the Tikhonov and the real GL regularization but the concavity on the right side is lost by the Tikhonov regularization.

4.2. Examples for C2. For C2 we take initial data as follows:

1. Tikhonov: $z_0(x) = 1$ for all $x \in \Omega$
2. complex GL: $\Re(v_0(x)) = 1$ and $\Im(v_0(x)) = 0$ for all $x \in \Omega$

In the experiment we want to reconstruct the same “8”-shaped inclusion as for C1, but now we have a slope in the conductivity, cf. the right picture in figure 4.1. Note that this conductivity is not in $H^1(\Omega)$ but the minimizers of the functionals necessarily are in $H^1(\Omega)$.

The results are plotted in figure 4.4. The Tikhonov regularization reconstructs the maximum values better, but there is a distinctive minimum of the values in the middle of the “8”. The slope of the conductivity is reconstructed by the complex GL method. Furthermore the reconstruction of the surrounding medium is better by the complex GL regularization.

Now we consider the same example, where we add 3 % gaussian noise to the data. The reconstructions are shown in figure 4.5. Again the maximum values in the object are found better by the Tikhonov regularizer. But the slope is not recognized

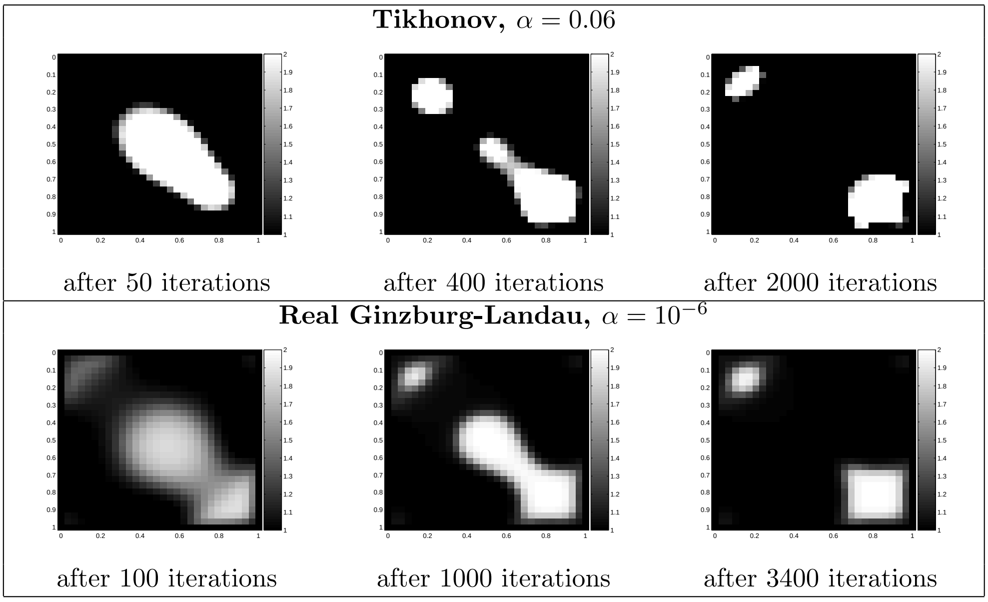

FIG. 4.2. *Course of evolution for the regularizers of C1*

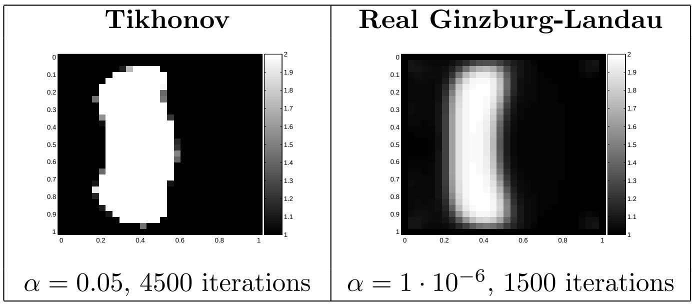

FIG. 4.3. *Stationary solutions of the regularizes of C1.*

satisfactory as well as the surrounding medium. The reconstruction with the complex GL regularizer of the surrounding medium is quite good. The constant slope of the conductivity in the object is found, but the values are not correct.

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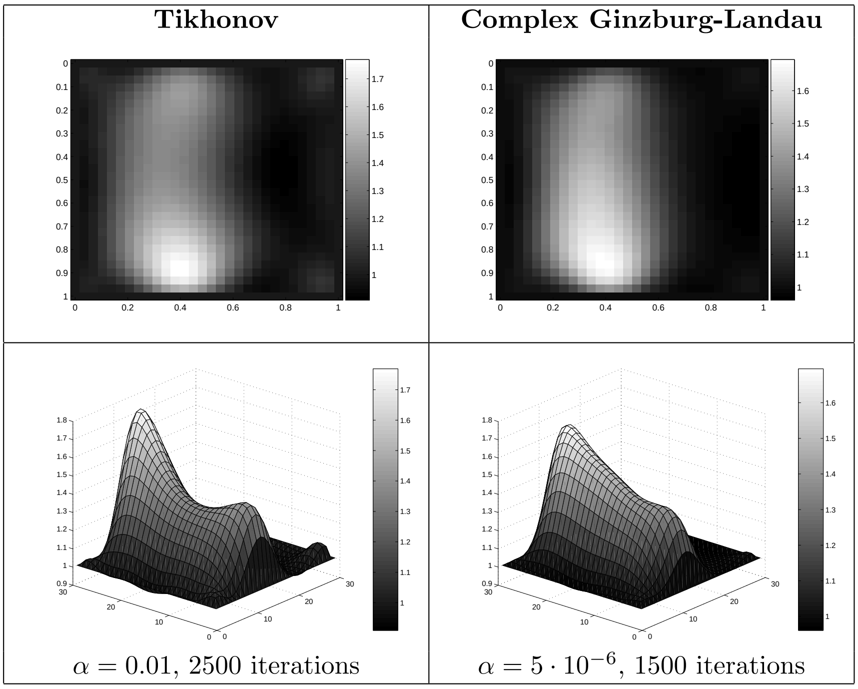

FIG. 4.4. *Stationary solutions of both regularizes for C2. Note that the results are differently scaled.*

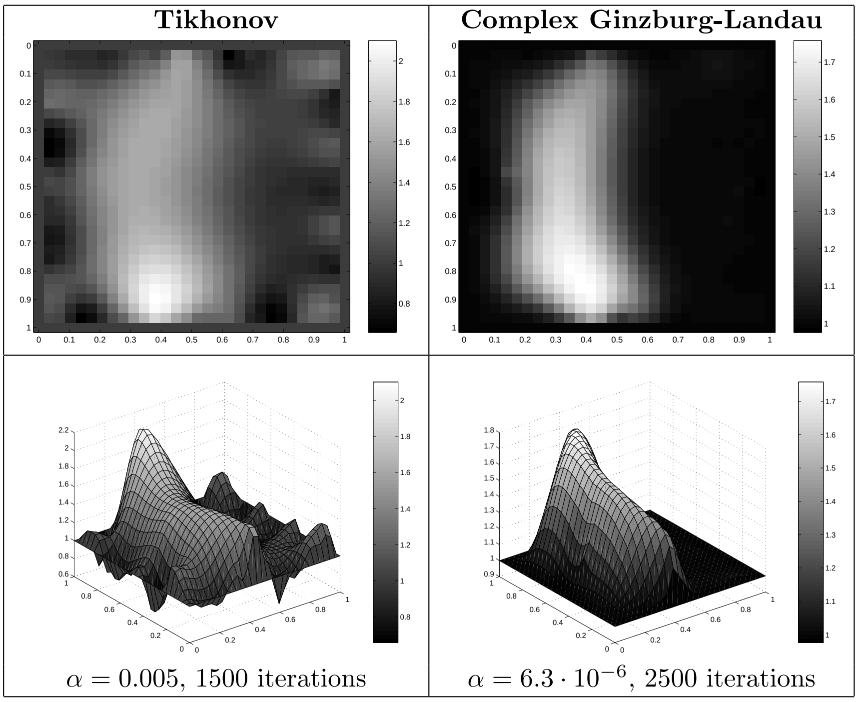

FIG. 4.5. *Stationary solutions of both regularizes for C2 with noise. Note that the results are differently scaled.*

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