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Convex Inverse Scale Spaces

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CONVEX INVERSE SCALE SPACES

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Abstract. Inverse scale space methods are derived as asymptotic limits of iterative regularization methods. They have proven to be efficient methods for denoising of gray valued images and for the evaluation of unbounded operators.

In the beginning, inverse scale space methods have been derived from iterative regularization methods with squared Hilbert norm regularization terms, and later this concept was generalized to Bregman distance regularization (replacing the squared regularization norms); therefore allowing for instance to consider iterative total variation regularization. We have proven recently existence of a solution of the associated *inverse* total variation flow equation. In this paper we generalize these results and prove existence of solutions of inverse flow equations derived from iterative regularization with general convex regularization functionals.

We present some applications to filtering of color data and for the stable evaluation of the DiZenzo edge detector.

1. Introduction

There are at least two evolutionary concepts based on partial differential equations for data filtering:

Scale space methods with time dependent partial differential equations approximate data u^δ (for instance images) by the solution of evolution equations (see e.g. [15]) at some time $t > 0$. The value of t controls the amount of filtering.

Inverse scale space methods as introduced in [12] are defined as the semigroups corresponding to iterative regularization

$$\begin{aligned} u_{k+1} &= \operatorname*{argmin}_{u \in H_1} \left\{ \frac{1}{2\alpha} \left\| u - u^\delta \right\|_1^2 + \frac{1}{2} \left\| L\left(u - u_k\right) \right\|_2^2 \right\} \\ &= \operatorname*{argmin}_{u \in H_1} \left\{ \frac{1}{2\alpha} \left\| u - u^\delta \right\|_1^2 + \frac{1}{2} \left\| L(u) \right\|_2^2 - \langle L^* L u_k, u \rangle_1 \right\} \end{aligned} \tag{1}$$

where $L : H_1 \to H_2$ is a linear and densely defined operator between two Hilbert spaces H_1 and H_2 and L^* denotes its adjoint. Here one typically initializes $u_0 = 0$ or $u_0 = \int_\Omega u^\delta \, \mathrm{d}x$ and u_{k+1} satisfies the Euler-Lagrange equation

$$u_{k+1} - u^\delta = \alpha \left(L^* L(u_{k+1}) - L^* L(u_k) \right). \tag{2}$$

Identifying the regularization parameter α and a time discretization Δt via $\Delta t = \alpha^{-1}$ equation (2) can be considered as an implicit time step of the following flow equation

$$\begin{aligned} u - u^\delta &= \left(L^* L u \right)', \\ u(0) &= u_0 \,. \end{aligned} \tag{3}$$

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For instance, for $L = \nabla$ we have that $L^*L = -\triangle$ and (3) becomes *Showalter's method* which has been used successively for image denoising and for the stable evaluation of gradients (see e.g. [10]).

Two different approaches for generating *nonlinear* inverse scale spaces have been considered that allow a consistent generalization of the linear case:

In [12] nonlinear evolution equations have been derived from variational regularization techniques on reflexive Sobolev spaces.

In [14] the flow according to the iterative Bregman distance of the total variation semi-norm has been derived, which has been analyzed in [5]. Iterative Bregman distance regularization reads as follows:

- The first step consists in computing a minimizer u_1 of ROF functional
$$u_1 := \operatorname{argmin} \frac{1}{2\alpha} \left\| u - u^\delta \right\|_{\mathrm{L}^2}^2 + |\mathrm{D}u|(\Omega).$$
- The $k+1$-th iterate is determined from
$$u_{k+1} = \operatorname*{argmin}_{u \in \mathrm{L}^2} \left\{ \frac{1}{2\alpha} \left\| u - u^\delta \right\|_{\mathrm{L}^2}^2 + |\mathrm{D}u|(\Omega) - \langle s, u \rangle \right\} , \tag{4}$$
where s is an element of the subgradient of the total variation semi norm at u_k.

Note that in the linear case (1) we always have that L^*Lu is an element of the subgradient of $\frac{1}{2} \|L(\cdot)\|_2$ at u. This shows that replacing the squared regularization norm in the iterative method (1) by the *Bregman distance* (of the total variation semi-norm) gives a consistent definition of nonlinear inverse scale spaces. It has been shown in [5] that for $\alpha \to \infty$, the functions $u_\alpha : [0, \infty) \to \mathrm{L}^2(\Omega)$ with values $u_\alpha(t) = u_k$ for $k - 1 \leq \alpha t < k$ converge to the unique solution of the flow equation

$$v'(t) = u^\delta - u(t), \quad |\mathrm{D}u(t)|(\Omega) = \langle u(t), v(t) \rangle . \tag{5}$$

In this paper we generalize the results of [14] and prove results for variational regularization with Bregman distances of arbitrary convex functionals. Moreover, we prove existence of solutions of according flow equations. In particular, the results give existence of solutions of flow equations for denoising of vector valued data such as color images. Moreover, the techniques can be applied for the stable evaluation of unbounded operators such as the DiZenzo edge detector.

2. Iterative Regularization with the Bregman Distance

In the sequel, without stating this explicitly, we always assume that $J : H \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous and proper functional defined on an Hilbert space H. The norm on H is denoted by $\|\cdot\|$ and is induced by the inner product $\langle \cdot, \cdot \rangle$.

The *domain* $D(J)$ of J denotes the set of all $u \in \Omega$ such that $J(u) < +\infty$.

In order to analyze iterative Bregman regularization and the according gradient flows we review basic results from convex analysis.

Definition 1 (Convex Analysis)**.** An element $s \in H$ is an element of the *subgradient* $\partial J(u)$ of J at $u \in H$ if

$$J(v) - J(u) - \langle s, v - u \rangle \geq 0 \text{ for all } v \in H .$$

The *Bregman* distance of $u, \tilde{u} \in H$ with respect to J and $s \in \partial J(\tilde{u}) \subseteq H$ is defined by

$$D_J^s(u, \tilde{u}) := J(u) - J(\tilde{u}) - \langle s, u - \tilde{u} \rangle . \tag{6}$$

Moreover, let

$$I(\alpha; u, \tilde{u}) := \left\{ \frac{1}{2\alpha} \left\| u - u^\delta \right\|^2 + D_J^{\tilde{v}}(u, \tilde{u}) \right\} \text{ for } \alpha > 0$$

With this notation iterative Bregman distance regularization reads as follows:

Algorithm 1. Let $u^\delta \in H$.

- Choose $u_0 \in D(J)$ and $v_0 \in \partial J(u_0)$.
- For $k = 0, 1, \ldots$

$$\begin{aligned} u_{k+1} &:= \operatorname*{argmin}_{u \in H} I\left(\alpha; u, u_k\right). \\ v_{k+1} &:= v_k + \frac{1}{\alpha}(u^\delta - u_{k+1}) . \end{aligned}$$

In the following we prove well-posedness of this algorithm and generalize some results in [6] for Bregman distance regularization with convex functionals J.

Theorem 1. *Assume that $u^\delta \in H$, $\alpha > 0$, $u_0 \in D(J)$ and $v_0 \in \partial J(u_0)$. Then for each $k \in \mathbb{N}$ there exists a unique minimizer $u_k \in H$ of $I(\alpha; \cdot, u_k)$ and a subgradient $v_k \in J(u_k)$ such that*

$$\alpha v_k + (u_k - u^\delta) = \alpha v_{k-1} \tag{7}$$

and

$$\left\| u_{k+1} - u^\delta \right\| \le \left\| u_k - u^\delta \right\| . \tag{8}$$

Proof. Let $\tilde{u} \in H$ and $s \in \partial J(\tilde{u})$. We show weak lower semicontinuity and coercivity of $I(\alpha; \cdot, u_k)$. Then, existence of a minimizer follows from [7, Chap. 3, Thm. 1.1]. Since both $u \to J(u)$ (J is convex) and $u \to \langle s, u \rangle$ are weakly lower semicontinuous on H, the Bregman distance

$$u \mapsto D_J^s(u, \tilde{u})$$

is weakly lower semicontinuous. Therefore $I(\alpha; \cdot, u_k)$ is weakly lower semicontinuous on H and proper. It remains to show that $I(\alpha; \cdot, u_k)$ is coercive on H, that is for every $k \in \mathbb{N}$ and every $\alpha > 0$ there exist constants $\lambda > 0$ and $\gamma \in \mathbb{R}$ such that

$$I(\alpha; u, u_k) > \lambda \left\| u \right\| + \gamma$$

for all u in H. We verify the assertion for the functional $I(\alpha; \cdot, u_0)$. For $k > 1$ the assertion can be proven analogously taking into account that $u_k \in D(J)$ and $v_k \in \partial J(u_k)$. Since J is convex we have that

$$J(u) \ge J(u_0) + \langle v_0, u - u_0 \rangle , \qquad \text{for all } u \in H.$$

Therefore,

$$\begin{aligned} \frac{1}{2\alpha} \left\| u - u^\delta \right\|^2 + J(u) - \langle v_0, u \rangle &\ge \frac{1}{2\alpha} \left\| u - u^\delta \right\|^2 + J(u_0) - \langle v_0, u_0 \rangle \\ &\ge \frac{1}{2\alpha} \left| \left\| u \right\| - \left\| u^\delta \right\| \right|^2 + J(u_0) - \langle v_0, u_0 \rangle . \end{aligned}$$

for all $u \in H$ and hence we conclude that $I(\alpha; \cdot, u_0)$ is coercive. Thus we can apply [7, Chap. 3, Thm. 1.1] to obtain existence of a minimizer u_1 satisfying the Euler-Lagrange equation

$$v_1 := \frac{u^\delta - u_1}{\alpha} \in \partial J(u_1).$$

Uniqueness follows from the strict convexity of $\left\| \cdot \right\|^2$.

Moreover since the Bregman distance is nonnegative we have

$$\begin{aligned}\frac{1}{2\alpha}\left\|u_{k+1}-u^\delta\right\|^2 &\le \frac{1}{2\alpha}\left\|u_{k+1}-u^\delta\right\|^2 + D_J^{v_{k+1}}(u_{k+1},u_k)\\ &= I(\alpha;u_{k+1},u_k) \le I(\alpha;u_k,u_k) = \frac{1}{2\alpha}\left\|u_k-u^\delta\right\|^2 .\end{aligned}$$

This shows (8) □

As we will see in Section 3, the dual formulation of Algorithm 1 turns out to be the key ingredient in order to establish the corresponding continuous inverse scale space. The dual formulation is based on the Fenchel transform defined by:

Definition 2. The *Legendre-Fenchel* conjugate of J is the functional $J^* : H \to \mathbb{R}\cup\{+\infty\}$ defined by

$$u^* \mapsto J^*(u^*) := \sup_{u\in H}\left\{\langle u^*,u\rangle - J(u)\right\} .$$

We consider the dual functional of I with respect to $v \in H$ which is given as follows:

$$I^*(\alpha;v,\tilde v) := \frac{\alpha}{2}\left\|v-\tilde v\right\|^2 + J^*(v) - \left\langle u^\delta, v\right\rangle , \quad \tilde v \in H. \tag{9}$$

Theorem 2. *Assume that $u^\delta \in H$, $\alpha > 0$, $u_0 \in D(J)$ and $v_0 \in \partial J(u_0)$. Then v_k as defined in Algorithm 1 satisfy*

$$v_k = \mathop{argmin}_{v\in H} I^*\left(\alpha;v,v_{k-1}\right) . \tag{10}$$

Proof. The functional I^* is strictly convex and weakly lower semicontinuous with respect to v and thus $I^*(\alpha;\cdot,v_{k-1})$ attains a unique minimizer $\tilde v_k$. It remains to show that $v_k = \tilde v_k$. From the definition of v_k in Algorithm 1 and Theorem 1 it follows that

$$v_k = v_{k-1} - \frac{1}{\alpha}(u_k - u^\delta) \in \partial J(u_k) . \tag{11}$$

Then, from the duality relation (see for instance [9])we have that

$$u_k \in \partial J^*\left(v_{k-1} - \frac{1}{\alpha}(u_k-u^\delta)\right) . \tag{12}$$

Moreover (11) is equivalent to $-\alpha(v_k - v_{k-1}) = u_k - u^\delta$ and this yields

$$(v_k - v_{k-1})\alpha - u^\delta = -u_k. \tag{13}$$

Combination of (11), (12) and (13) shows that

$$0 \in \alpha(v_k - v_{k-1}) - u^\delta + \partial J^*(v_k) = \partial I^*(\alpha;v_k,v_{k-1}) . \tag{14}$$

Therefore, v_k minimizes the functional $I^*(\alpha;\cdot,v_{k-1})$, which together with the fact that the minimizer is unique implies that $v_k = \tilde v_k$. □

For the inverse total variation flow equation – i.e. $J(u) = |\mathrm{D}u|(\Omega)$ – J^* is a barrier function. That is $J^*(v) = 0$ if and only if the G-norm (see [13], [3]) of $u \in \mathrm{BV}(\Omega)$ such that $|\mathrm{D}u|(\Omega) = \langle u, v\rangle$ is less than 1 (see e.g. [5]) and $+\infty$ else.

3. Continuous Inverse Scale Space Flow

In this section we show that the sequences $\{u_k\}$ and $\{v_k\}$ in Algorithm 1 can be considered as discrete approximations of the unique solution (u,v) of

$$v'(t) = u^\delta - u(t), \qquad v(t) \in \partial J(u(t)), \tag{15a}$$

$$v(0) = v_0 \qquad u(0) = u_0. \tag{15b}$$

The following analysis uses results from [2] where a theory of gradient flows in metric spaces has been established. Moreover, we show that the solution (u,v) of (15) satisfies the *inverse fidelity* axiom, that is

$$\lim_{t\to\infty} \left\| u(t) - u^\delta \right\| = 0$$

provided that $u^\delta \in \overline{D(J)}$. The last relation justifies to call (15) *inverse scale space method*.

For $\alpha > 0$, initial data $u_0 \in D(J)$ and $v_0 \in \partial J(u_0)$ and $k \in \mathbb{N}$ the Bregman iterates u_k and v_k are extended to piecewise constants functions $\overline{U}_\alpha(t)$ and $\overline{V}_\alpha(t)$ satisfying

$$\overline{U}_\alpha(t) = u_k, \quad \overline{V}_\alpha(t) = v_k, \qquad \text{for } \frac{k}{\alpha} \le t < \frac{k+1}{\alpha} \tag{16}$$

Convergence of the functions $\left\{\overline{V}_\alpha(t)\right\}$ for $\alpha \to \infty$ follows from [2, Thm. 4.2.2] and reads as follows:

Theorem 3. *Assume that $u^\delta \in H$, $u_0 \in D(J)$ and $v_0 \in \partial J(u_0)$. Then there exists a absolutely continuous function $v : [0,\infty) \to H$ such that*

$$\lim_{\alpha\to\infty} \overline{V}_\alpha(t) = v(t)$$

uniformly on every bounded $[0,T]$.

Proof. The assertion is a consequence of [2, Thm. 4.2.2]. In order to apply this theorem the following two assumptions have to be verified.

- **(A1)** The functional $\phi(v) = J^*(v) - \left\langle u^\delta, v\right\rangle$ is proper, lower semicontinuous and there exists $\tilde{v} \in D(\phi)$, and $\tilde{r} > 0$ such that

$$\inf\left\{\phi(v) : \|v - \tilde{v}\| \le \tilde{r}\right\} > -\infty. \tag{17}$$

- **(A2)** For every v_0, v_1 and w in $D(\phi)$ we have that

$$I^*(\alpha; (1-t)v_0 + tv_1, w) \\ \le (1-t)I^*(\alpha; v_0, w) + tI^*(\alpha; v_1, w) - \frac{\alpha}{2} t(1-t)\left\|v_0 - v_1\right\|^2$$

  for all $t \in [0,1]$ and $\alpha \ge 0$.

Since by assumption J is lower semicontinuous and proper the same holds for J^*. To verify the coercivity of J^* (assumption (17)) set $\tilde{v} = 0 \in D(\phi)$ and $w = \operatorname{argmin}_{v\in H} I^*(\alpha; v, 0)$. Then

$$-\infty < \frac{\alpha}{2}\left\|w\right\|^2 + \phi(w) - \frac{\alpha\tilde{r}^2}{2} \le \phi(v),$$

for all $v \in H$ such that $\|v - 0\| \le \tilde{r}$. This shows **(A1)**. Moreover, for all v_0, v_1 and w in H we have

$$\left\|(1-t)v_0 + tv_1 - w\right\|^2 = (1-t)\left\|v_0 - w\right\|^2 + t\left\|v_1 - w\right\|^2 - t(1-t)\left\|v_0 - v_1\right\|^2.$$

This implies that for $v_0, v_1, w \in D(\phi)$

$$
\begin{aligned}
\bar{I}^*(\alpha;(1-t)v_0+tv_1,w) &= \frac{\alpha}{2}\left\|(1-t)v_0+tv_1-w\right\|^2 - \langle(1-t)v_0+tv_1,w\rangle \\
&= \frac{\alpha}{2}\Big((1-t)\left\|v_0-w\right\|^2 + t\left\|v_1-w\right\|^2\Big) \\
&\quad - \frac{\alpha}{2}t(1-t)\left\|v_0-v_1\right\|^2 - (1-t)\left\langle v_0,u^\delta\right\rangle - t\left\langle v_1,u^\delta\right\rangle \\
&= (1-t)\bar{I}^*(\alpha;v_0,w) + t\bar{I}^*(\alpha;v1,w) - \frac{\alpha}{2}t(1-t)\left\|v_0-v_1\right\|^2.
\end{aligned}
$$

Therefore assumption **(A2)** holds. □

In [2] actually the stronger result has been shown, that for each partition of the interval $[0,\infty)$ the piecewise constant function converges to v provided that the supremum of the stepsizes tend to zero.

In order to show that the function v in Theorem 3 satisfies a gradient flow equation associated with ϕ we introduce the operator $\partial^0\phi : D(\phi) \subseteq H \to H$ defined by

$$
\begin{aligned}
\partial^0\phi(v) &= \operatorname*{argmin}_{u\in H}\{\|u\| \;:\; u\in\partial\phi\} \\
&= \operatorname*{argmin}_{u\in H}\left\{\left\|u-u^\delta\right\| \;:\; u\in\partial J^*(v)\right\}
\end{aligned}
$$

and the *slope function*

$$
|\partial\phi| : v \mapsto \begin{cases} \left\|\partial^0\phi(v)\right\| & \text{if } v\in D(\phi) \\ +\infty & \text{else.} \end{cases}
$$

Since the norm in the Hilbert space H is strictly convex, it follows that for every $v \in D(\phi)$ there exists a unique element u in $\partial\phi$ with minimal norm. Thus the operator $\partial^0\phi(v)$ is single valued and well defined.

Theorem 4. *Under the assumptions of Theorem 3 the function v is the unique solution of the gradient flow equation*

$$
v'(t) = -\partial^0\phi(v(t)), \quad \textit{for a.e. } t\in[0,\infty) \tag{18a}
$$

$$
v(0) = v_0. \tag{18b}
$$

Proof. The assertion the v solves (18) follows directly from [2, Thm. 2.3.3] and [2, Prop. 1.4.1]. The results are applicable if **(A1)** and **(A2)** hold and the slope function $|\phi|(v)$ is lower semicontinuous on H. The later follows from the strong-weak closedness of $\partial\phi \subseteq H\times H$ ([2, Lem. 2.3.6]). Uniqueness of v can be proven with standard arguments as for instance from [4]. The mapping $v \to \phi(v)$ is convex and therefore $\partial\phi$ is monotone, that is

$$
\langle u_1-u_2, v_1-v_2\rangle \geq 0 \text{ for all } v_i\in H, u_i\in\partial\phi(v_i),\ i=1,2.
$$

Assume that there exist two solutions $v, \hat{v}$ of (18), i.e.

$$
-v'(t)\in\partial\phi(v(t)), \quad -\hat{v}'(t)\in\partial\phi(\hat{v}(t)).
$$

From the monotonicity it follows that $\langle -v'(t)+\hat{v}'(t), v(t)-\hat{v}(t)\rangle \geq 0$ and therefore

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|v(t)-\hat{v}(t)\right\|^2 = \langle v'(t)-\hat{v}'(t), v(t)-\hat{v}(t)\rangle \leq 0.
$$

This shows that $v=\hat{v}$. □

From definition it follows that the right hand side of (18) can be written as $u(t)-u^\delta$ for a (unique) function $u : [0,\infty)\to H$ satisfying $u(t)\in\partial J^*(v(t))$ for all $t\in[0,\infty)$. This is equivalent to $v(t)\in\partial J(u(t))$ such that (u,v) is the unique solution of (15). In order to show the *inverse fidelity* property of the solution (u, v) we cite one assertion of [2, Thm. 4.3.2]

Theorem 5. *The solution (u, v) of (15) satisfies*

$$\left\|u(t) - u^\delta\right\|^2 \leq |\partial\phi|^2(\tilde{v}) + \frac{1}{t^2}\|\tilde{v} - v_0\| \tag{19}$$

for all $\tilde{v} \in D(\phi)$.

Corollary 1. *Let $u^\dagger$ denote the orthogonal projection of u^δ on to the closure of $D(J)$. Then we have that*

$$\limsup_{t\to\infty}\left\|u(t) - u^\delta\right\| \leq \left\|u^\dagger - u^\delta\right\|.$$

In particular for $u^\delta \in \overline{D(J)}$ it follows that

$$\lim_{t\to\infty}\left\|u(t) - u^\delta\right\| = 0\ .$$

Proof. Let $\varepsilon > 0$. Then there exists $u^\dagger_\varepsilon \in D(J)$ such that $\left\|u^\dagger_\varepsilon - u^\dagger\right\| < \varepsilon$. For $\tilde{v} \in \partial J(u^\dagger_\varepsilon)$ we have $u^\dagger_\varepsilon \in \partial J^*(\tilde{v})$ and consequently

$$|\partial\phi|(\tilde{v}) = \inf_{u\in H}\left\{\left\|u - u^\delta\right\|\ :\ u \in \partial J^*(\tilde{v})\right\} \leq \left\|u^\dagger_\varepsilon - u^\delta\right\| \leq \left\|u^\dagger - u^\delta\right\| + \varepsilon.$$

Using this inequality in (19) it follows that

$$\left\|u(t) - u^\delta\right\|^2 \leq \left(\left\|u^\dagger - u^\delta\right\| + \varepsilon\right)^2 + \frac{1}{t^2}\|\tilde{v} - v_0\|\ .$$

Taking into account that $\varepsilon > 0$ is arbitrary and taking $t \to \infty$ gives the assertion. □

The previous results show that v_k as in Algorithm 1 approximates the function $v(t)$ in the flow equation (18). In the sequel we show that $\overline{U}_\alpha(t)$ approximates the primal function u in (18) as well. We skip the proof since it uses exactly the same techniques as [5, Thm. 7].

Theorem 6. *Let $\overline{U}_\alpha(t)$ and u be as in* (16) *and* (15)*, respectively. Then*

$$\lim_{\alpha\to\infty}\overline{U}_\alpha(t) = u(t), \qquad \textit{almost everywhere in } [0,\infty)\ . \tag{20}$$

Corollary 2 (Monotonicity)**.** *Let $u^\delta \in H$. If (u, v) is the solution of (15) we have*

$$\left\|u(s) - u^\delta\right\| \leq \left\|u(t) - u^\delta\right\| \tag{21}$$

for almost all s, t in $[0,\infty)$ satisfying $s > t$.

Proof. Let $\{\alpha_l\}_{l\in\mathbb{N}} \subseteq \mathbb{R}^+$ $\lim_{l\to\infty}\alpha_l = \infty$ and that (20) holds for s and t. Then there exists an index l_0 such that for all $l > l_0$ we have that $\frac{1}{\alpha_l} < s - t$ and it consequently follows from (8) that

$$\left\|\overline{U}_{\alpha_l}(s) - u^\delta\right\| \leq \left\|\overline{U}_{\alpha_l}(t) - u^\delta\right\|, \qquad \text{for all } l > l_0.$$

With this we obtain the estimate

$$\begin{aligned}
\left\|u(s) - u^\delta\right\| &\leq \left\|\overline{U}_{\alpha_l}(s) - u^\delta\right\| + \left\|\overline{U}_{\alpha_l}(s) - u(s)\right\| \\
&\leq \left\|\overline{U}_{\alpha_l}(t) - u^\delta\right\| + \left\|\overline{U}_{\alpha_l}(s) - u(s)\right\| \\
&\leq \left\|\overline{U}_{\alpha_l}(t) - u(t)\right\| + \left\|\overline{U}_{\alpha_l}(s) - u(s)\right\| + \left\|u(t) - u^\delta\right\|.
\end{aligned}$$

Taking the limit $l \to \infty$ shows (21). □

4. Applications

In this Section we highlight some applications of inverse scale spaces and iterative Bregman distance regularization.

4.1. **Linear Inverse Scale Space.** In [12] we introduced inverse scale spaces as methods for evaluation of unbounded operators. Let $L : D(L) \subseteq H_1 \to H_2$ be a linear, closed, densely defined and unbounded operator between two Hilbert spaces H_1 and H_2. Since L is unbounded, computing $y = L(u)$ for a given $u \in H_1$ is ill posed. That is, for small perturbation u^δ of u it might be that $u^\delta \notin D(L)$ or that $\left\|L(u^\delta) - L(u)\right\|$ might be significantly large.

In order to provide a stable method for evaluation of L the *iterative Tikhonov-Morozov regularization* (1) can be used. In [12] it is shown that the discrepancy principle provides termination after a finite number of iterations. The discrepancy principle stops the iteration when for the first time the iteration error $\left\|u_k - u^\delta\right\|_1$ is below a given upper bound δ for the data error. As we have explained in Section 1 iterative Tikhonov Morozov regularization is equivalent to Bregman regularization with

$$J : u \mapsto \frac{1}{2}\left\|L(u)\right\|_2^2 .$$

Note that linearity of L implies convexity of J and lower semicontinuity follows from the closedness of L. From the analysis presented in Section 3 it follows that iterative Bregman distance regularization can be considered the solution of an implicit time step of

$$(L^*Lu)' = u^\delta - u, \quad u(0) = 0 \tag{22}$$

with time step size $\frac{1}{\alpha}$. Moreover since L is densly defined we see from Corollary 1 that

$$\lim_{t\to\infty} u(t) = u^\delta .$$

Let $\Omega \subseteq \mathbb{R}^n$ and $m \geq 1$.

If L denotes the gradient $D : \mathrm{H}^1(\Omega)^m \subseteq \mathrm{L}^2(\Omega)^m \to \mathrm{L}^2(\Omega)^{nm}$, then

$$L^*L(\vec{u}) = -\triangle\vec{u} := -\left(\triangle u_i\right)_{1\leq i\leq m} .$$

The according inverse scale space is the flow equation

$$(\triangle\vec{u})' = \vec{u} - \vec{u}^\delta, \quad \vec{u}(0) = 0 . \tag{23}$$

It can be used for stable evaluation of the derivative of a function $\vec{u} \in \mathrm{H}^1(\Omega)^m$ from noisy data $\vec{u}^\delta \in \mathrm{L}^2(\Omega)^m$. This is for instance useful for approximating the *diZenzo* edge detector [8] defined by

$$z(\vec{u})(x) = \left(D\vec{u}(x)\right)\left(D\vec{u}(x)\right)^T \in \mathbb{R}^{n\times n} . \tag{24}$$

The eigensystem of $z(\vec{u})$ at a point $x \in \Omega$ describes the geometry of the data. For $m = 1$ it is well known that the eigenvectors of $z(u)(x)$ are perpendicular and tangential to the level set of u at x and the corresponding eigenvalues are $\left|\nabla u\right|^2$ and 0 respectively.

In the numerical experiments we have calculated the DiZenzo edge detector with the inverse scale space method consisting in solving (23). Figure 1 shows the data (right column), the larger eigenvalue λ_1 (middle column) and the eigenvectors of $z(x)$ for a small subset of Ω (left column). The upper and middle row shows the original and noisy data respectively (additive Gaussian noise). The bottom row shows the solution of (22) at time $t = 0.9$.

4.2. **Inverse Total Variation Denoising.** Minimization of total variation based regularization functionals has turned out to be effective in image denoising applications.

Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set with piecewise Lipschitz boundary. We take $H = \mathrm{L}^2(\Omega)^m$, $m \geq 1$ and define for a vector valued function $u \in \mathrm{L}^2(\Omega)^m$ the

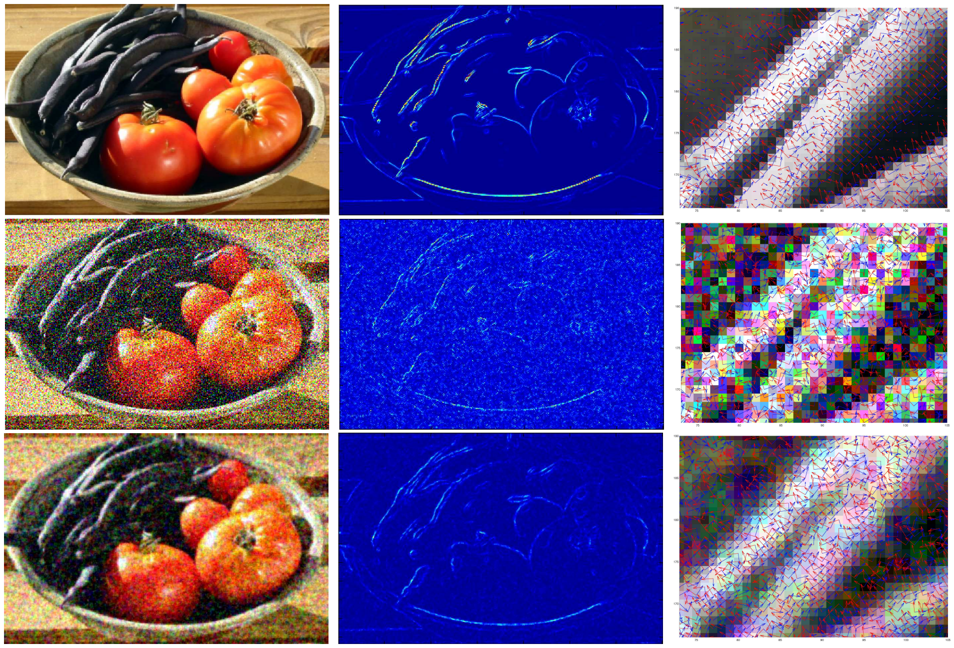

FIGURE 1. **Left column.** top: data, middle: noisy data, bottom: solution of (22) at $t = 0.9$. **Middle column.** corresponding edge detectors λ_1 **right column.** corresponding eigenvectors of z; zoom in some detail).

total variation semi-norm:

$$
(25)\quad J(u) := \left(\sum_{i=1}^{m} \left(\sup_{\substack{\phi \in \mathcal{C}_c^1(\Omega) \\ |\phi(x)|_2 \leq 1}} \int_\Omega \mathrm{div}\phi \; u_i \; \mathrm{d}x \right)^2 \right)^{\frac{1}{2}}
= \begin{cases} \left(\sum_{i=1}^{m} |\mathrm{D}u_i|(\Omega)^2 \right)^{\frac{1}{2}} & \text{if } u \in \mathrm{BV}(\Omega)^m \\ +\infty & \text{else} \end{cases} .
$$

J is convex and proper (since $\mathrm{BV}(\Omega)^m \cap \mathrm{L}^2(\Omega)^m \neq \emptyset$). Moreover J is lower semi-continuous w.r.t. $\mathrm{L}^2(\Omega)^m$ norm (see e.g. [11, Chap. 5.2, Thm.1] or [1, Thm. 2.3]). The numerical experiments in Figure 2 show the multi-scale evolution of a color image by the inverse total variation flow with J as in (25) and Figure 3 shows the texture part of the images.

5. Conclusion

In this paper we have generalized the analysis of inverse Bregman distance total variation regularization to Bregman distance regularization of arbitrary convex functionals. Moreover, we have derived the according flow equations and analyzed them using general results from [2]. We applied the results for filtering of color data and for the stable evaluation of DeZenzo's edge detector.

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FIGURE 2. Inverse TV scale space: original image (upper left), solutions of (15) at times $t = 1, 5$ and 20 (from left to right).

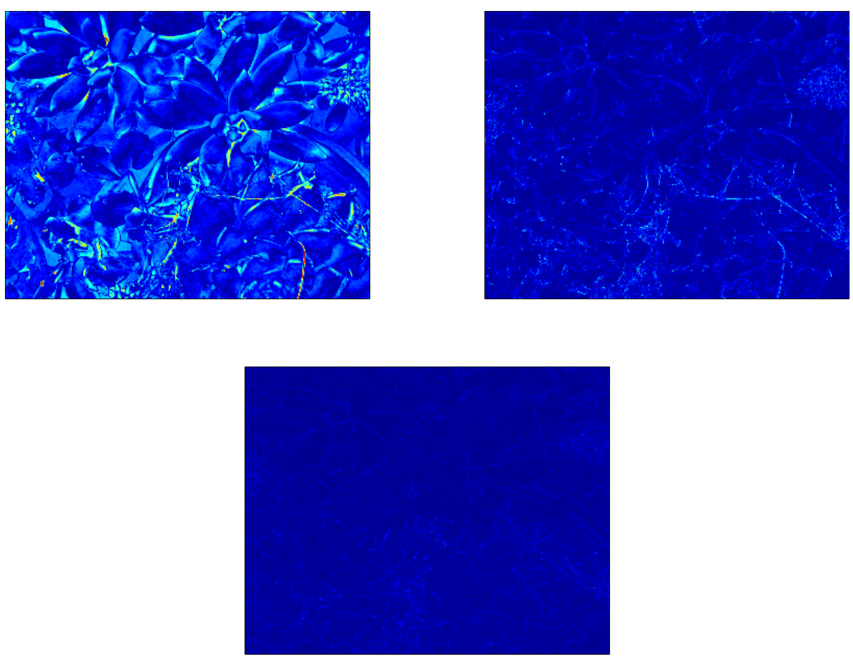

FIGURE 3. Difference images $|u^\delta - u(t)|_2$ for $t = 1, 5$ and 20.

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